Semi-stationary subdivision operators in geometric modeling*

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Abstract Based on the view of operator, a novel class of uniform subdivision construction method is proposed for free form curve and surface design. This method can give an appropriate manner of parameter change in the subdivision iteration with less parameters and better shape control, such as building local revolving surfaces. The convergent property of order 2 subdivision surface is elegantly analyzed using computing techniques of matrix. This method is promised to be much valuable in Computer Aided Design and computer graphics, due to the simplicity both in mathematical theory and practical implementation, the similarity to the B-spline curve and surface, G^1 continuity, the affine invariability and local flexible control.

Keywords: surface modeling, subdivision operator, subdivision surface, geometric continuity.

The subdivision surface methods[1~4],1) are now widely used in computer animation and geometric modeling because of their robust working in arbitrary topological mesh and successful theoretical analysis[4~8]. The common characteristic of these methods is

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1) Loop, C. Smooth subdivision surfaces based on triangle, Master's thesis. Dept Math, Univ of Utah, August 1987.

that the used parameters are fixed in each step of subdivision operation, which is called stationary subdivision. Since the shape of stationary subdivision surface is totally determined by control meshes, it is not convenient for user to adding further changes except mesh modification. Moreover the stationary subdivision surface can not accurately represent some ordinary surfaces used in engineering, such as cylinder, cone, sphere or revolving surfaces. The method proposed in Ref.[9] is suffered from the problem introducing heavy knot information and weight parameters. And the drawback of Level of Detail (LOD) solution[10] is the redundant storage for emplacing detail of hierarchy. Thus to find a subdivision method to uniformly represent quadric surface, revolving surface and traditional subdivision surface is an urgent issue. To remedy just about the weakness of the traditional ones, a new method called semi-stationary subdivision is proposed in this paper in which the parameters are changed by following rules. As an example, the C-subdivision surface can exactly represent a revolving surface in the local parts of a limit subdivision surface. Different from traditional methods, we define the subdivision schemes in operator forms which correspond to the geometric and topologic modeling manipulations and can be easily extended to general cases.

The convergent property is an important part in the research programs about subdivision surfaces. As the introduced subdivision matrix is not fixed, it is impossible for us to use the classic criterion[8,9] for subdivision convergence. Using the matrix computing techniques, we derive out the convergent condition and continuous conditions.

1 α-subdivision curve

In the following, we first decompose the curve subdivision procedure into a series of basic operators, and then to produce a new combined subdivision method; which can greatly extend the existed subdivision method.

Definition 1 (Linear operator over spatial point sequence). Let $\boldsymbol{p}_i, \boldsymbol{p}'_i \in \mathbb{R}^3, i \in \mathbb{Z}$. The linear operator over spatial data points $\boldsymbol{P} = \{\boldsymbol{p}_i\}_{i\in Z}$ to $\boldsymbol{P}' = \{\boldsymbol{p}'_i\}_{i\in Z}$ is

$$A: \boldsymbol{p}'_i = \sum_{j\in Z} a_{ij}\boldsymbol{p}_j \, , \tag{1}$$

where $\sum_{j\in\mathbb{Z}} a_{ij} = 1, a_{ij} \in \mathbb{R}$ and only finite number of a_{ij} is not zero.

It is obvious that the operator A defined in (1) possesses the linear property and compact support property. The former can ensure the affine invariance. The later can guarantee the local control of the limit curve. For convenient, we introduce a backward shifting $B: \boldsymbol{p}'_i = \boldsymbol{p}_{i+1}$, and convolving $C: \boldsymbol{p}'_i = (\boldsymbol{p}_i + \boldsymbol{p}_{i+1})/2$. Then we define following special linear operators.

Definition 2 (Second order bisect operator). Let $\boldsymbol{P}^{(n)} = \{\boldsymbol{p}_i^{(n)}\}_{i\in Z}$ be the nth steps of subdivision control data points. If the $(n+1)$th subdivision procedure satisfies

$$S_2: \boldsymbol{P}^{(n)} \mapsto \boldsymbol{P}^{(n+1)} := \begin{cases} \boldsymbol{p}_{2i}^{(n+1)} = \dfrac{1+2f_n}{2+2f_n}\boldsymbol{p}_i^{(n)} + \dfrac{1}{2+2f_n}\boldsymbol{p}_{i+1}^{(n)}, \\ \boldsymbol{p}_{2i+1}^{(n+1)} = \dfrac{1}{2+2f_n}\boldsymbol{p}_i^{(n)} + \dfrac{1+2f_n}{2+2f_n}\boldsymbol{p}_{i+1}^{(n)}, \end{cases} \tag{2}$$

where $f_n = f(\alpha; n) = f(2^{-n}\alpha)$ is a C^1 real function, we call the $\boldsymbol{P}^{(n)}$ to be processed by a second order bisect operator, denoted by $S_2(f;n)$, or shortly denoted by

S_2. The function f is called the kernel function of operator S_2, and $s := f(0)$ is the stiffness.

Definition 3 (k-order bisect operator).

$$\begin{cases} S_{2k}(f;n) := BCS_{2k-1}(f;n), & k = 2.3\ldots; \\ S_{2k+1}(f;n) := CS_{2k}(f;n), & k = 1,2\ldots. \end{cases} \tag{3}$$

Recursively applying S_k to it, the control data points will convergent to a limit curve with appreciate parameters and kernel function. Since the parameters are changed in special rules during the subdivision operation, the above procedure is called semi-stationary subdivision procedure. We have the following theorem.

Theorem 1. If the stiffness $s \in (-1/2, \infty)$, the subdivision scheme defined by $S_k(f;n)$ is convergent; and if $s \in (1/2, \infty)$, the convergent curve is GC^{k-1}.

Proof. Since $k = 2$ is a special case of Chapter 18.2 in Ref.[11], we only need to prove the operator C improves the smoothness. Given $\boldsymbol{P}^{(0)} = \left\{\boldsymbol{p}_i^{(0)}\right\}_{i \in Z}$, let curve $\boldsymbol{r}_k(t)$ be produced by $S_k(f;n)$. We define that $B_k(t-m)$ is produced by S_k when $\boldsymbol{P}_m^{(0)} = 1, \boldsymbol{P}_j^{(0)} = 0$ $(j \neq m)$. Thus $\boldsymbol{r}_k(t) = \sum\limits_{j \in \mathbb{Z}} \boldsymbol{p}_j^{(0)} B_k(t-j) = \sum\limits_{j \in \mathbb{Z}} \boldsymbol{p}_j^{(1)} B_k(2t-j)$, where the point sequence $\boldsymbol{P}^{(1)} = \boldsymbol{S}_k \boldsymbol{P}^{(0)}$ is computed by (2)and(3) ($\boldsymbol{S}_k$ represents the computing matrix). We denote the convolution of two functions is $f \otimes g(t) = \int_{-\infty}^{\infty} f(s) g(t-s) \mathrm{d}s$ and define $\overline{\boldsymbol{r}}(t) := \boldsymbol{r}(t) \otimes B_0(t) = \sum\limits_{j \in \mathbb{Z}} \boldsymbol{p}_j^{(0)} (B_k \otimes B_0)(t-j)$. Here $B_0(t)$ is the box function, whose value is equal to 1 in [0,1] and 0 outside. We have the following identical equation:

$$B_k(t) \otimes B_0(t) = \frac{1}{2}(B_k(2t) \otimes B_0(2t) + 2B_k(2t) \otimes B_0(2t-1) + B_k(2t-2) \otimes B_0(2t-2)).$$

Then one can verify $\overline{\boldsymbol{r}}_k(t) = \sum_{j \in \mathbb{Z}} \frac{1}{2}(\boldsymbol{p}_j^{(1)} + \boldsymbol{p}_{j+1}^{(1)})(B_k \otimes B_0)(2t-j)$. This means that

$\overline{\boldsymbol{r}}_k(t) \equiv \boldsymbol{r}_{k+1}(t)$ is produced by subdivision CS_k over $\boldsymbol{P}^{(0)} = \left\{\boldsymbol{p}_i^{(0)}\right\}_{i\in\mathbb{Z}}$.

Let $f \equiv 1$ in operator S_k, the limit curve is a *k*-order uniform B-spline curve, and the case of $k = 2$ is Chaikin's[12] algorithm. Note the convolving operator will enhance the smoothness of the subdivision surface. Let $f(\alpha;n) := \cos(2^{-(n+1)}\alpha)$ in operator S_k, then we can get extended C-B-spline subdivision method[13], which has similar local properties as standard B-spline and can precisely represent arc and ellipse arc. Similarly, we can choose $f(\alpha) = 1 - \alpha^2/2$ or $f(\alpha) = (e^{\alpha} + e^{-\alpha})/2$.

Since a family of curves can be defined by parameter α with fixed f and k in $S_k(f;n)$, we call them α-subdivision curves. In the same manner, we can also define corresponding tensor product form, α-subdivision surface. Furthermore we are extending it into the following quad-based subdivision scheme.

2 α-subdivision surface

As same as tensor product-derived schemes[1,4,9], which are derived from curve midpoint insertion subdivision method, a new class of subdivision surface can be defined based on the method described in the last section. And we still denote the derived subdivision method by $S_k(f;n)$.

2.1 Doo-Sabin type subdivision scheme

Doo-Sabin subdivision surface is the extension of the tensor product form quadric uniform B-spline. Since $S_2(f;n)$ can be viewed as a generalized version of quadric B-spline curve subdivision method, based on the traditional Doo-Sabin subdivision

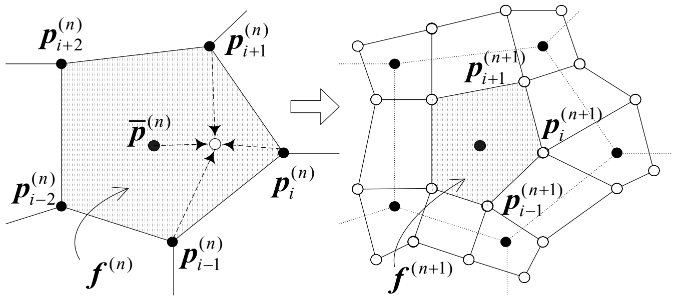

Fig.1. Doo-sabin type subdivision.

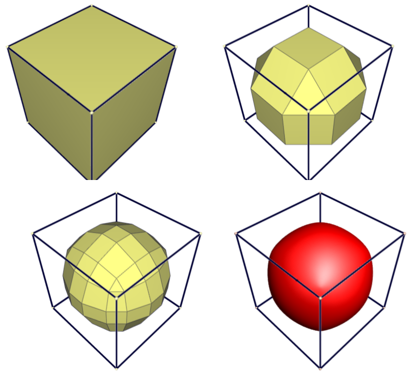

Fig.2. Applying subdivision operators on a cube.

topologic rules[4,9] (Fig.1), we can obtain a new scheme by changing the computing weights for new vertices only. Fig.2 is an example of this scheme. The formula of the new vertex geometric position for the n-th subdivision operation is

$$\boldsymbol{p}_i^{(n+1)} = \frac{f_n}{(1+f_n)}\boldsymbol{p}_i^{(n)} + \frac{f_n}{2(1+f_n)^2}(\boldsymbol{p}_{i-1}^{(n)} + \boldsymbol{p}_{i+1}^{(n)}) + \frac{1}{N(1+f_n)^2}\sum_{i=1}^{N}\boldsymbol{p}_i^{(n)}. \tag{3}$$

As the coefficients will be changed after each subdivision operation by (3), we introduce the following elegant lemma of circulant matrix.

Lemma 1. Let

$$\boldsymbol{A} := \mathrm{Cir}_n(a_1, a_2, \cdots, a_n) = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}$$

be a circulant matrix, its eigenvalues are $\lambda_j = \sum_{i=1}^{n} a_i \omega_j^{i-1}$, $j = 1, 2, \cdots n$; and corresponding eigenvector is $\boldsymbol{W}_j = (1, \omega_j, \omega_j^2, \quad \cdots, \omega_j^{n-1})^{\mathrm{T}}$, here $\omega_j = \cos\varphi_{n,j} + \mathrm{i}\sin\varphi_{n,j}$ is the j-th root of $z^n = 1$, with $\varphi_{n,j} = 2\pi(j-1)/n$.

Note the circulant matrices have the same group of eigenvectors. By this invariance, we have the following theorem.

Theorem 2. (Doo-Sabin subdivision scheme convergence and continuity theorem). Given an arbitrary mesh M without superposition vertices, applying subdivision operator $S_2(f;n)$, on any face $\boldsymbol{P}^{(0)}=\{\boldsymbol{p}_i^{(0)}\}_{i=1}^N$ in M, if the stiffness $s\in(0,\infty)$, the subdivision scheme is convergent, and the vertex $\bar{\boldsymbol{p}}=\sum_{i=1}^n \boldsymbol{p}_i^{(0)}/n$ lies on the limit surface; if $s\in(1/2,\infty)$, the limit surface is tangent continuous.

Proof. Same as in the classical analysis method, we consider an N-side face $\boldsymbol{f}^{(0)}=\{\boldsymbol{p}_i^{(0)}\}_{i=1}^N$ on the original mesh, here $\boldsymbol{p}_i^{(0)}$ is a vertex of the face. Let $\boldsymbol{f}^{(n)}=\{\boldsymbol{p}_i^{(n)}\}_{i=1}^N$ be a face corresponding to $\boldsymbol{f}^{(0)}$ after n times of subdivision, and we denote it by $\boldsymbol{P}^{(n)}=[\boldsymbol{p}_1^{(n)},\ \boldsymbol{p}_2^{(n)},\ ...,\ \boldsymbol{p}_k^{(n)}]^{\mathrm{T}}$. Then by equation (2), we have $\boldsymbol{P}^{(n+1)}=\boldsymbol{S}_2\boldsymbol{P}^{(n)}$.

Here

$$\boldsymbol{S}_2^{(n)}(\alpha)=\mathrm{Cir}_N\left[\frac{\frac{1}{N}+f_n+f_n^2}{(1+f_n)^2},\frac{\frac{1}{N}+\frac{f_n}{2}}{(1+f_n)^2},\frac{1}{N(1+f_n)^2},\ldots,\frac{1}{N(1+f_n)^2},\frac{\frac{1}{N}+\frac{f_n}{2}}{(1+f_n)^2}\right].$$

is the subdivision matrix. (For shorting the notation, in the following prove we omit α). By Lemma 1, we can find a common group of eigenvectors for $\boldsymbol{S}_2^{(n)}$. Therefore we obtain the uniform diagonal decomposition for $\mathbf{S}_2^{(n)}$ as follows:

$$\boldsymbol{S}(\alpha)=\bar{\boldsymbol{W}}^{\mathrm{T}}\mathrm{diag}(\lambda_1^{(n)},\lambda_2^{(n)},\ldots,\lambda_N^{(n)})\boldsymbol{W},$$

where $\boldsymbol{W}=\left(w_{ij}\right)_{N\times N}$, $w_{ij}=\omega_j^{i-1}/\sqrt{N}$. It is obvious that $\boldsymbol{W}$ is a U-matrix, satisfying $\bar{\boldsymbol{W}}^{\mathrm{T}}=\boldsymbol{W}^{-1}$, which is independent of α and subdivision level n. Denote $\delta_j^{(n)}=\prod_{i=0}^{n-1}\lambda_j^{(i)}$, then after n times of subdivision, the control data points are

$$\boldsymbol{P}^{(n)}=\bar{\boldsymbol{W}}^{\mathrm{T}}\mathrm{diag}(\delta_1^{(n)},\delta_2^{(n)},\ldots,\delta_N^{(n)})\boldsymbol{W}\boldsymbol{P}^{(0)}.$$

And we have

$$\lambda_j^{(n)} = \begin{cases} 1, & j=1; \\ \dfrac{f_n}{1+f_n} + \dfrac{2f_n}{(1+f_n)^2}\cos(2\pi(j-1)/N), & j \neq 1; \end{cases}$$

so $\forall n > M$, if $f_n > 0$, then $\lambda_j^{(n)} = \lambda_{N-j+2}^{(n)} < \lambda_1^{(n)} \equiv 1, j = 2,3,\ldots,N$. It follows

$$\boldsymbol{P}^{(\infty)} = \lim_{n\to\infty} \boldsymbol{P}^{(n)} = \bar{\boldsymbol{W}}^{\mathrm{T}} \mathrm{diag}(1,0,\ldots,0)\boldsymbol{W}\boldsymbol{P}^{(0)} = \sum_{i=1}^{k} \boldsymbol{p}_i^{(0)} (1,1,\ldots,1)^{\mathrm{T}} / N .$$

Thus the subdivision scheme is convergent, and any face in the original mesh converges to its barycenter

$$\bar{\boldsymbol{p}}^{(n)} = \sum_{i=1}^{N} \boldsymbol{p}_i^{(n)} / N = \sum_{i=1}^{N} \boldsymbol{p}_i^{(n-1)} / N = \bar{\boldsymbol{p}}^{(n-1)} = \cdots = \bar{\boldsymbol{p}}^{(0)} .$$

Furthermore, for the limit surface's first order continuity, we only need to verify that all vertices in $\boldsymbol{f}^{(n)}$ converge to a common tangent plan. Note the point vector sequence $\boldsymbol{P}^{(0)}$ in the linear space expanded by the eigenvectors of $\boldsymbol{S}_2^{(n)}$ can be represented as

$$\boldsymbol{P}^{(0)} = (\boldsymbol{x}^{(0)}, \boldsymbol{y}^{(0)}, \boldsymbol{z}^{(0)}) = \left(\sum_{j=1}^{N} \boldsymbol{W}_j \boldsymbol{d}_{1j}, \sum_{j=1}^{N} \boldsymbol{W}_j \boldsymbol{d}_{2j}, \sum_{j=1}^{N} \boldsymbol{W}_j \boldsymbol{d}_{3j} \right) = \sum_{j=1}^{N} \boldsymbol{W}_j \otimes \boldsymbol{d}_j ,$$

where $\boldsymbol{d}_j = (d_{1j}, d_{2j}, d_{3j})$ is the weight of eigenvector with respect to three components. After *n* times of subdivision, the point sequence is

$$\boldsymbol{P}^{(n)} = \boldsymbol{S}_2^{(n-1)} \boldsymbol{S}_2^{(n-2)} \cdots \boldsymbol{S}_2^{(0)} \boldsymbol{P}^{(0)} = (\sum_{j=1}^{N} \delta_j^{(i)} \boldsymbol{W}_j d_{1j}, \sum_{j=1}^{N} \delta_j^{(i)} \boldsymbol{W}_j d_{2j}, \sum_{j=1}^{N} \delta_j^{(i)} \boldsymbol{W}_j d_{3j}) = \sum_{j=1}^{N} \delta_j^{(n)} \boldsymbol{W}_j \otimes \boldsymbol{d}_j .$$

So the different vector of *l*-th vertex on $\boldsymbol{f}^{(n)}$ and barycenter is

$$\boldsymbol{p}_l^{(n)} - \bar{\boldsymbol{p}}^{(n)} = \boldsymbol{p}_l^{(n)} - \bar{\boldsymbol{p}}^{(0)} = \sum_{j=1}^{N} \delta_j^{(n)} \omega_j^l \boldsymbol{d}_j - \boldsymbol{d}_1 = \sum_{j=2}^{N} \delta_j^{(n)} \omega_j^l \boldsymbol{d}_j$$

Since $\delta_2^{(n)} = \delta_N^{(n)} > \delta_j^{(n)}, j = 3,4,\ldots,N-1$, we have

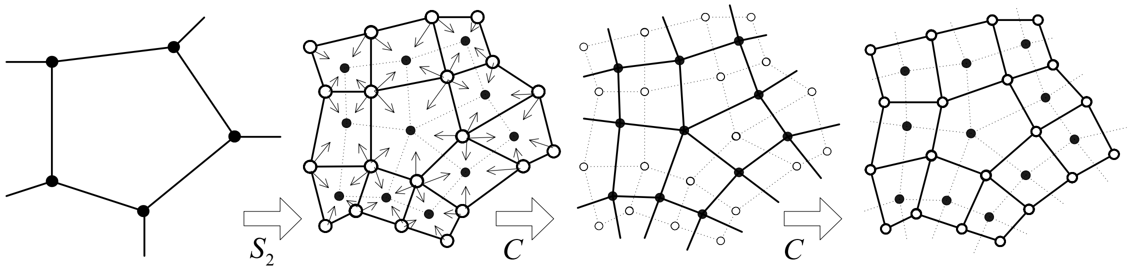

Fig.3 Higher order subdivision schemes.

$$\lim_{n\to\infty}\frac{\boldsymbol{p}_l^{(n)}-\overline{\boldsymbol{p}}^{(n)}}{\left\|\boldsymbol{p}_l^{(n)}-\overline{\boldsymbol{p}}^{(n)}\right\|}=\lim_{n\to\infty}\frac{\omega_2^l\boldsymbol{d}_2+\omega_N^l\boldsymbol{d}_N+\sum_{j=3}^{N-1}(\delta_j^{(n)}/\delta_2^{(n)})\omega_j^l\boldsymbol{d}_j}{\left\|\omega_2^l\boldsymbol{d}_2+\omega_N^l\boldsymbol{d}_N+\sum_{j=3}^{N-1}(\delta_j^{(n)}/\delta_2^{(n)})\omega_j^l\boldsymbol{d}_j\right\|}=\frac{\omega_2^l\boldsymbol{d}_2+\omega_N^l\boldsymbol{d}_N}{\left\|\omega_2^l\boldsymbol{d}_2+\omega_N^l\boldsymbol{d}_N\right\|}.$$

Thus the limit vector is perpendicular to the vector $\boldsymbol{d}_2\times\boldsymbol{d}_N$. Thus $\boldsymbol{d}_2$ and $\boldsymbol{d}_N$ expand the commonality tangent plane at the point $\overline{\boldsymbol{p}}$.

With increasing times of subdivision, the parameters of semi-subdivision schemes tend to zero. So in most cases, the characteristic map[8] of Doo-Sabin type subdivision surface is regular, which guarantees the scheme is GC^1. Note let $f(\alpha)=s\cos(\alpha/2)$ in S_2, the subdivision schemes defined by S_2 can be used to create revolving surface and can be viewed as a quadric case of Ref.[15]. We have the following corollary.

Corollary 1. If $s\in(1/2,\infty)$ and $\alpha\in(-\pi,\pi)$, the Doo-Sabin type C-subdivision surface is GC^1.

2.2 High order subdivision

Similar to Definition 3 and Ref.[16], we define $S_k=C^{k-2}S_2$ to obtain higher order semi-stationary subdivision schemes. In surface case, C is a dual-average operation. That is, we create a new vertex on each face with its barycenter, then connect the vertices whose corresponding faces are connected to create a new mesh(See Fig.3). It is

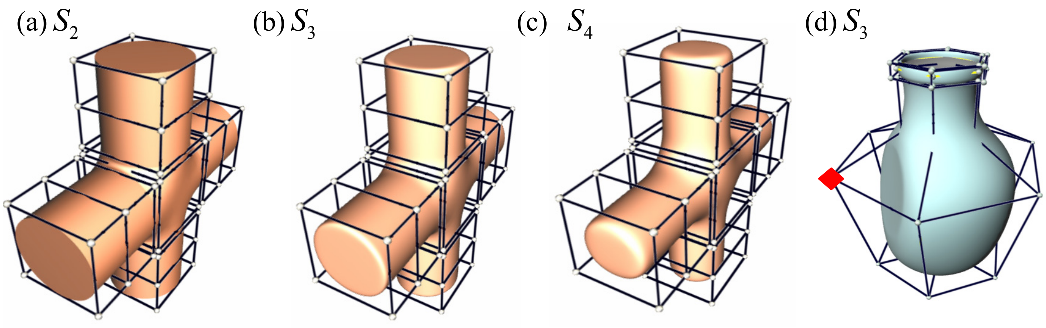

Fig.4. Semi-stationary subdivision examples. (a~c) are examples employing different subdivision operators on the same mesh. (d) is an example employing the tag technique.

easy to verify that $S_3 := CS_2$ is a variant of Catmull-Clark subdivision schemes[1,15]. Since in the regular case the subdivision procedure creates a tensor product surface, by Theorem 1, the surface generated by S_k is GC^{k-1} everywhere except the neighborhood around the irregular case which should be seriously analyzed and will be discussed it in a future paper. Fig.4 shows several examples.

3 Discussions

When the subdivision operator $S_k(f;n)$ and initial control mesh $M^{(0)}$ are given, the smoothness of the limit surface is determined by the stiffness $s := f(0)$, and the details are listed in Table 1. Since the basic shape or the lower frequency of mesh is mainly determined by the first several subdivision operations, it is related to the choice of α. When $f(\alpha) < 0$, some weights become negative, the surface has no convex closure property, which is benefit to create special bumps or burrs.

Table 1. Relationship between stiffness and shape type

	Case 1	*Case 2*	*Case 3*	*Case 4*
$s = f(0) \in$	$(-\infty, -1]$	$(-1, 0)$	$(0, \infty)$	$(1/2, \infty)$
Continuity type	Diverge	Fractal surface	GC^0	GC^1

In the above discussion, we implicitly assume that the same α is used for the whole mesh. However, due to the local support of the subdivision operator, this constraint is not necessary. A straightforward extension is to tag every point a parameter when k is odd and every face a parameter when k is even for given S_k. We employ a parameter update strategy similar to that of Ref.[9]. An example is shown in Fig.4(d), we set the vertex(tagged with a diamond) $\alpha = 9.0$, and zero otherwise. In fact, we obtain a class of non-uniform subdivision surface through this technique.

4 Conclusion

In a uniform view of subdivision operators, a construction framework of semi-stationary subdivision curves and surfaces is proposed in this paper. And the continuous condition of quadric subdivision case is analyzed. As a special case, the proposed scheme can exactly represent revolving surface and show bump effect. Our method can get abundant detail effects only based on simple linear combination and flexible selection of kernel function f and parameter α. The method can be slightly modified to fit the triangle mesh case. Additionally, the complex combination of basic subdivision operator and convolution operator can create higher order subdivision operators. We will study their properties in future work.

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