Level Set (I)

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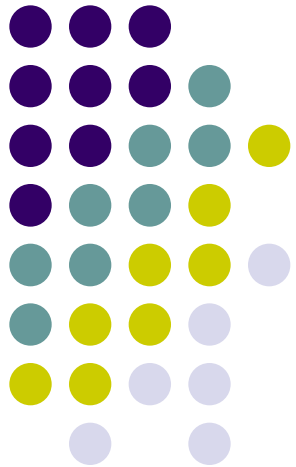

outline

- 数学基础
- Level Set方法
- Level Set方法数值解法
- Level Set在图像图形中的应用

survey

- Level set 与 Poisson方程的比较
  - 一般的讲，level set的应用和效果都要更好！
  - 但是level set由于需要较多数学，所以工程上不被普遍接受。
  - Poisson方程是线性的，而level set方程一般都是非线性的，需要稳定的算法求解。

Http://cermics.enpc.fr/~paragios/book/book.html

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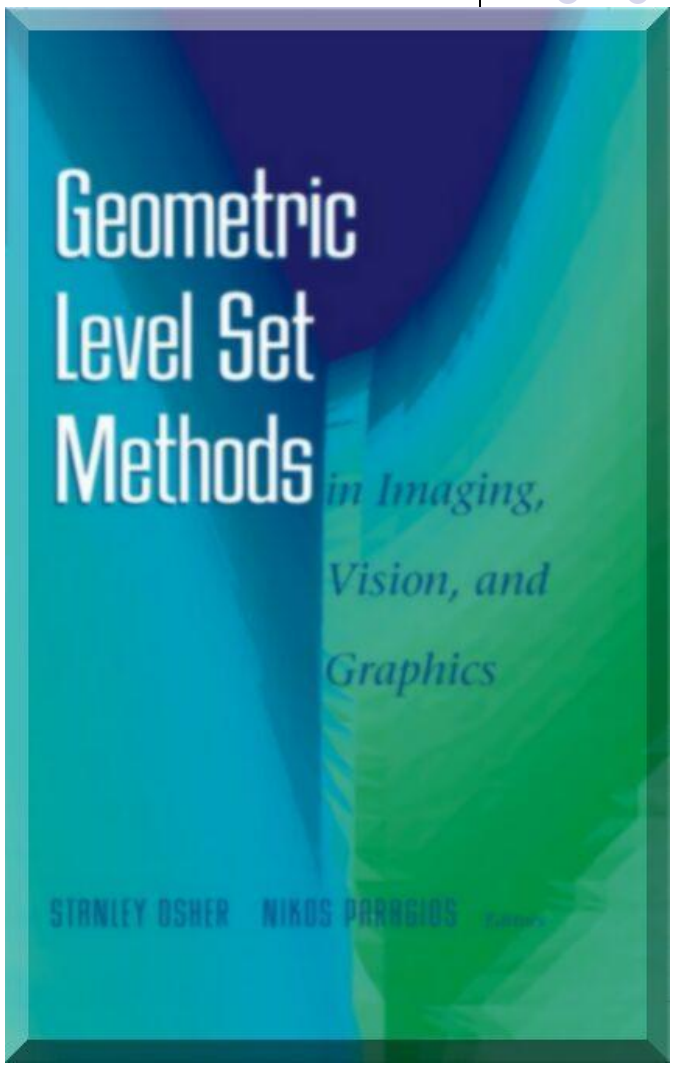

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  - 微分几何简介
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  - 隐函数，距离函数
- Level Set 方法概念
- 数值解法
  - 向上差分法
  - Hamilton-Jacobi ENO
  - Hamilton-Jacobi WENO
  - TVD Runge-Kutta
- Level set 与变分方程的关系

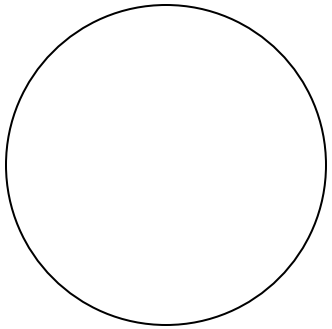
数学基础—曲线的微分几何

- 映射$C(p)$: $[a, b]$ in $\mathbf{R}\rightarrow\mathbf{R}^2$
- 定义了一个平面的曲线，p是参数，对每一个p_0 in $[a, b]$, 我们得到曲线上的一点

$$C(p_0)=[x(p_0), y(p_0)]$$

- **正则曲线：** 如果 $C'(p)=[x'(p), y'(p)]\neq 0$
  - 例 $C(p)=[\cos(p), \sin(p)], \quad p\in[0,2\pi]$

圆

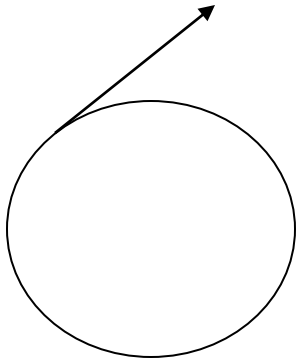


数学基础—曲线的微分几何

- 曲线的切线

$$C'(p)=[x'(p),y'(p)]=\frac{\partial C}{\partial p}$$

- 弧长参数
  - 如果曲线的参数满足 $\left\|\frac{\partial C}{\partial p}\right\|=1$



p表示曲线上以某一点为标准的弧长

数学基础—曲线的微分几何

- 弧长

$$L(t_0,t_1)=\int_{t_0}^{t_1}[(x'(t))^2+(y'(t))^2]^{1/2}\,dt$$

- 弧长参数: s

$$\langle C'(s),C'(s)\rangle=1$$

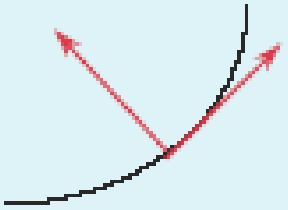
求导 ↓

$$\langle C'(s),C''(s)\rangle=0$$

数学基础—曲线的微分几何

- 曲率

$$\kappa = \|C''(s)\|$$



- 假设T表示切线，N表示法线，则

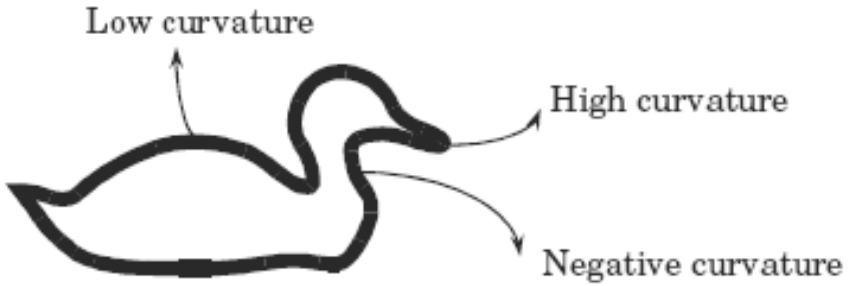
$$\frac{dC}{ds} = T \qquad \frac{d^2C}{ds^2} = \kappa N$$

$$\left\| \frac{\partial C(s)}{\partial s} \right\| = 1$$

$$< C_s, C_{ss} > = 0$$

$$C_s \perp C_{ss}$$

$$\kappa := \|C_{ss}\|$$



数学基础—曲线的微分几何

- **Frenet** 公式

$$\frac{dT}{ds} = \kappa N$$

$$\frac{dN}{ds} = -\kappa T$$

数学基础—曲线的微分几何

- 曲率的其他定义
  - 假设θ 为切线*T*与*x*轴之间的夹角，则

$$\kappa = \frac{d\theta}{ds}$$

- 隐式曲线的曲率 $C \equiv \{(x,y) : u(x,y) = 0\}$

$$\kappa = \frac{u_{xx}u_y^2 - 2u_x u_y u_{xy} + u_{yy}u_x^2}{(u_x^2 + u_y^2)^{3/2}}$$

数学基础—曲线的微分几何

- 曲率的性质
  - 旋转，平移不变
  - 缩放要变化

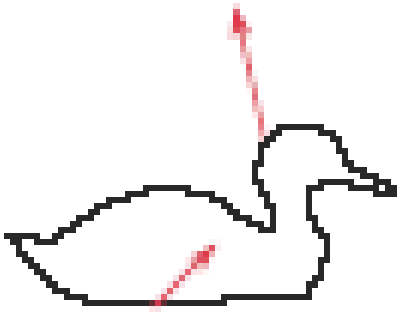
数学基础—曲线的微分几何

- 隐式曲线的法向量 $u(x,y)=0$

$$N=+(-)\frac{\nabla u}{\|\nabla u\|}$$

- 因为T和N互相垂直，所以平面上任何曲线都可以用曲线上任何一点的T和N的线性组合来表示

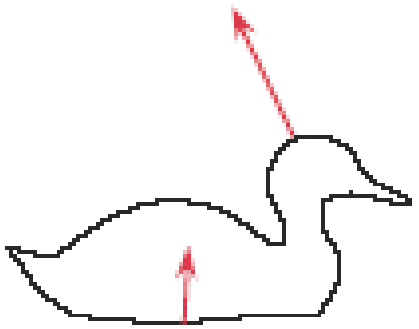
$$\frac{\partial C}{\partial t}=\alpha T+\beta N$$



数学基础—曲线的微分几何

- 如果只考虑**几何形状**的变化，那其变化只跟法线方向的变化有关系，则有

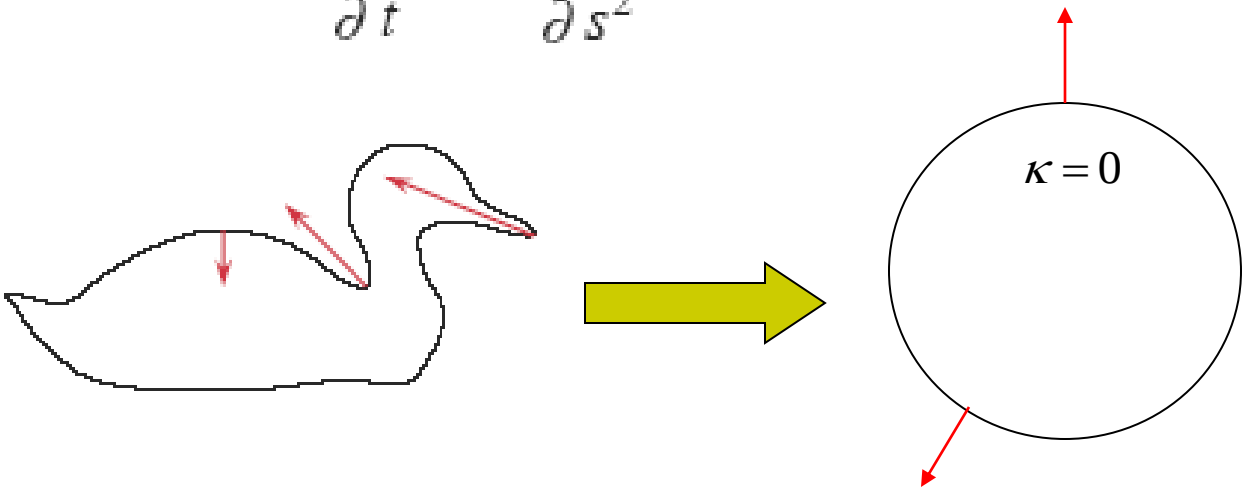
$$\frac{\partial C}{\partial t} = \beta N$$



数学基础—曲线的微分几何

- 例：沿着曲率变化最大方向的曲线变形

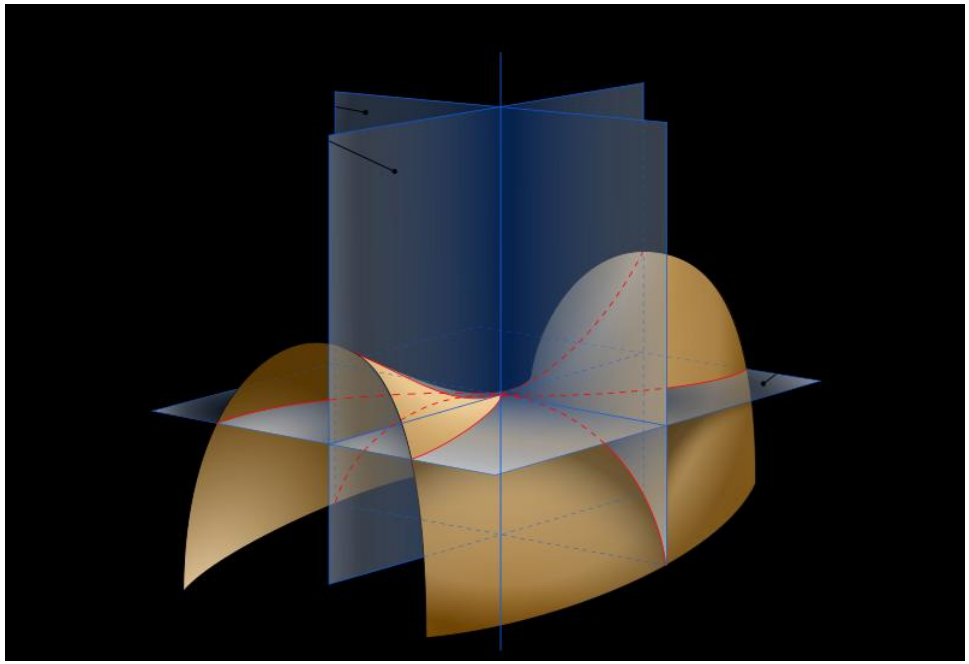
$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial s^2} = \kappa \vec{N}$$



最后变化为曲率都为常数的曲线停止，即圆

数学基础—曲面的微分几何

- 平均曲率和高斯曲率
  - 每个正则曲面都有两个主曲率。
  - 这两个的平均值就是*平均曲率*，两个的积是*高斯曲率*。



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数学基础—数学形态学

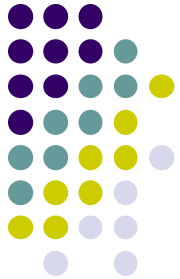
- 是一个经典的基于几何的理论
- 广泛应用于图像处理

形态算子

- 一组空间滤波操作
- 用于改变二值区域的形状
  - 腐蚀：减少物体边界的象素数
  - 膨胀：增加物体边界的象素数
  - 复合方法
    - 开：腐蚀，然后膨胀
    - 闭：膨胀，然后腐蚀

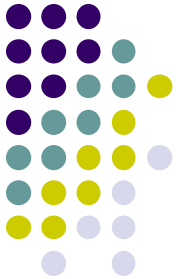
Original Image

膨胀与腐蚀（Dilation, Erosion）



- 数学形态学里面最重要的操作
- 腐蚀将图像的尺寸减少
- 膨胀增加图像的尺寸
- 可以用来消除图像上小的亮斑噪声和不规则的边

腐蚀（续）

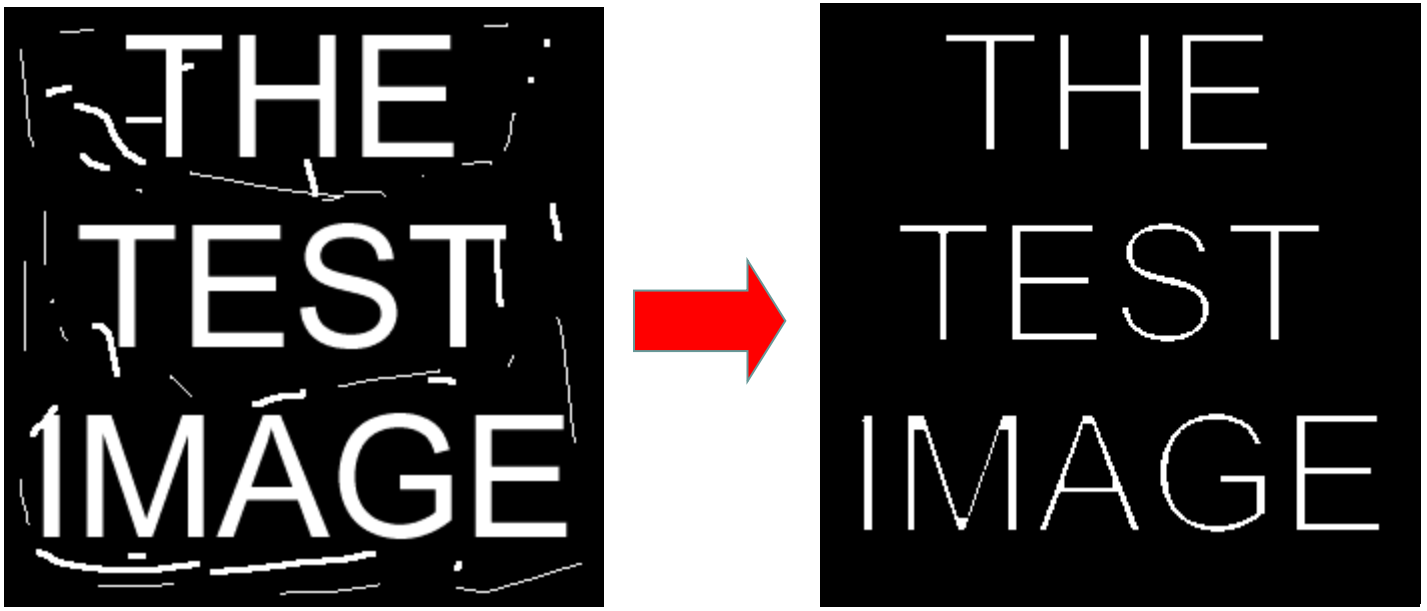


- 定义：物体的颜色是白，背景是黑
- 定义腐蚀模板为
  - 1　1　1
  - 1　1　1
  - 1　1　1
- 将模板与图像进行加操作
- 如果有，则结果为1，否则为0

腐蚀（续）

- 模板的效果相当于去掉物体边界处的单个象素
- 4种情况:
  - 当前处理象素为1，邻域象素为1—》1
  - 当前处理象素为0，邻域象素为1—》0
  - 当前处理象素为0，邻域象素为1、0的混合—》0
  - 当前处理象素为1，邻域象素为1 、0的混合—》1

腐蚀（续）



原始图像　　腐蚀后的图像　腐蚀两次

膨胀

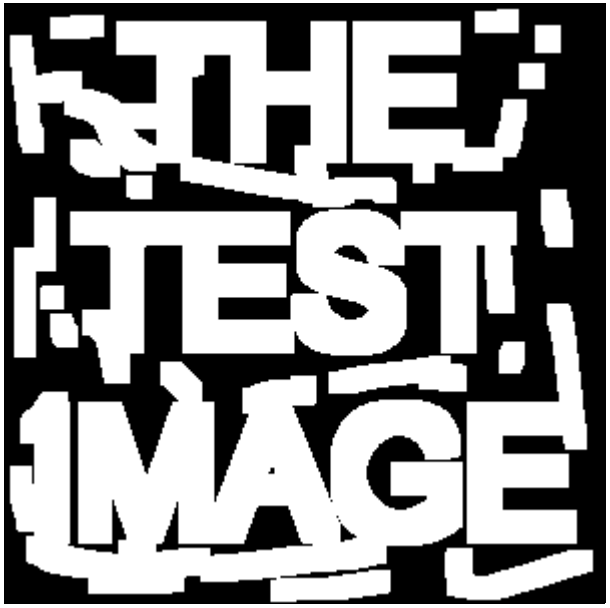
- 膨胀是腐蚀的逆操作
- 模板文件是
  - 0 0 0
  - 0 0 0
  - 0 0 0
- 其效果相当于在物体的边界添加单个象素

膨胀（续）

- 4种情况
  - 当前处理象素为0，邻域象素为0—》0
  - 当前处理象素为1，邻域象素为1—》1
  - 当前处理象素为1，邻域象素为1、0的混合—》1
  - 当前处理象素为0，邻域象素为1 、0的混合—》1
- 逻辑操作算子是Or

膨胀（续）

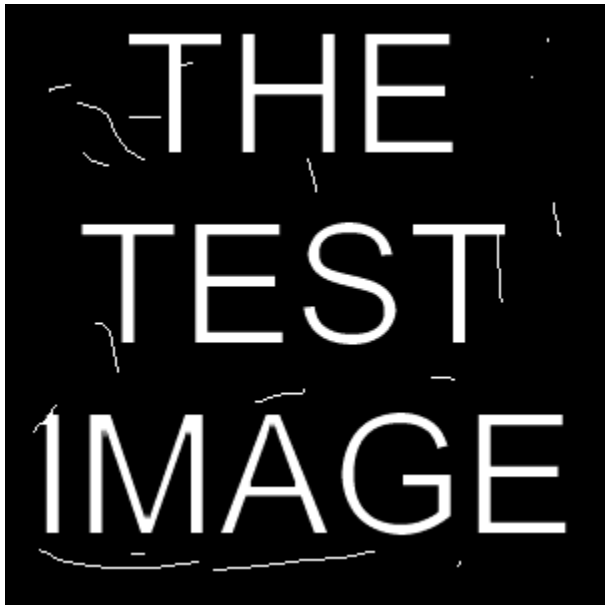
原始图像



膨胀膨胀图像

开操作

- 开操作相当于先做腐蚀操作，再做膨胀操作
- 效果相当于去掉单个象素，但是保留原来的形状何尺寸。



原始图像

腐蚀两次，然后膨胀两次（开操作）

闭操作

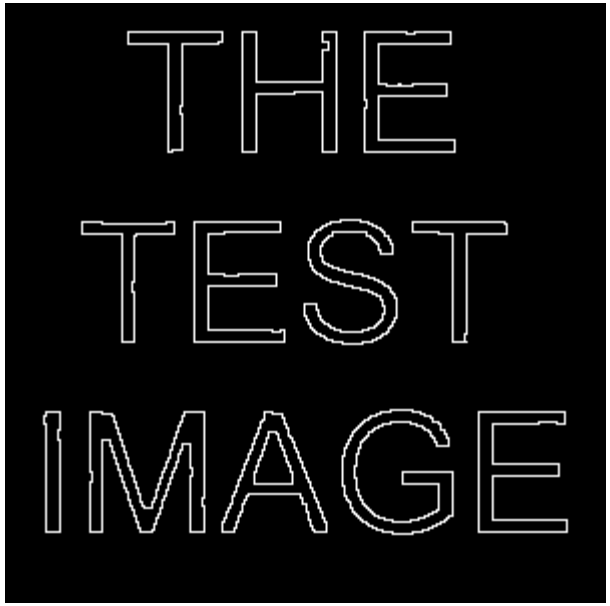
- 闭操作是开操作的相互操作
- 先膨胀，然后腐蚀
- 它可以用来填补一些小洞

原始图像

闭操作结果

轮廓抽取

- 先做腐蚀操作， 再将腐蚀结果图像减去原始图像



数学基础—数学形态学

$$A \oplus B := \{a + b, a \in A, b \in B\} = \bigcup_{b \in B} A_b$$

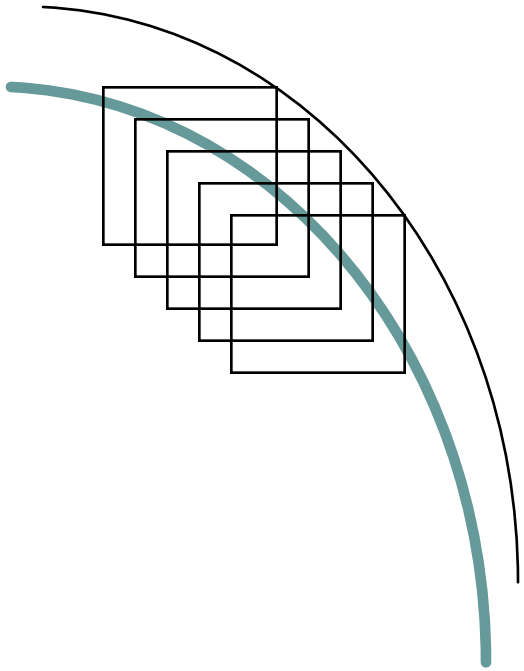
$$A \ominus B := (A^c \oplus B)^c = \bigcap_{b \in B} A_b$$

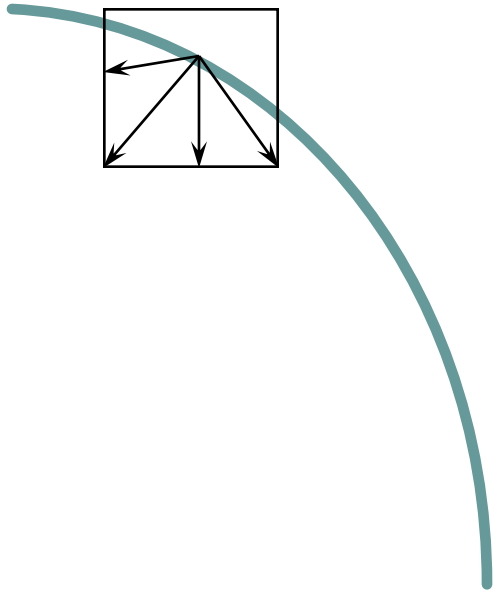
$$A \circ B := (A \ominus B) \oplus B = \bigcup_{\{y : B_y \subseteq A\}} A_y$$

$$A \bullet B := (A \oplus B) \ominus B$$

数学基础—形态学与曲线演化

- $$A \oplus (B \oplus C) = (A \oplus B) \oplus C$$
  $$A \oplus (rB) = A \oplus r_1 B \oplus r_2 B \oplus \ldots$$





数学基础—形态学与曲线演化

- 法线速度的一般表示

$$\beta = \sup_{\theta}\{r(\theta)\cdot N\}$$

  - 例子

$$\beta = N \qquad B = disk$$

$$\beta = \max\{N_x, N_y\} \qquad B = diamond$$

$$\beta = |N_x| + |N_y| \qquad B = square$$

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数学基础—隐函数

- 隐函数(implicit function):　　自变量和因变量之间的法则是由一个方程式所确定

$$F(x,y)=0$$

$$y=y(x)$$

- 例子

$$x^2+y^2-1=0$$

数学基础—距离场函数

- 距离函数定义

$$d(x) = \min(|x - x_l|) \quad for\ all \quad x_l \in \partial\Omega$$

- 距离函数的性质

$$|\nabla d| = 1$$

数学基础—距离场函数

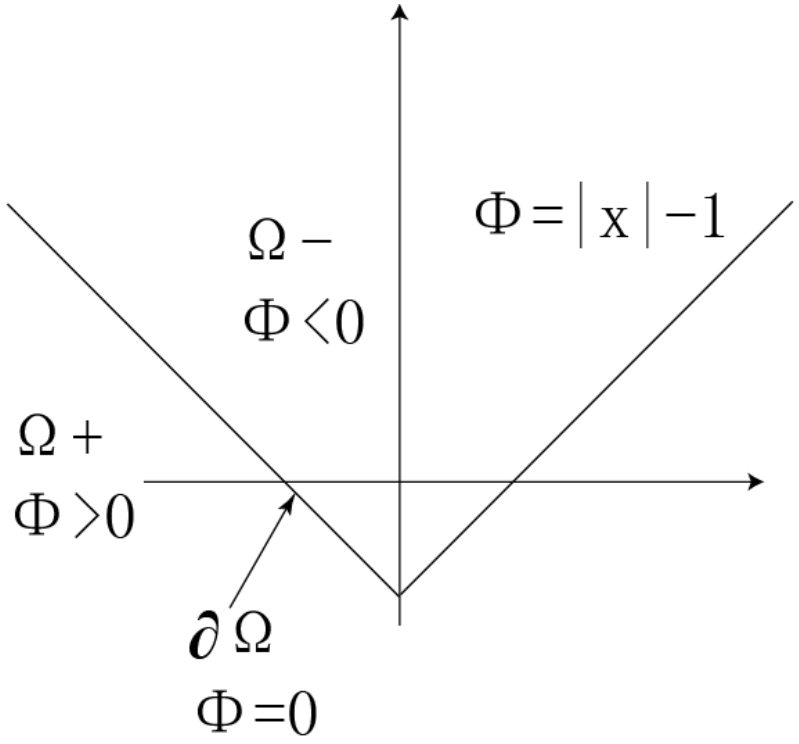
- 带符号的距离场隐函数 $\Phi(x)$

$$|\Phi(x)| = d(x)$$

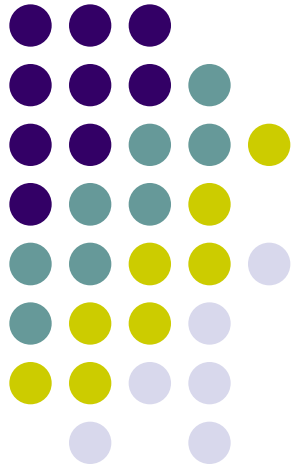
$$\Phi(x) = \begin{cases} -d(x) & for\ x \in \Omega^- \\ 0 & for\ x \in \partial\Omega \\ d(x) & for\ x \in \Omega^+ \end{cases}$$

数学基础—距离场函数

- 例子 $\Phi(x) = |x| - 1$

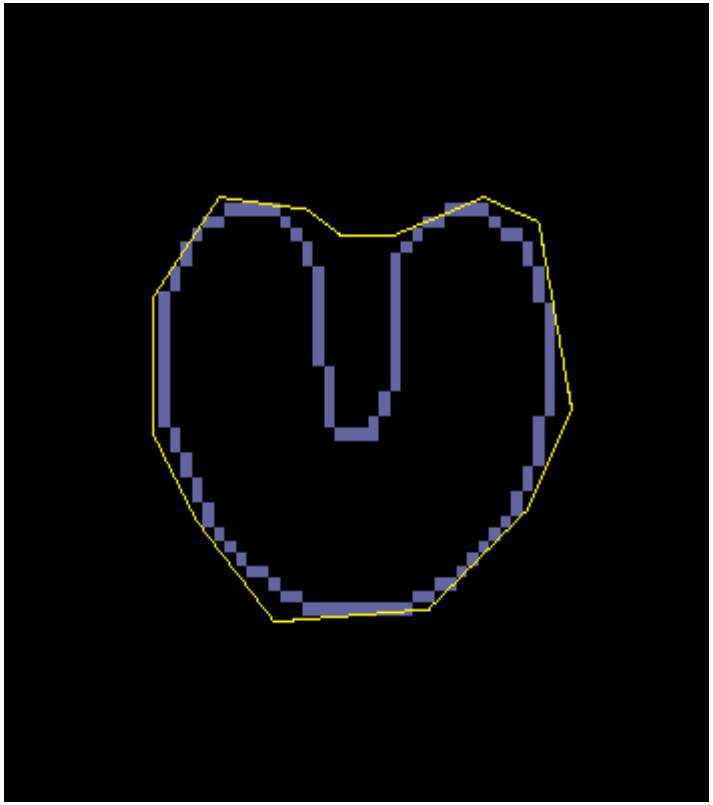


动态可变形模型



What is Snake?

- Active contour model; parametric model
- Result from Kass,Witkin,and Terzopoulos,1987
- Energy minimizing formulation
- Depends on its shape and location within image



- Example: cell nucleus
- Initialization: external
- Fitting: iterative
  - Using the current geometry
    ("Contour influence")
    and the image contents
    ("Image influence")
  - "Influence":
    scalar = energy
    vector = force
  - Until the energy is minimum, or the forces are null

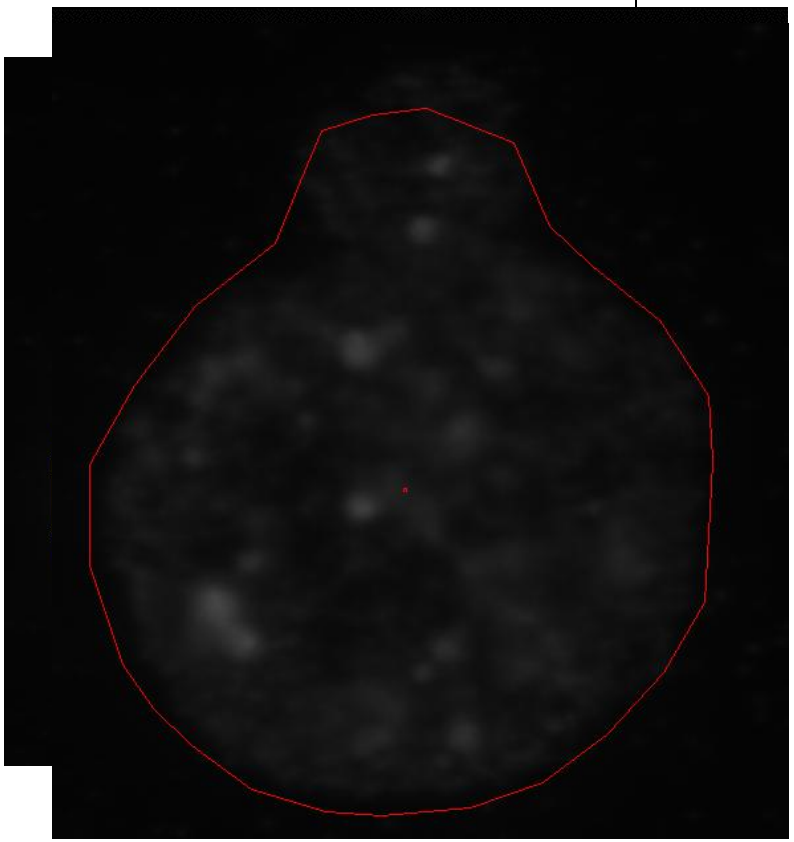


Image: Ian Diblik 2004

Traditional Snake model

- Snake Model (1987) [Kass-Witkin-Terzopoulos]
  - Planar parameterized curve C:R-->RxR
  - A cost function defined along that curve

$$E[(C)(p)] = \alpha \int_0^1 E_{int}(C(p))dp + \beta \int_0^1 E_{img}(C(p))dp + \gamma \int_0^1 E_{con}(C(p))dp$$

  - The internal term stands for regularity/smoothness along the curve and has two components (resisting to stretching and bending)
  - The image term guides the active contour towards the desired image properties (strong gradients)
  - The external term can be used to account for user-defined constraints, or prior knowledge on the structure to be recovered
  - The lowest potential of such a cost function refers to an equilibrium of these terms

Active Contour Components

- The internal term…

$$E_{int}(C(p)) = w_{tension}(C(p)) \left| \frac{\partial C}{\partial p}(p) \right|^2 + w_{stiffness}(C(p)) \left| \frac{\partial^2 C}{\partial p^2}(p) \right|^2$$

  - The first order derivative makes the snake behave as a membrane
  - The second order derivative makes the snake act like a thin plate

- The image term…

$$E_{img}(C(p)) = w_{line} E_{line}(C(p)) + w_{edge} E_{edge}(C(p)) + w_{term} E_{term}(C(p))$$

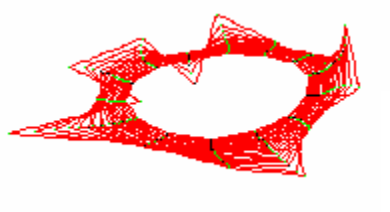
  - Can guide the snake to
    - Iso-photes $E_{line}(C(p)) = I(C(p))$ edges $E_{edge}(C(p)) = |\nabla I(C(p))|^2$
    - and terminations

- Numerous Provisions…: balloon models, region-snakes, etc…

Traditional snake model Internal force

- Elastic force (alpha force)—pull inward

(a) Initial contour



(b) after executing 20 times

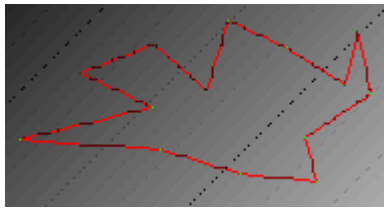
- Bending force (beta force) —smooth rather than shrinkage

(a) Initial contour

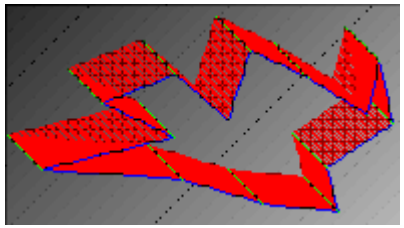
(b) after executing 20 times

Traditional snake model External force

- Image gradient force
- The main difference among snake models
- F_{ext} in traditional snake model: $F_{ext}^{(p)} = -\nabla E_{ext}$
- How to choose E_{ext} : take on smaller value at the boundaries



(a) Initial contour



(b) after executing 20 times

$$E_{ext}^{(1)}(x,y) = -\left|\nabla I(x,y)\right|^2 \qquad E_{ext}^{(2)}(x,y) = -\left|\nabla [G(x,y) * I(x,y)]\right|^2$$

Optimizing Active Contours

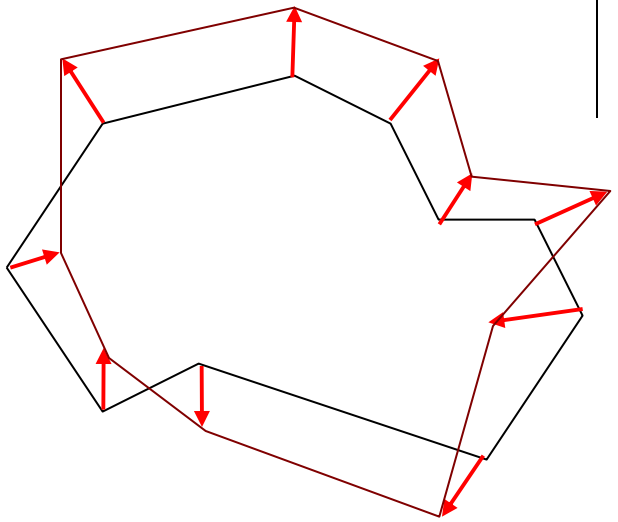
- Taking the Euler-Lagrange equations:

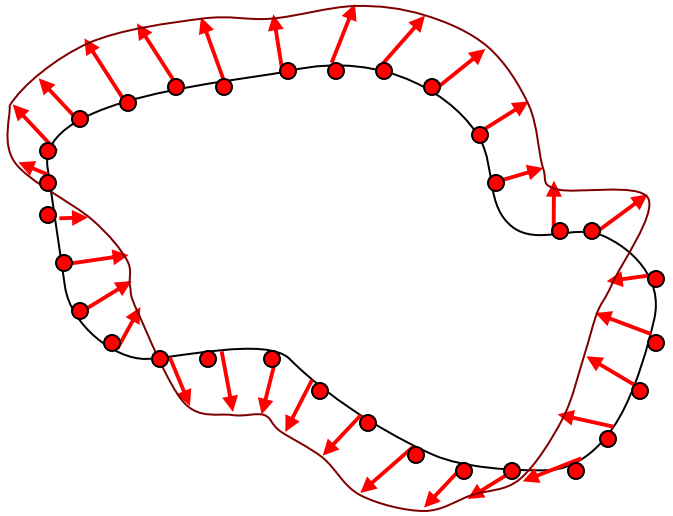
$$\alpha \left(w_{tension} \frac{\partial^2 C}{\partial p^2}(p) - w_{stiffness} \frac{\partial^4 C}{\partial p^4}(p) \right) - \beta \nabla E_{img}(C(p) = 0$$

- That are used to update the position of an initial curve towards the desired image properties

  - Initial the curve, using a certain number of control points as well as a set of basic functions,
  - Update the positions of the control points by solving the above equation
  - Re-parameterize the evolving contour, and continue the process until convergence of the process…

- Geometric AC:
  - Stored as vertices
  - Each vertex is moved iteratively
- Parametric AC:
  - Stored as coefficients
  - Sampled before each iteration
  - Each sample is moved
  - New coefficients are computed (interpolation)

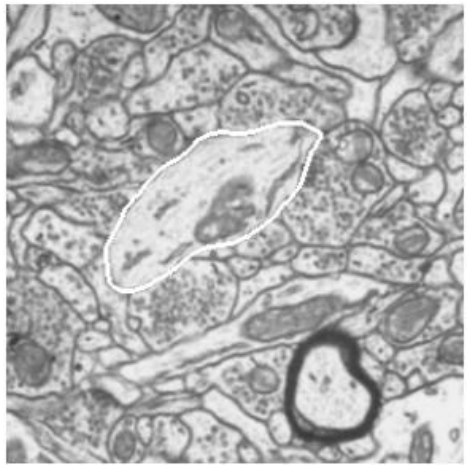




动态可变形模型

- 经典的可变形模型—snakes
  - 假设：物体在图像中的边界是分段连续或光滑的，并且用参数表示为 $v(s)=(x(s),y(s)),s\in[0,1]$
  - 轮廓的能量约束表示为 $E(v)=S(v)+P(v)$
    - 其中S表示对边界线的内部约束，P表示对边界线的外部约束
    - S可以定义为 $S(v)=\frac{1}{2}\int_0^1 w_1(s)\left|\frac{\partial v}{\partial s}\right|^2+w_2(s)\left|\frac{\partial^2 v}{\partial s^2}\right|^2 ds$ 分别表示对曲线的张量和硬度的限制
    - P可以定义为 $P(x,y)=-c|\nabla[G_\sigma * I(x,y)]|$ 表示曲线被吸引到梯度大（可能是边界）的区域
  - 上式的Euler-Lagrange方程为

$$-\frac{\partial}{\partial s}(w_1\frac{\partial v}{\partial s})+\frac{\partial^2}{\partial s^2}(w_2\frac{\partial^2 v}{\partial s^2})+\nabla P(v(s))=0$$



动态可变形模型

- 动态snake
  - 经典的snake是一个静态问题，如果将问题的求解看作是曲线的演化，曲线变为 $v(s,t)=(x(s,t),y(s,t))$，其中t是演化参数，则上面问题可以化为

$$\mu\frac{\partial^2 v}{\partial t^2}+\gamma\frac{\partial v}{\partial t}-\frac{\partial}{\partial s}(w_1\frac{\partial v}{\partial s})+\frac{\partial^2}{\partial s^2}(w_2\frac{\partial^2 v}{\partial s^2})=-\nabla P(v(s,t))$$

该式的稳定解就是原静态问题的解，其中前两项可以看作曲线的惯性力和粘滞力。

动态可变形模型

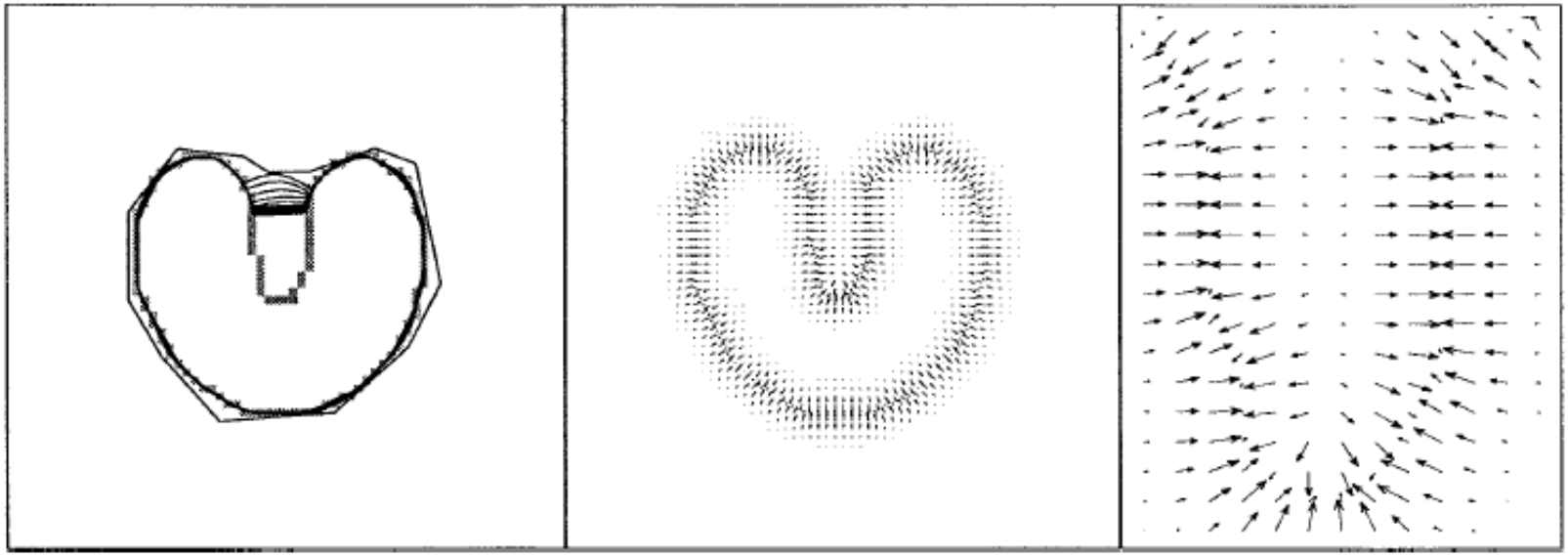
- 动态可变形模型的求解
  - 一般的方法式将几何模型v表示为局部或全局基函数的线性组合（利用有限元法，有限差分法，或者样条函数法等），然后将问题转化为求解线性参数。
  - 一般的形式为 $$E(u) = \frac{1}{2} u^T K u + P(u)$$

    其中K叫做刚性矩阵。最后的方程可以化为一个二次常微分形式

$$M\ddot{u} + D\dot{u} + Ku = f$$

Problem of traditional snake

- Capture range
- Poor convergence

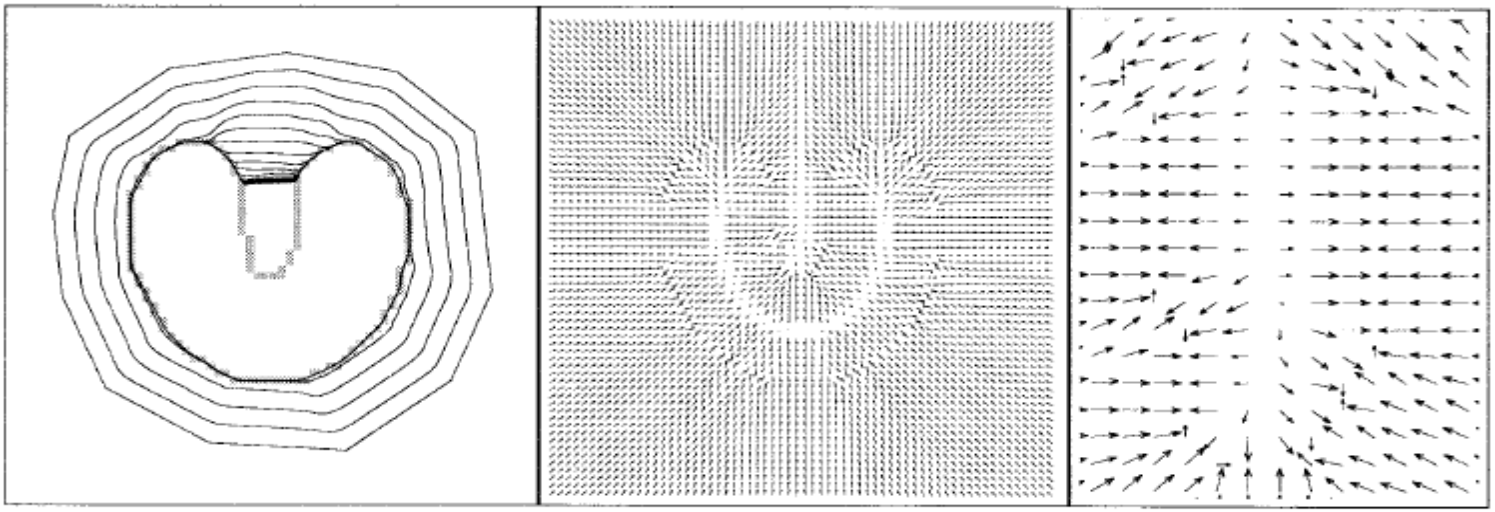


(a) (b) (c)

Fig. 1. (a) Convergence of a snake using (b) traditional potential forces, and (c) shown close-up within the boundary concavity.

External force---balloons

- L.D.Cohen and I.Cohen,1993
- Push the curve outward $F = k_1\vec{n}(s) - k\frac{\nabla P}{\|\nabla P\|}(v(s)) \quad (0<k_1<k<1)$
- Problem: Poor convergence



(a) (b) (c)

Fig. 2. (a) Convergence of a snake using (b) distance potential forces, and (c) shown close-up within the boundary concavity.

External force---GVF

- C Xu and J.L.Prince,1998
- GVF(gradient vector flow):

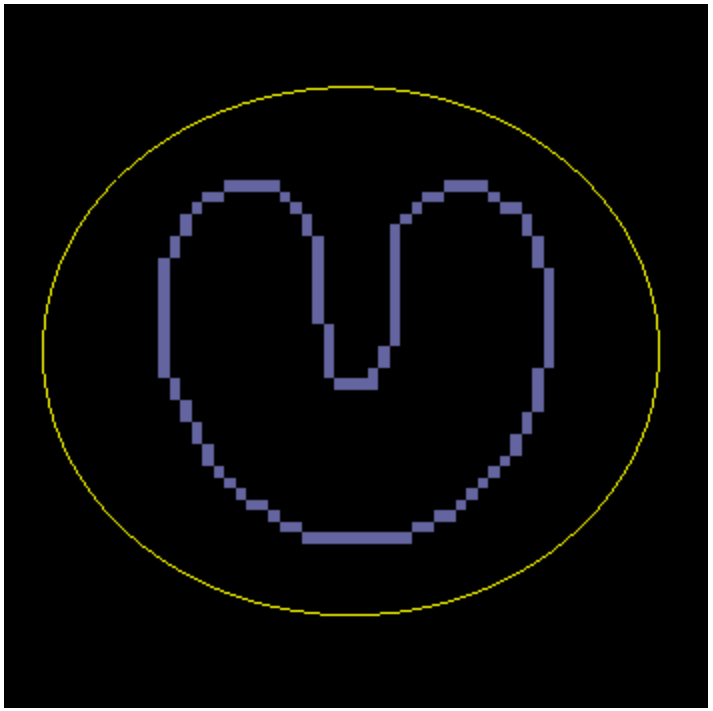
$$F_{ext}^{(g)} = v(x, y)$$

- Edge map:

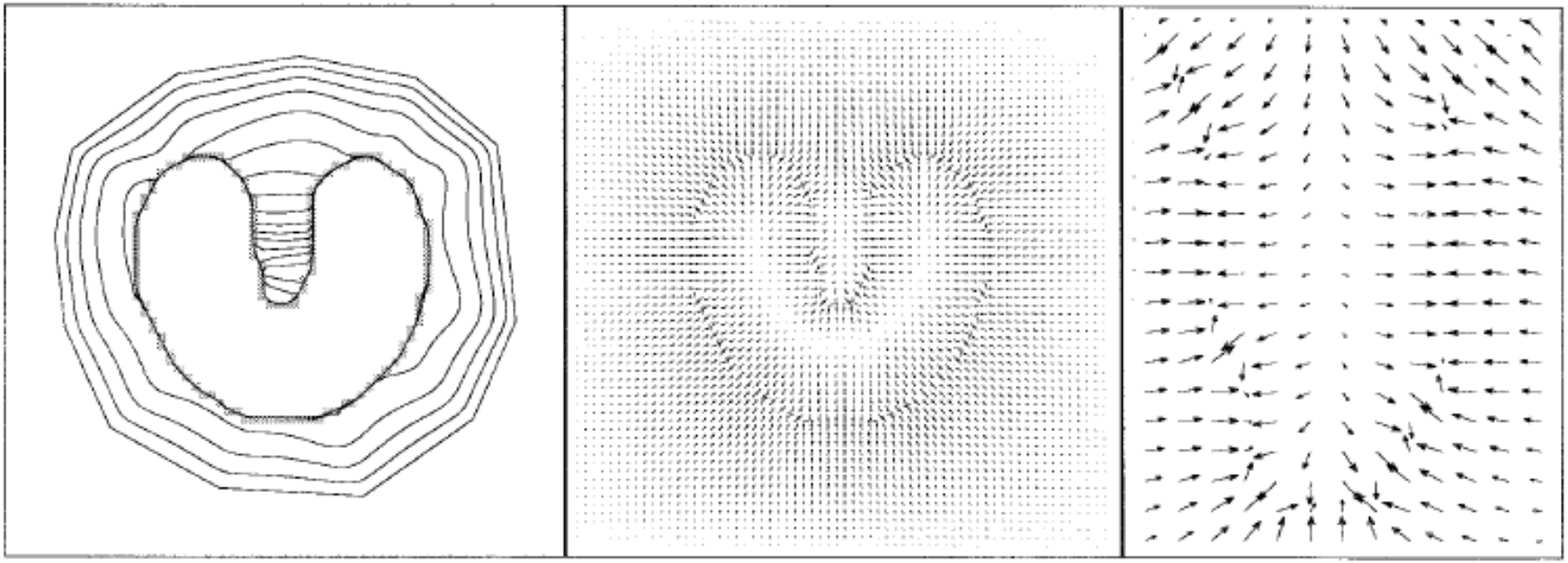
$$f(x, y) = -E_{ext}^{(g)}(x, y)$$

- Energy minimize:

$$E = \iint \mu(u_x^2 + u_y^2 + v_x^2 + v_y^2) + |\nabla f|^2 |v - \nabla f|^2 \, dxdy$$



Result



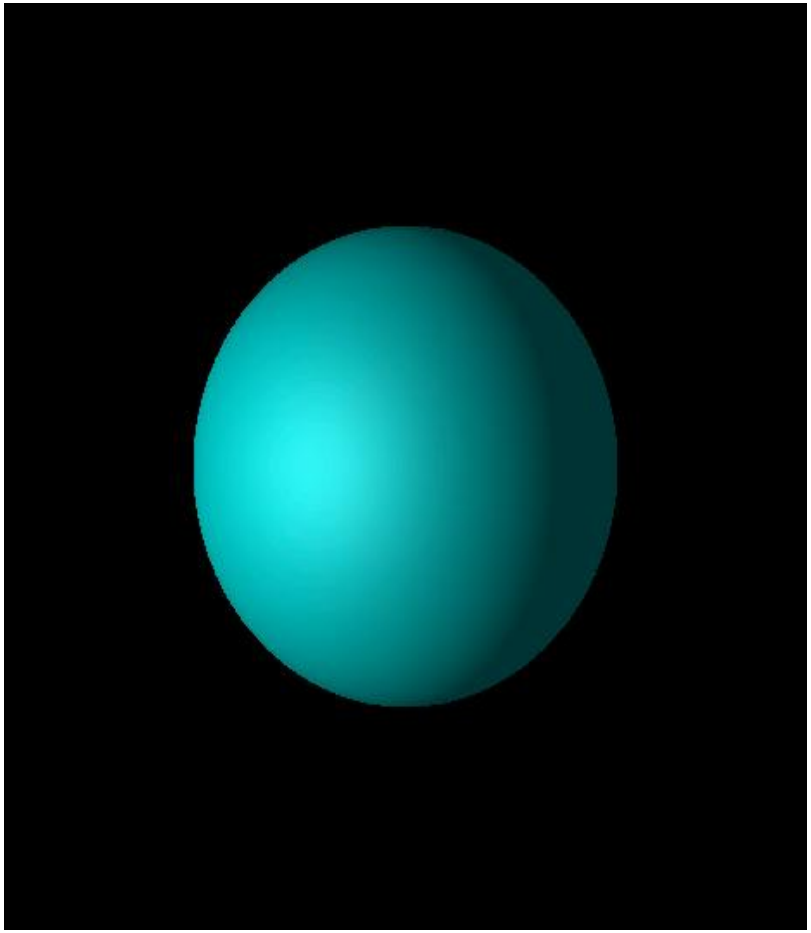
(a) (b) (c)

Fig. 3. (a) Convergence of a snake using (b) GVF external forces, and (c) shown close-up within the boundary concavity.

More about snake…

- Can’t change topology
- Other model:
  - **Snakes with Topology Control**

  -------*Stephan Bischoff, Leif Kobbelt*
- Higher dimension:
  - GVF—C.Xu,98
  - L.Cohen,95
  - …



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- Level set 与变分方程的关系

Level Set —水平集

- Level set 的数学定义

  假设隐函数 $\varphi(x,t)$ 表示一个高维空间的方程，其在低维空间上的接触面为 $\varphi(x,t)=0$ ，其中

  $x=(x_1,x_2,...x_n)\in R^n$

  则level set 方程 $\Gamma(t)$ 有如下的性质，其中接触面表示为

$$\varphi(x,t)<0 \quad for \quad x\in\Omega$$

$$\varphi(x,t)>0 \quad for \quad x\notin\overline{\Omega}$$

$$\varphi(x,t)=0 \quad for \quad x\in\partial\Omega=\Gamma(t)$$

The Level Set Method

- Let us consider in the most general case the following form of curve propagation:

$$C(p,t) = F(\mathcal{K})\mathcal{N}$$

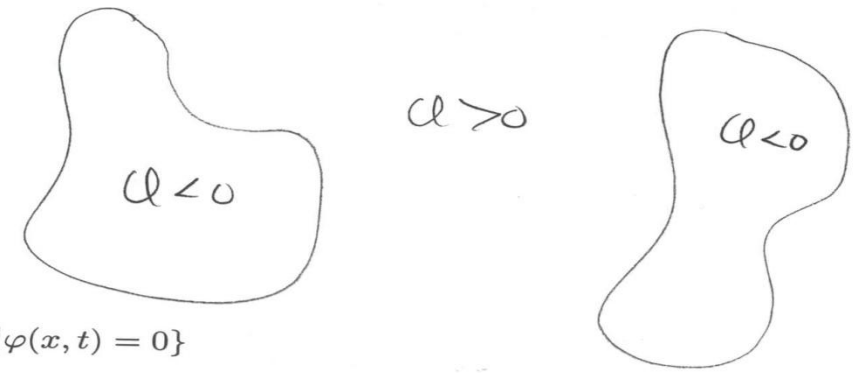
- Addressing the problem in a higher dimension…

- The level set method represents the curve in the form of an implicit surface:

$$\varphi(x,y,t) : \mathcal{R}^2 \times [0,T) \rightarrow \mathcal{R}$$

- That is derived from the initial contour according to the following condition:

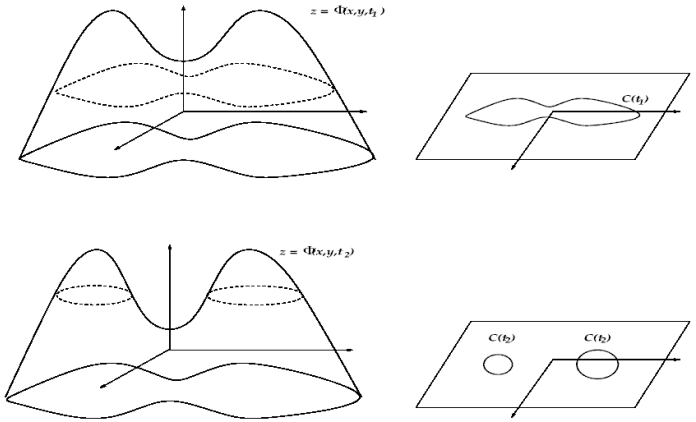
$$C(p,0) = \{(x,y) : \varphi(x,y,0) = 0\}$$



$\{x|\varphi(x,t) = 0\}$ defines $\Gamma(t)$.

The Level Set Method

- Construction of the implicit function



$$C(p,0) = \{(x,y) : \varphi(x,y,0) = 0\}$$
$$C(p,t) = \{(x,y) : \varphi(x,y,t) = 0\}\,,\; C(t) = \varphi^{-1}(0)$$

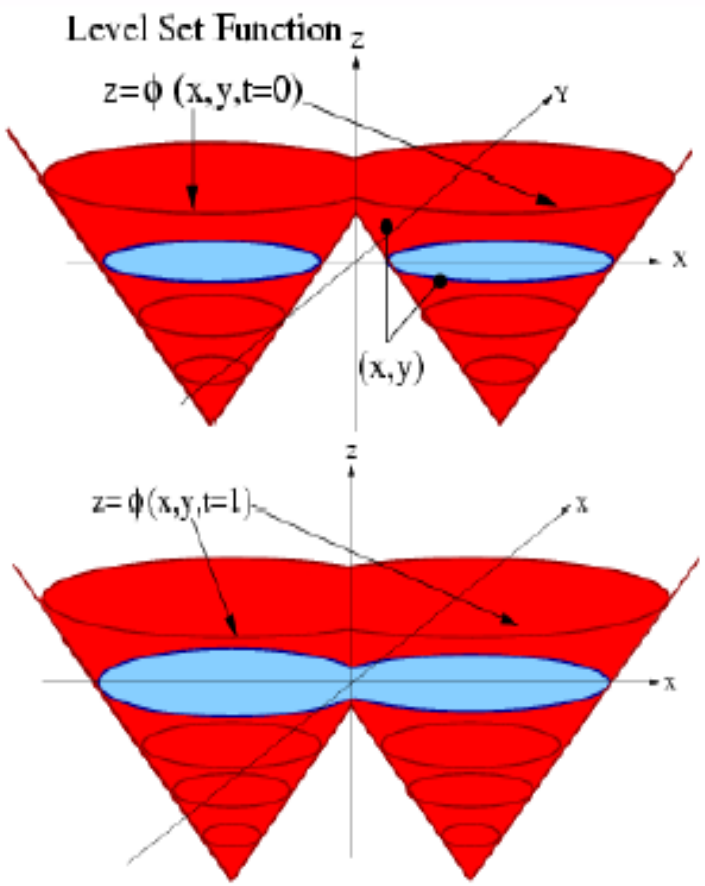
- And taking the derivative with respect to time (using the chain rule)

$$\varphi(C(t),t) = 0 \quad \Rightarrow \quad \frac{\partial \varphi}{\partial C} \cdot \underbrace{\frac{\partial C}{\partial t}}_{FN} + \underbrace{\frac{\partial \varphi}{\partial t}}_{\varphi_t} = 0 \qquad (1)$$

- And we are DONE…

What is Level Set

- Adding an extra dimension to the problem



The level set function:

$$z = \Phi(x,y,t)$$

Contour at time *t:*

$$0 = \Phi(x,y,t)$$

The level set PDE:

$$\Phi_t + F|\nabla\Phi| = 0$$

given $\Phi(x,y,t=0)$

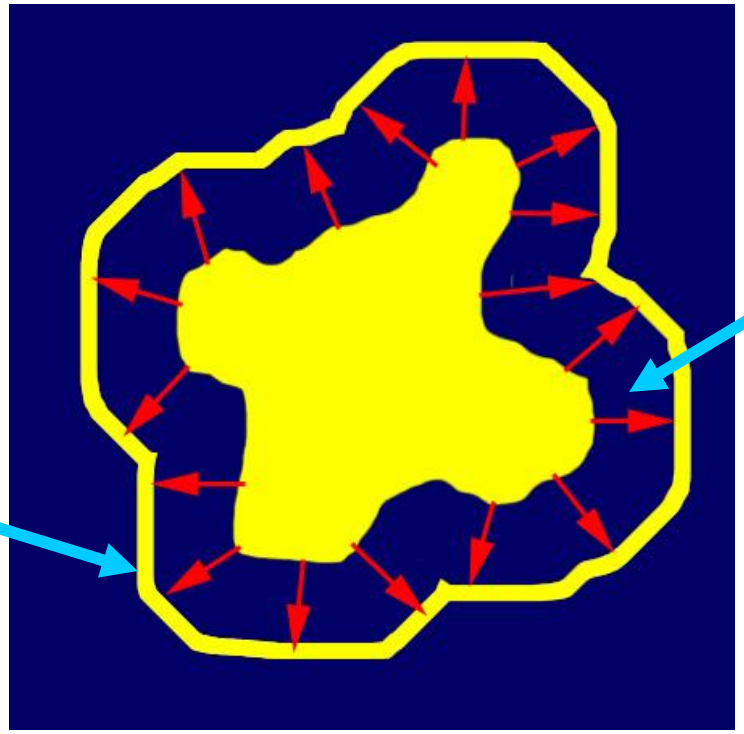
Evolving the level set

- Initialization of Ø:

$$\psi(x, t = 0) = \pm d$$

$$\begin{aligned} \varphi(x,t) &> 0 \text{ for } x \in \Omega \\ \varphi(x,t) &< 0 \text{ for } x \notin \bar{\Omega} \end{aligned}$$

- Pr $\varphi(x,t) = 0 \text{ for } x \in \partial\Omega = \Gamma(t)$



Level Set —水平集

- Level set 的运动可以表示为

$$\frac{\partial \varphi}{\partial t} + v \cdot \nabla \varphi = 0$$

- 假设 $v_N = v \cdot \frac{\nabla \varphi}{|\nabla \varphi|}$ 则上式成为

$$\frac{\partial \varphi}{\partial t} + v_N |\nabla \varphi| = 0$$

Level Set —水平集

- Level set 方法的几何意义很直观：
  - 给定一个高维空间在低维空间（n维）定义上的接触面，分析和计算其边界在速度v下的运动轨迹
  - 速度v是与位置，时间和接触面的几何形状有关的（如平均曲率，法向），还有外部的物理作用力。

(UCLA的Osher和Sethian首先提出了这个方法)

Level Set —水平集

- 小结
  - Level set 方法实际上就是求解一个随时间变化的偏微分方程

$$\frac{\partial \varphi}{\partial t} + v_N |\nabla \varphi| = 0$$

  - 其中 v_N 表示可以是任何关于时间，位置，几何等量的函数。

  - 有时提到 Level set 方法是指其数值解方法
  - 应用Level set方法需要解决两个问题
    - 如何列出有意义的方程求解实际问题
    - 如何能快速、稳定地求出方程的数值解

Level Set—水平集

- 通常，演化速度F含有三种成份：

$$\frac{\partial \varphi}{\partial t} = \mathbf{g}(\alpha + \beta k)\mathbf{n}, \qquad \alpha, \beta \in R, \beta \geq 0$$

1. 与曲率相关的所谓扩散项起到保持曲线的光滑性的作用
2. 对流项为曲线演化提供动力支持
3. 速度衰减因子使速度在边缘轮廓处停止

边界检测和轮廓线提取

- 隐式动态轮廓模型
  - 用隐式模型可以跟踪拓扑变化的轮廓
  - 隐式轮廓线的微分方程表达为

$$\frac{\partial \varphi}{\partial t} = g(x)|\nabla \varphi|(\nabla \cdot (\frac{\nabla \varphi}{|\nabla \varphi|}) + k)$$

$$\varphi(x,0) = \varphi_0(x)$$

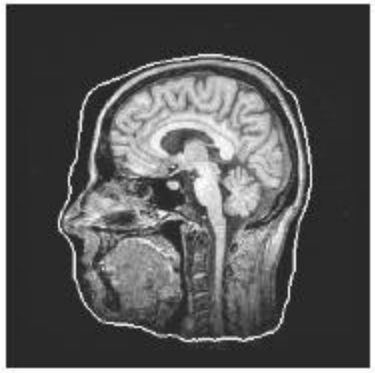
  - 注意轮廓不但被梯度驱动，而且被曲率驱动

边界检测和轮廓线提取—隐式动态轮廓模型

- 试验结果








Level Set —水平集

- Level set 的主要优点就是其考虑了物体几何的一些更本质的特征（如曲率，梯度等），所以得到的结果能够比已有的一些其他方法要好。

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Level set 的数值解法

- 几种差分方法的稳定性分析
  - 假设求解一阶一维方程 $u_t + au_x = 0,\ u(x,0) = u_0(x)$
  - 向前差分格式 $u_j^{n+1} = u_j^n - ar(u_j^n - u_{j-1}^n)$
  - 向后差分格式 $u_j^{n+1} = u_j^n - ar(u_{j+1}^n - u_j^n)$
  - 中心差分格式 $u_j^{n+1} = u_j^n - ar(u_{j+1}^n - u_{j-1}^n)/2$

Level set 的数值解法

- 向前差分格式的稳定性条件 $0 \le ar \le 1$
- 向后差分格式的稳定性条件 $-1 \le ar \le 0$
- 中心差分格式的稳定性条件 恒不稳定

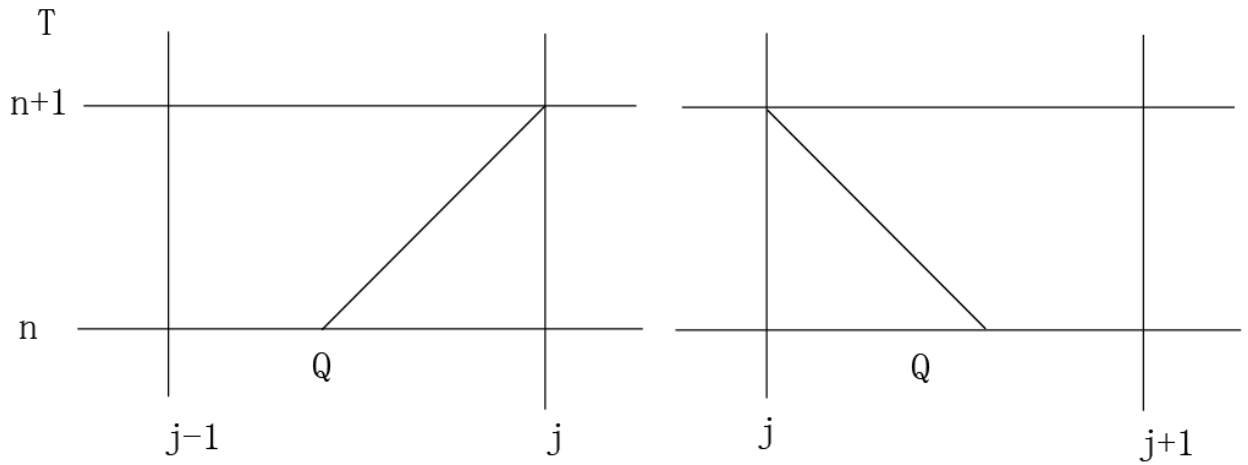
（稳定性条件的分析方法，如Von Neumann方法，参见数值分析的教材）

Level set 的数值解法—CIR格式

- 所以选择差分格式要根据a的符号来判断

解为
$$u_j^{n+1}=\begin{cases}u_j^n-ar(u_j^n-u_{j-1}^n), a\geq 0\\ u_j^n-ar(u_{j+1}^n-u_j^n), a<0\end{cases}$$

或
$$\frac{u_j^{n+1}-u_j^n}{\tau}+a^+\frac{u_j^n-u_{j-1}^n}{h}-a^-\frac{u_{j+1}^n-u_j^n}{h}=0$$



Level set 的数值解法—CIR格式

- 其中

$$a^{\pm} = \frac{1}{2}(|a| \pm a)$$

上式称为Courant-Isaacson-Rees格式（CIR格式）

在求解其它类型的偏微分方程时也要用到类似的格式，不过分析其稳定性要复杂很多。

Level set 的数值解法—LF格式

- 如果将中心差分格式的 u_j^n 替换为 $\frac{1}{2}(u_{j+1}^n + u_{j-1}^n)$ 则得到Lax-Friedrichs格式

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{ar}{2}(u_{j+1}^n - u_{j-1}^n)$$

其稳定性条件为 $|ar| \le 1$

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Upwind 差分法

- 假设 $t^n = n\Delta t$

$$\frac{\partial \varphi}{\partial t} + v \cdot \nabla \varphi = 0 \quad \Rightarrow \quad \frac{\varphi^{n+1} - \varphi^n}{\Delta t} + v^n \cdot \nabla \varphi^n = 0$$

根据CIR格式，先考虑一维的情况，当$v^n > 0$ 时，曲线从左往右移动，所以要用到 φ_i^n左边的值，即用向后差分来估计$\nabla \varphi^n$. 同理，当 $v^n < 0$时，用向前差分。

Upwind 差分法

- 算法的精度

$$O(\Delta x)$$

- 算法的稳定条件

$$\Delta t \max\{\frac{|v_x|}{\Delta x}+\frac{|v_y|}{\Delta y}+\frac{|v_z|}{\Delta z}\} \in (0,1)$$

可见*upwind*差分法虽然简单，但是精度不高，计算速度慢

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Hamilton-Jacobi ENO 方法

- ENO: Essentially Nonoscillatory（不波动，不摇动）
- 基本思想：用尽量光滑的多项式插值 φ 然后再求解 φ_x。用HJ ENO方法可以更精确地估计 φ_x^+ 或者 φ_x^- 。

$$D_i^0 = \varphi_i$$

- 定义算子

$$D_{i+1/2}^1 \varphi = \frac{D_{i+1}^0 \varphi - D_i^0 \varphi}{\Delta x}$$

$$D_i^2 \varphi = \frac{D_{i+1/2}^1 \varphi - D_{i-1/2}^1 \varphi}{2\Delta x}$$

$$\cdots$$

Hamilton-Jacobi ENO 方法

- 上面的分步差分可以用来重建如下形式的插值多项式

$$\varphi(x) = Q_0(x) + Q_1(x) + Q_2(x) + Q_3(x)$$

然后通过对$\varphi(x)$的求导可以得到 φ_x^- 或 φ_x^+ 的估计值。

$$\varphi_x(x_i) = Q_1'(x_i) + Q_2'(x_i) + Q_3'(x_i)$$

- 例如，为了估计φ_x^-，应该从k=i-1开始，为了估计φ_x^+，应该从k=i开始。

然后定义

$$Q_1(x) = (D^1_{k+1/2}\varphi)(x - x_i)$$

Hamilton-Jacobi ENO 方法

即 $Q_1'(x) = D^1_{k+1/2}\varphi$

得到 φ_x 的一阶精度估计（其形式是和upwind方法得到的形式是一样的）

- 同样可以得到二阶和更高阶精度的估计。

  二阶精度估计

$$Q_2'(x_i) = D^2_{k+1/2}\varphi(2(i-k)-1)\Delta x$$

  三阶精度估计$k^*=k-1$

$$Q_3'(x_i) = D^3_{k+1/2}\varphi(3(i-k^*)^2 - 6(i-k^*)^2 + 2)(\Delta x)^2$$

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Hamilton-Jacobi WENO 方法

- WENO: Weighted ENO
- 当计算$(\varphi_x^-)_i$时，三阶精度的HJ ENO算法需要知道$\{\varphi_{i-3},\varphi_{i-2},\varphi_{i-1},\varphi_i,\varphi_{i+1},\varphi_{i+2}\}$的值，共有3种HJ ENO估计$(\varphi_x^-)_i$的方法。定义

$$v_1 = D^-\varphi_{i-2}$$

$$v_2 = D^-\varphi_{i-1}$$

$$v_3 = D^-\varphi_i$$

$$v_4 = D^-\varphi_{i+1}$$

$$v_5 = D^-\varphi_{i+2}$$

则三种估计为

$$\varphi_x^1 = \frac{v_1}{3} - \frac{7v_2}{6} + \frac{11v_3}{6}$$

$$\varphi_x^2 = -\frac{v_2}{3} + \frac{5v_3}{6} + \frac{v_4}{3}$$

$$\varphi_x^3 = \frac{v_3}{3} + \frac{5v_4}{6} - \frac{v_5}{6}$$

Hamilton-Jacobi WENO 方法

- HJ ENO方法的目的就是从上面3个估计中选出一个最光滑的多项式逼近。
- 后来有人提出这种HJ ENO方法可以进一步通过将3个式子加权来提高精度。
- WENO的形式可以写成

$$\varphi_x = w_1\varphi_x^1 + w_2\varphi_x^2 + w_3\varphi_x^3,$$

$$w_1 + w_2 + w_3 = 1, \quad 0 \le w_i \le 1$$

$$w_1 = 0.1, w_2 = 0.6, w_3 = 0.3$$

可以证明对光滑区域可以达到5阶精度！

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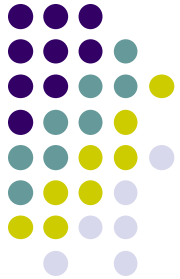
TVD Runge-Kutta 方法



- 前面提到的HJ ENO 和 HJ WENO可以达到5阶精度，向前Euler算法（Upwind算法）只能达到1阶精度。
- 用TVD RK方法可以达到更高阶的精度
- TVD: total variation diminishing
- 一阶TVD RK就是向前Euler算法。
- 二阶TVD RK和二阶RK算法相似。

TVD Runge-Kutta 方法

- 三阶TVD RK算法

$$\frac{\varphi^{n+1}-\varphi^{n}}{\Delta t}+v^{n}\cdot\nabla\varphi^{n}=0$$

$$\frac{\varphi^{n+2}-\varphi^{n+1}}{\Delta t}+v^{n+1}\cdot\nabla\varphi^{n+1}=0$$

$$\varphi^{n+1/2}=\frac{3}{4}\varphi^{n+1}+\frac{1}{4}\varphi^{n+2}$$

$$\frac{\varphi^{n+3/2}-\varphi^{n+1/2}}{\Delta t}+v^{n+1/2}\cdot\nabla\varphi^{n+1}=0$$

$$\varphi^{n+1}=\frac{1}{3}\varphi^{n}+\frac{2}{3}\varphi^{n+3/2}$$

TVD Runge-Kutta 方法

- 虽然4阶以上的TVD RK方法存在，但在实际应用中在时间上的精度的提高并不会对结果有太大的改进，而且从计算复杂性上考虑会得不偿失。

求解Level set的数值方法小结

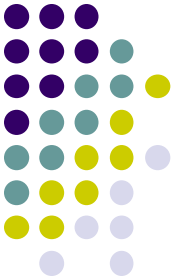
- Level set 的一般形式可以表达为

$$\frac{\partial \varphi}{\partial t} + v \cdot \nabla \varphi = 0$$

$$v = a(x) \cdot \nabla \varphi - \mu(x) \nabla \cdot \left(\frac{\nabla \varphi}{|\nabla \varphi|} \right)$$

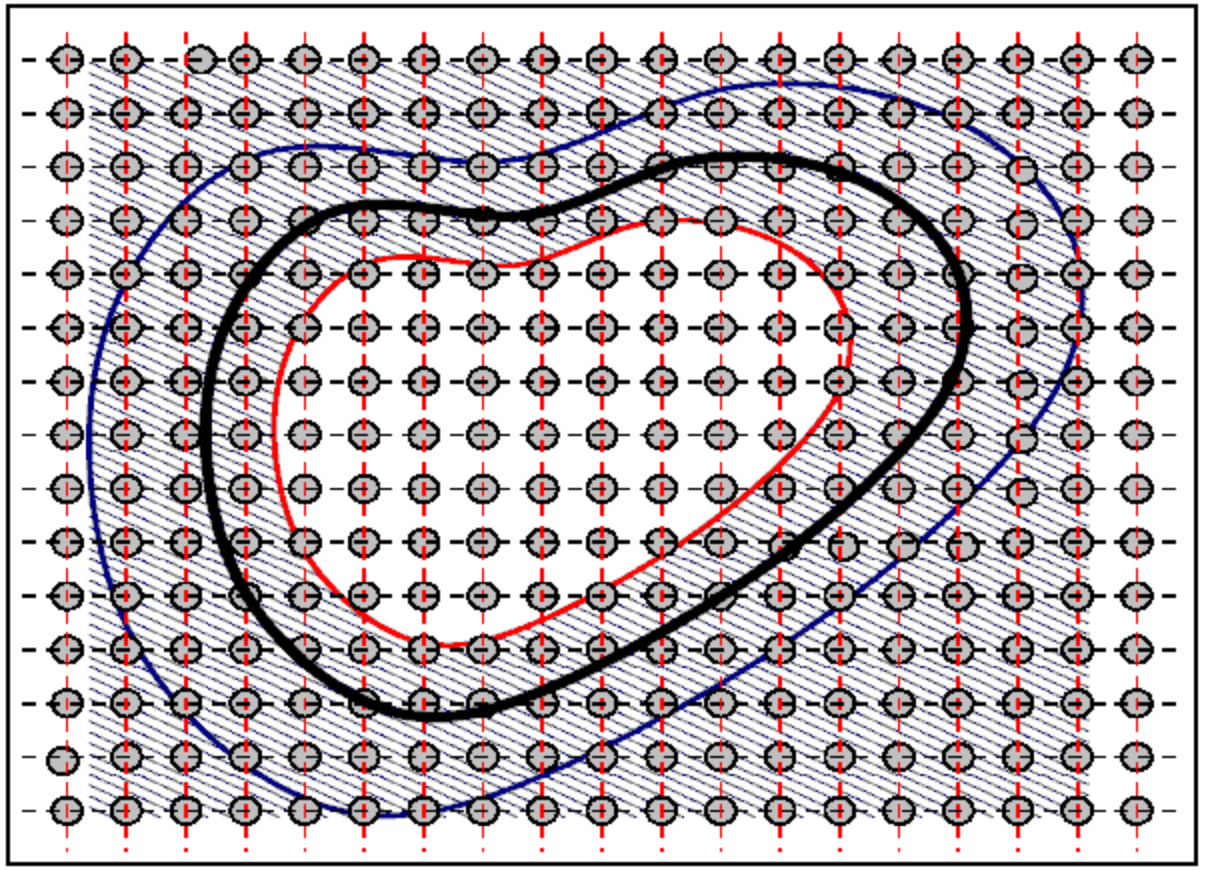
其中 $a(x) \cdot \nabla \varphi$ 称为对流项，$\nabla \cdot \left(\frac{\nabla \varphi}{|\nabla \varphi|} \right)$ 为曲率。

求解Level set的数值方法小结



- 求解该方程一般分为3步
  - 用ENO,WENO或upwind方法求解对流项。
  - 用中心差分的方法估算曲率。
  - 用TVD RK方法来求解。

更高级的解法？！？



Outward Band

$\Phi(s) = +d$

Front Position

$\Phi(s) = 0$

Inward Band

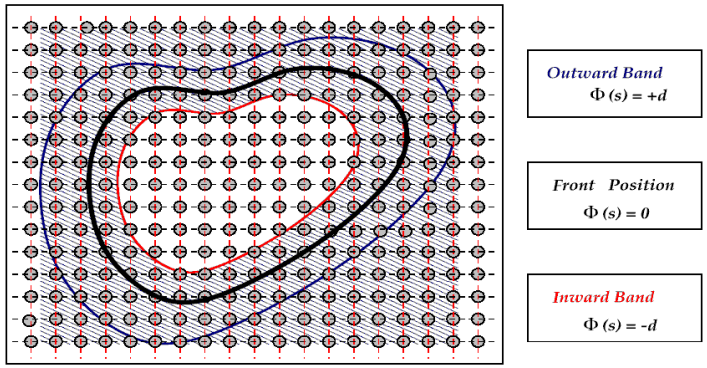
$\Phi(s) = -d$

From theory to Practice (Narrow Band)

[Chop:93, Adalsteinsson-Sethian:95]

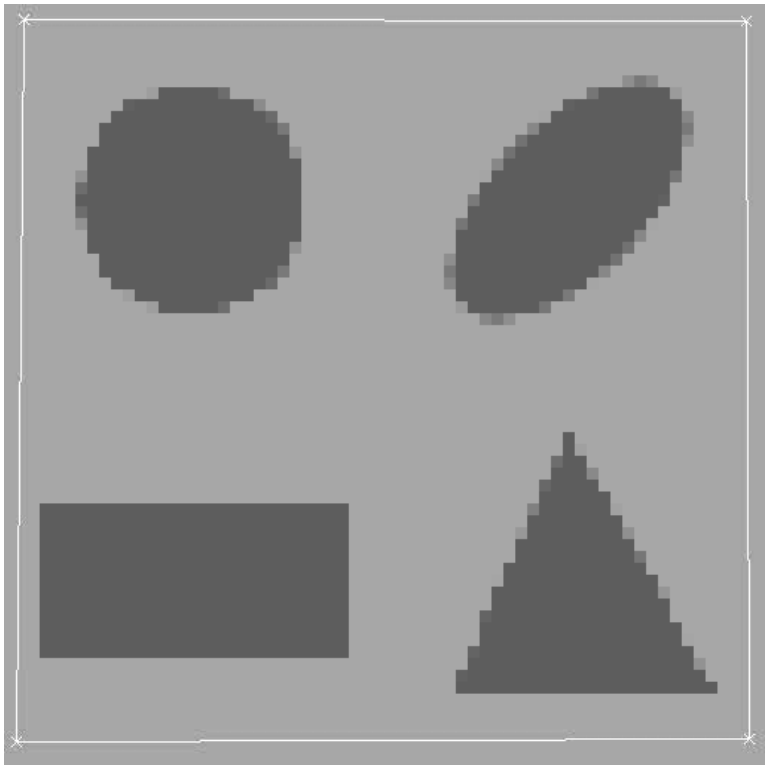
- Central idea: we are interested on the motion of the zero-level set and not for the motion of each iso-phote of the surface

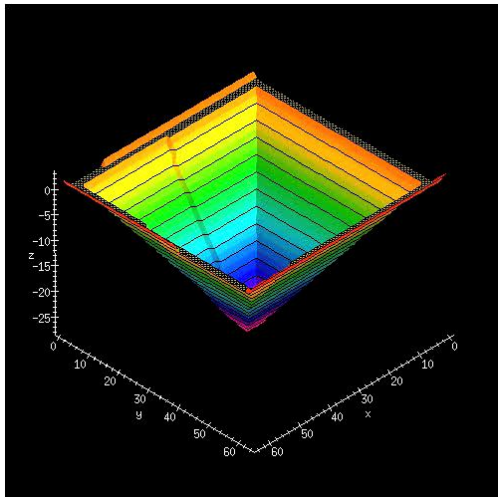
  - Extract the latest position
  - Define a band within a certain distance
  - Update the level set function
  - Check new position with respect the limits of the band
  - Update the position of the band regularly, and re-initialize the implicit function

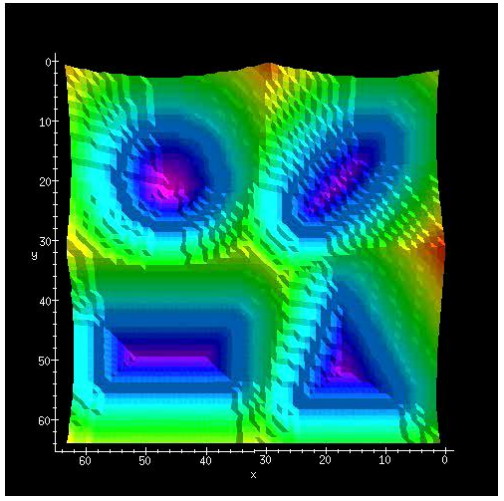


- Significant decrease on the computational complexity, in particular when implemented efficiently and can account for any type of motion flows

Narrow Band (the basic derivation)







Results are courtesy: R. Deriche

Handling the Distance Function

- The distance function has to be frequently re-initialized…
  - Extraction of the curve position & re-initialization:
    - Using the marching cubes one can recover the current position of the curve, set it to zero and then re-initialize the implicit function: the Borgefors approach, the Fast Marching method, explicit estimation of the distance for all image pixels…
  - Preserving the curve position and refinement of the existing function (Susman-smereka-osher:94)

$$\frac{d}{d\tau}\phi_m = \mathrm{sgn}\left(\phi_m^0\right)\left(1 - |\nabla\phi_m|\right)$$

  - Modification on the level set flow such that the distance transform property is preserved (gomes-faugeras:00)
    - Extend the speed of the zero level set to all iso-photes, rather complicated approach with limited added value?

From theory to Practice (Fast Marching)

[Tsitsiklis:93,Sethian:95]

- Central idea: "move" the curve one pixel in a progressive manner according to the speed function while preserving the nature of the implicit function

- Consider the stationary equation $F\ |\nabla T| = 1 \cdot$

- Such an equation can be recovered for all $\left[\frac{\partial C}{\partial t} = F\mathcal{N}\right]$ flows where the speed function has one sign (either positive or negative), propagation takes place at one direction

- If T(x,y) is the time when the implicit function reaches (x,y):

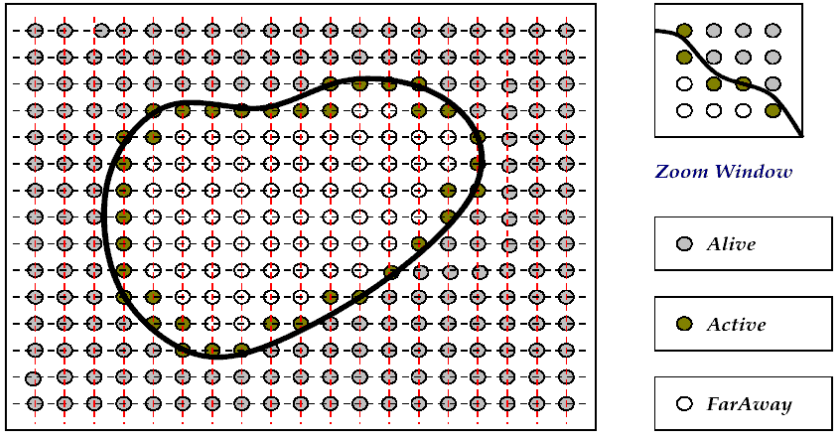
$$\begin{aligned} T(C(p,t)) \stackrel{\triangle}{=} t \quad &\Rightarrow \quad \nabla T \cdot C_t = 1 \\ &\Rightarrow \quad \nabla T \cdot \left(F \frac{\nabla T}{|\nabla T|} \right) = 1 \\ &\Rightarrow \quad F\ |\nabla T| = 1 \end{aligned}$$

Fast Marching (continued)

- Consider the stationary equation $F\ |\nabla T| = 1$ · in its discrete form:

$$\frac{1}{F_{\{i,j\}}^2} = \max\left(D_{\{i,j\}}^{-x}T,0\right)^2 + \min\left(D_{\{i,j\}}^{+x}T,0\right)^2 + \max\left(D_{\{i,j\}}^{-y}T,0\right)^2 + \min\left(D_{\{i,j\}}^{+y}T,0\right)^2$$

- And using the assumption that the surface propagates in one direction, the solution can be obtained by outwards propagation from the smallest T value...



  - active pixels, the curve has already reached them
  - alive pixels, the curve could reach them at the next stage
  - far away pixels, the curve cannot reach them at this stage

Fast Marching (continued)

- INITIAL STEP
  - Initialize $[T = 0]$ for the all pixels of the front (active), their first order neighbors alive and the rest far away
  - For the first order neighbors, estimate the arrival time according to: $\left[T_{\{i,j\}} = \frac{1}{F_{\{i,j\}}}\right]$
  - While for the rest the crossing time is set to infinity $[T_{\{i,j\}} = \infty]$

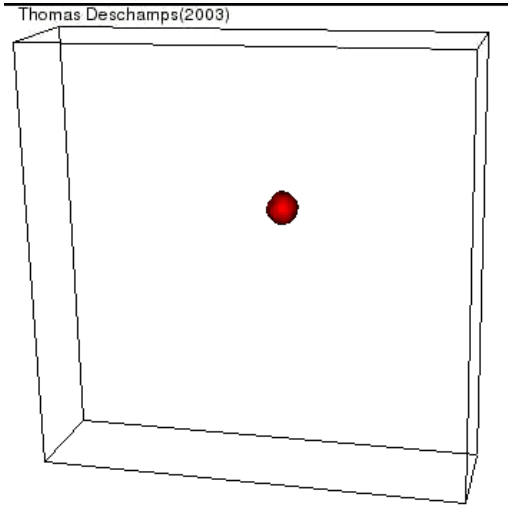
- PROPAGATION STEP
  - Select the pixel with the lowest arrival time from the alive ones
  - Change his label from alive to active and for his first order neighbors:
    - If they are alive, update their T value according to

$$\frac{1}{F^2_{\{i,j\}}} = \max\left(D^{-x}_{\{i,j\}}T, 0\right)^2 + \min\left(D^{+x}_{\{i,j\}}T, 0\right)^2 + \max\left(D^{-y}_{\{i,j\}}T, 0\right)^2 + \min\left(D^{+y}_{\{i,j\}}T, 0\right)^2$$

    - If they are far away, estimate the arrival time according to: $\left[T_{\{i,j\}} = \frac{1}{F_{\{i,j\}}}\right]$

Fast Marching Pros/Cons, Some Results

- Fast approach for a level set implementation
- Very efficient technique for re-setting the embedding function to be distance transform
- Single directional flows, great importance on initial placement of the contours
- Absence of curvature related terms or terms that depend on the geometric properties of the curve…
- Results are courtesy: J. Sethian, R. Malladi, T. Deschamps, L. Cohen



目录

- 数学基础
  - 微分几何简介
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  - 隐函数，距离函数
- Level Set 概念
- Level set 的数值解法
  - Upwind 差分法
  - Hamilton-Jacobi ENO
  - Hamilton-Jacobi WENO
  - TVD Runge-Kutta
- Level set 与变分方程的关系

Level set 与变分方程的关系

- 考虑Poisson方程的Dirichlet问题

$$\begin{cases} -\Delta u = f(x), & x \in \Omega \\ u = \varphi(x), & x \in \partial\Omega \end{cases}$$

定义能量函数

$$I[w] = \int_{\Omega} (\frac{1}{2} |Dw|^2 - wf) dx$$

- Dirichlet原理：

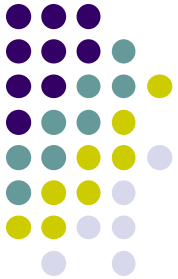
$$I(u) = \min_{w \in A} I[w]$$

Level set 与变分方程的关系

- 一般的，假定 $Au = f(x)$ 是线性偏微分方程，则 $F(u) = (Au, u) - 2(u, f)$ 的极小函数为 $Au = f(x)$ 的解。其中 $(*,*)$ 表示任何Banach空间算子。

- 对于很多偏微分方程，一般都可以找到一个与之对应的变分方程，其中边界值可以通过Green公式化简到内部积分。

Level set 与变分方程的关系



- 对于大多数变分方程，也可以通过Lagrange公式化归为相应的偏微分方程求解。

- 在后面的应用中我们可以看到很多问题都是先列出变分方程，再求解对应的偏微分方程得到Level set方程的。

Level set 与变分方程的关系

- 例(Total Variation模型)：

$$F(u)=\int|\nabla u|dx+\lambda\int|f-u|^2\,dx$$

用Euler-Lagrange方程化为偏微分方程为

$$u=f+\frac{1}{2\lambda}\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)$$

问题的求解可以化为Level set演化的形式，即

$$u_t=-(u-f)+\frac{1}{2\lambda}\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)$$

小结

- Level Set 方法可以看作是一类带时间的偏微分方程框架，其意义不仅是其有很多的应用，而且对这一类方程的求解的方法也是其受到广泛重视的重要原因。

参考资源

- Level set 的Matlab 工具箱
  http://www.cs.ubc.ca/~mitchell/ToolboxLS/
  用matlab实现了三种数值算法

- 有好的C++语言实现的Level Set工具箱吗？

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- [1] Stanley Osher and Ronald Fedkiw. *Level Set Methods and Dynamic Implicit Surfaces*. Springer-Verlag (2002).

- [2] James A. Sethian. *Level Set Methods and Fast Marching Methods*. Cambridge University Press (1999).

- [3] J.A. Sethian. Level Set Methods: Evolving interfaces in geometry, fluid mechanics, computer vision, and materials science. Cambridge University Press(1996).