Level Set (II)

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outline

- Level Set的基本方法
- Level Set的数值解法
- Level Set的建模方法与应用举例
- PDE方法的未来

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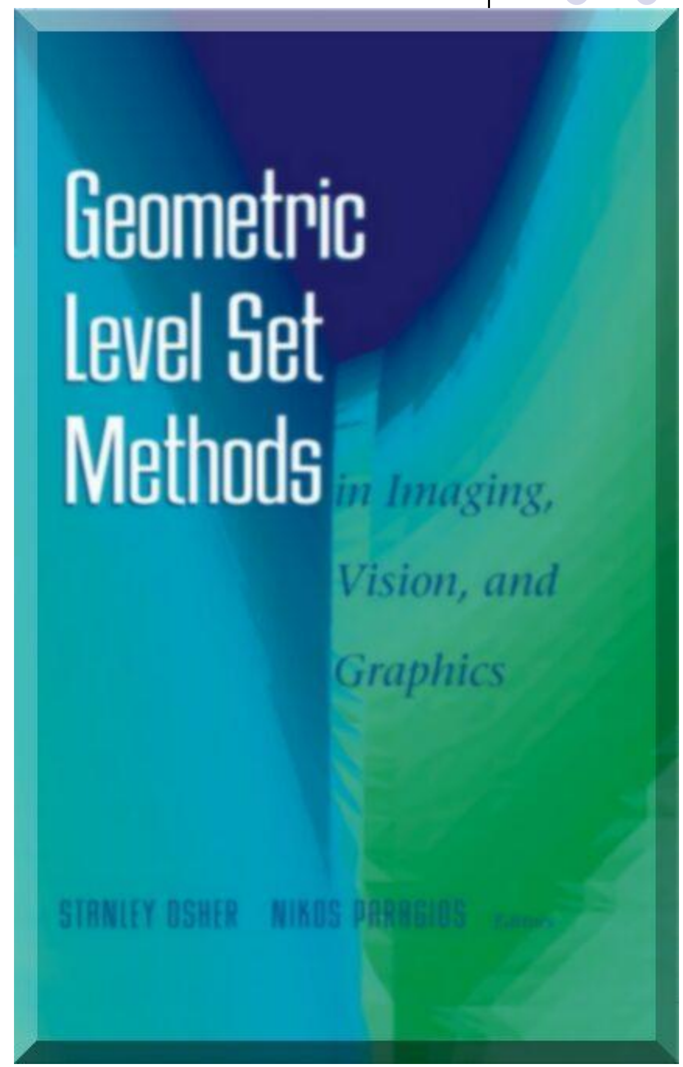


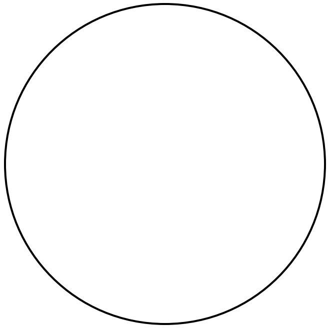
曲线演化的直观解释

- 映射$C(p)$: $[a, b]$ in $\mathbf{R}\rightarrow\mathbf{R}^2$定义了一个平面的曲线，$p$是参数， 对每一个$p_0$ in $[a, b]$, 我们得到曲线上的一点

$$C(p_0)=[x(p_0), y(p_0)]$$

- 正则曲线：如果 $C'(p)=[x'(p), y'(p)]\neq 0$
  - 例：单位圆圆

$$C(t)=[\cos(t),\sin(t)],\quad t\ \ in\ \ [0,2\pi]$$

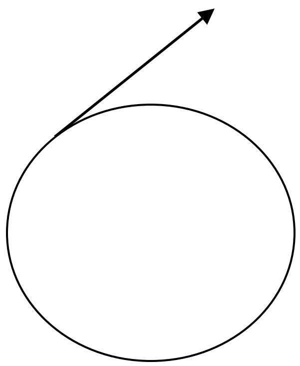


曲线演化的直观解释

- 曲线的切线

$$C'(p)=[x'(p),y'(p)]=\frac{\partial C}{\partial p}$$

- 弧长参数
  - 如果曲线的参数满足 $\left\|\frac{\partial C}{\partial p}\right\|=1$



p表示曲线上以某一点为标准的弧长

曲线演化的直观解释

- 弧长

$$L(p_0, p_1) = \int_{p_0}^{p_1} [(x'(p))^2 + (y'(p))^2]^{1/2} dp$$

- 对弧长参数

$$\langle C'(p), C'(p) \rangle = 1$$

$$\Big\downarrow \text{求导}$$

$$\langle C'(p), C''(p) \rangle = 0$$

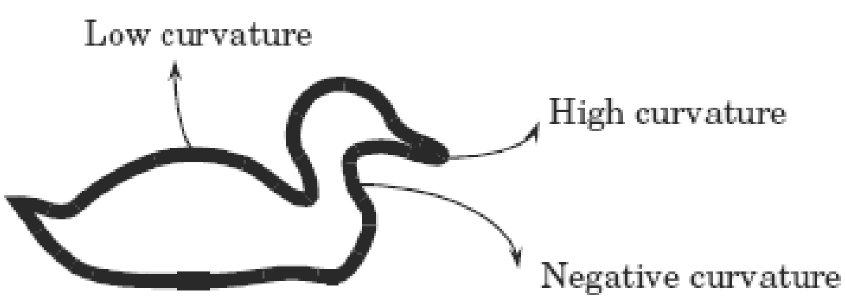
曲线演化的直观解释

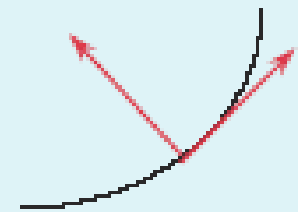
- 曲率

$$\kappa = \left\| C''(p) \right\|$$

- 假设*T*表示切线，*N*表示法线，则

$$\frac{dC}{dp} = T \qquad \frac{d^2C}{dp^2} = \kappa N$$





$$\left\| \frac{\partial C(s)}{\partial s} \right\| = 1$$

$$< C_s, C_{ss} > = 0$$

$$C_s \perp C_{ss}$$

$$\kappa := \left\| C_{ss} \right\|$$

曲线演化的直观解释

- **Frenet** 公式

$$\frac{dT}{dp} = \kappa N$$

$$\frac{dN}{dp} = -\kappa T$$

数学基础—曲线的微分几何

- 曲率的其他定义
  - 假设 θ 为切线T与x轴之间的夹角，则

$$\kappa = \frac{d\theta}{ds}$$

- 隐式曲线的曲率

$$C \equiv \{(x, y) : u(x, y) = 0\}$$

$$\kappa = \frac{u_{xx}u_y^2 - 2u_x u_y u_{xy} + u_{yy}u_x^2}{(u_x^2 + u_y^2)^{3/2}}$$

数学直观

- 隐式曲线的法向量 $u(x,y)=0$

$$N=+(-)\frac{\nabla u}{\|\nabla u\|}$$

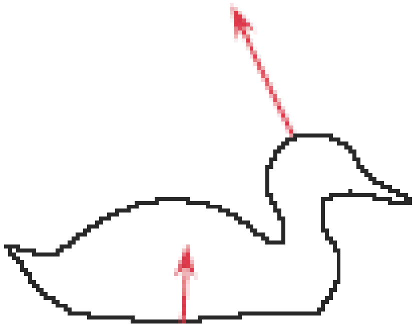
- 因为切向量***T***和法向量***N***互相垂直，所以平面上任何曲线都可以用曲线上任何一点的T和N的线性组合来表示

$$\frac{\partial C}{\partial t}=\alpha T+\beta N$$

曲线演化的直观解释

- 如果只考虑**几何形状**的变化，那其变化只跟法线方向的变化有关系，则有

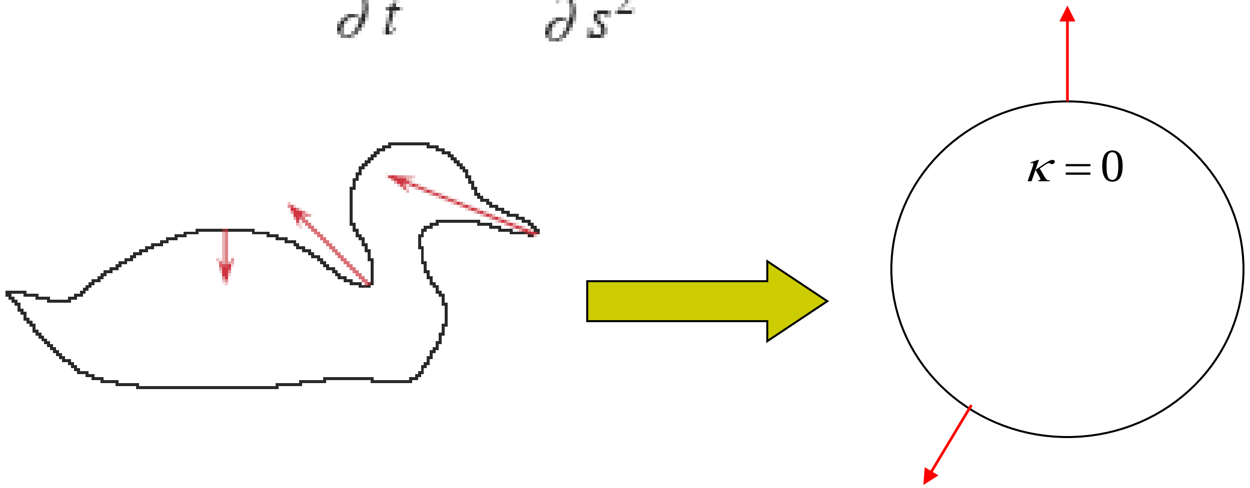
$$\frac{\partial C}{\partial t} = \beta N$$



曲线演化的直观解释

- 例：沿着曲率变化最大方向的曲线变形

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial s^2} = \kappa \vec{N}$$



最后变化为曲率都为常数的曲线停止，即圆

曲面演化的直观解释

- 平均曲率和高斯曲率
  - 每个正则曲面都有两个主曲率。
  - 两个主曲率的平均值就是**平均曲率**
  - 两个主曲率的积是**高斯曲率**

数学基础—隐函数

- 隐函数(implicit function):　　自变量和因变量之间的法则是由一个方程式所确定

$$F(x, y) = 0$$

$$y = y(x)$$

- 例子

$$x^2 + y^2 - 1 = 0$$

数学基础—距离场函数

- 距离函数定义

$$d(x) = \min(|x - x_l|) \quad for\ all \quad x_l \in \partial\Omega$$

- 距离函数的性质

$$|\nabla d| = 1$$

数学基础—距离场函数

- 带符号的距离场隐函数

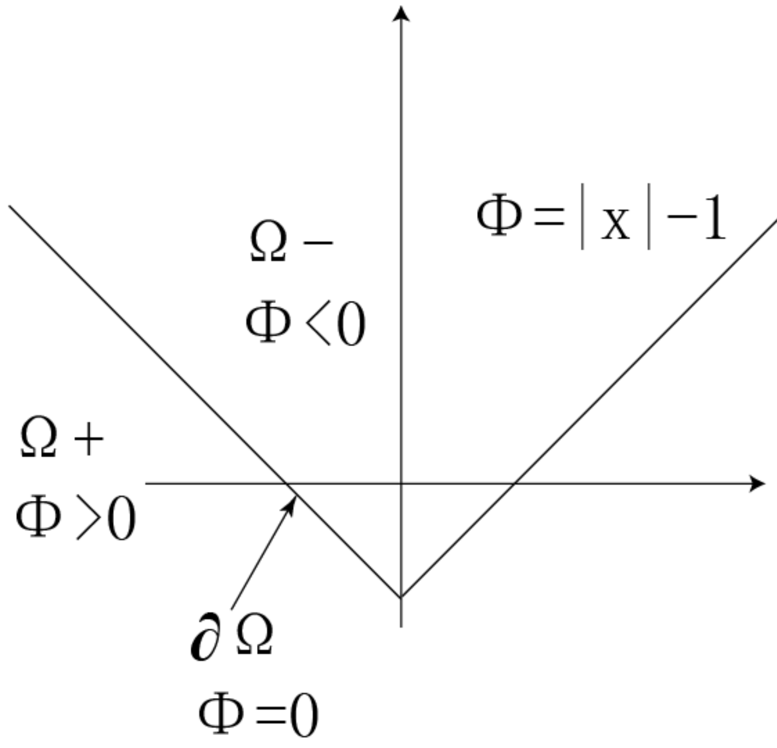
$$\Phi(x)$$

$$|\Phi(x)| = d(x)$$

$$\Phi(x) = \begin{cases} -d(x) & for \ x \in \Omega^- \\ 0 & for \ x \in \partial\Omega \\ -d(x) & for \ x \in \Omega^+ \end{cases}$$

数学基础—距离场函数

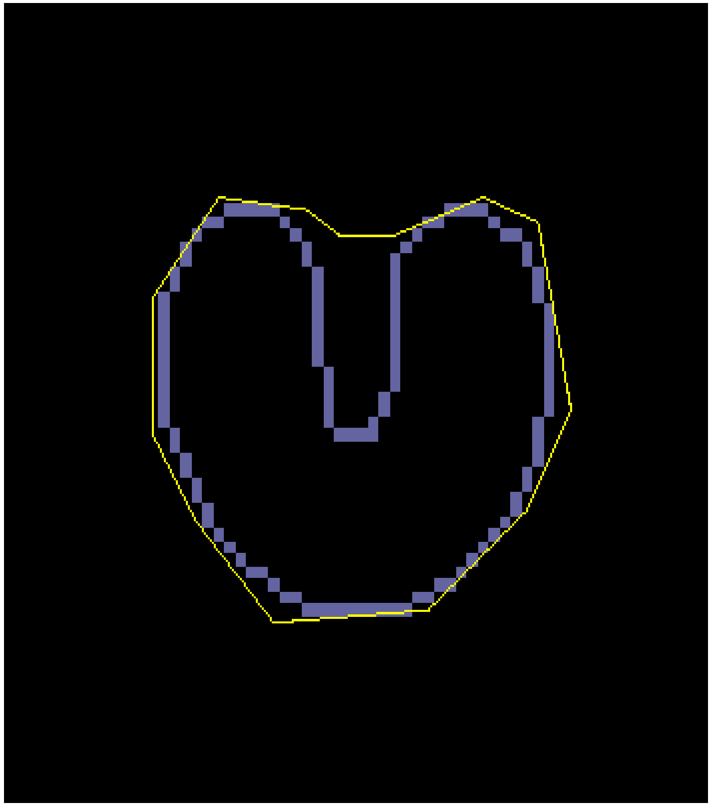
- 例子 $\Phi(x)=|x|-1$



动态可变形模型

What is Snake?

- Active contour model; parametric model
- Result from Kass,Witkin,and Terzopoulos,1987
- Energy minimizing formulation
- Depends on its shape and location within image



Traditional Snake model

- Snake Model (1987) [Kass-Witkin-Terzopoulos]
  - Planar parameterized curve C:R-->RxR
  - A cost function defined along that curve

$$E[(C)(p)] = \alpha \int_0^1 E_{int}(C(p))dp + \beta \int_0^1 E_{img}(C(p))dp + \gamma \int_0^1 E_{con}(C(p))dp$$

  - The internal term stands for regularity/smoothness along the curve and has two components (resisting to stretching and bending)
  - The image term guides the active contour towards the desired image properties (strong gradients)
  - The external term can be used to account for user-defined constraints, or prior knowledge on the structure to be recovered
  - The lowest potential of such a cost function refers to an equilibrium of these terms

Active Contour Components

- The internal term…

$$E_{int}(C(p)) = w_{tension}(C(p)) \left|\frac{\partial C}{\partial p}(p)\right|^2 + w_{stiffness}(C(p)) \left|\frac{\partial^2 C}{\partial p^2}(p)\right|^2$$

  - The first order derivative makes the snake behave as a membrane
  - The second order derivative makes the snake act like a thin plate

- The image term…

$$E_{img}(C(p)) = w_{line}E_{line}(C(p)) + w_{edge}E_{edge}(C(p)) + w_{term}E_{term}(C(p))$$

  - Can guide the snake to
    - Iso-phote $E_{line}(C(p)) = I(C(p))$, edges $E_{edge}(C(p)) = |\nabla I(C(p))|^2$
    - and terminations

- Numerous Provisions…: balloon models, region-snakes, etc…

Optimizing Active Contours

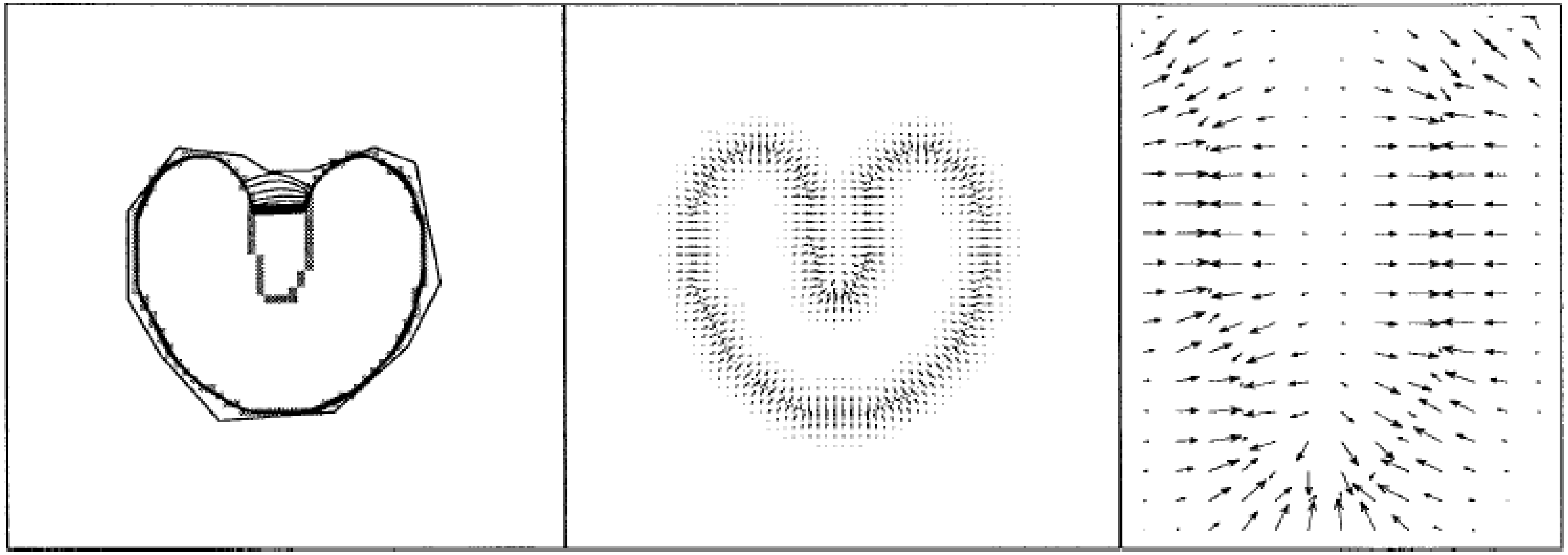
- Taking the Euler-Lagrange equations:

$$\alpha \left(w_{tension} \frac{\partial^2 C}{\partial p^2}(p) - w_{stiffness} \frac{\partial^4 C}{\partial p^4}(p) \right) - \beta \nabla E_{img}(C(p) = 0$$

- That are used to update the position of an initial curve towards the desired image properties
  - Initial the curve, using a certain number of control points as well as a set of basic functions,
  - Update the positions of the control points by solving the above equation
  - Re-parameterize the evolving contour, and continue the process until convergence of the process…

Problem of traditional snake

- Capture range
- Poor convergence



(a) (b) (c)

Fig. 1. (a) Convergence of a snake using (b) traditional potential forces, and (c) shown close-up within the boundary concavity.

External force---balloons

- L.D.Cohen and I.Cohen,1993
- Push the curve outward

$$F = k_1 \vec{n}(s) - k \frac{\nabla P}{\|\nabla P\|}(v(s)) \quad (0<k_1<k<1)$$

- Problem: Poor convergence

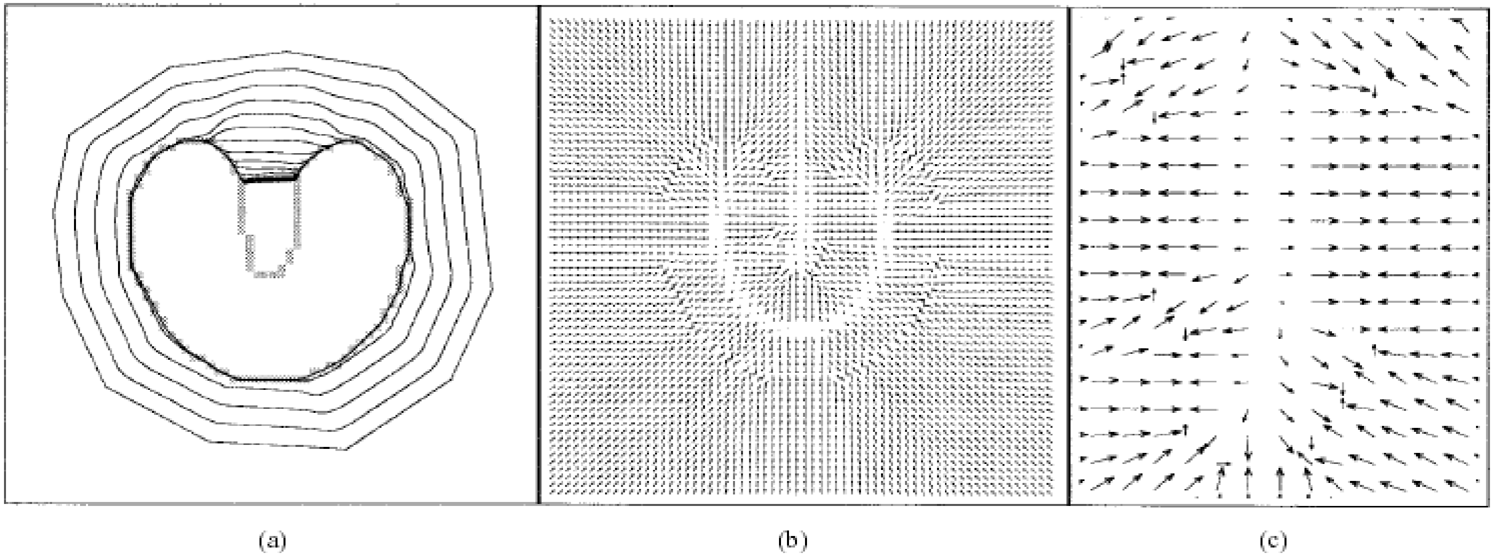


Fig. 2. (a) Convergence of a snake using (b) distance potential forces, and (c) shown close-up within the boundary concavity.

Level Set ?!?

Level Set —水平集

- Level set 的数学定义

  假设隐函数 $\varphi(x,t)$ 表示一个高维空间的方程，其在低维空间上的接触面为 $\varphi(x,t)=0$ ，其中 $x=(x_1,x_2,\ldots x_n)\in R^n$

  则level set 方程 $\Gamma(t)$ 有如下的性质，其中接触面表示为

$$\varphi(x,t)<0 \qquad for \quad x\in\Omega$$

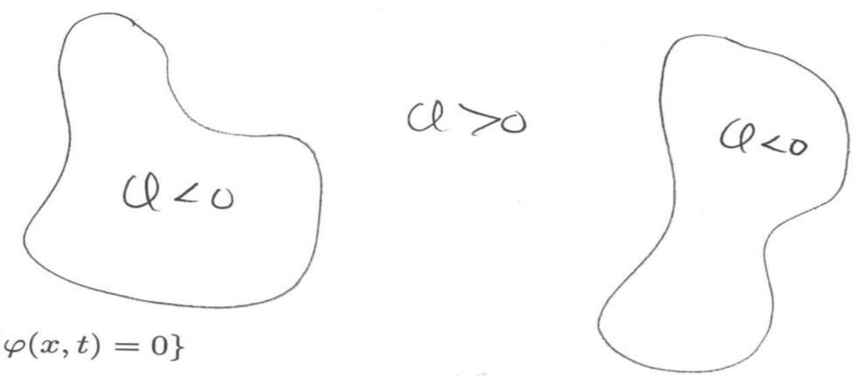
$$\varphi(x,t)>0 \qquad for \quad x\notin\overline{\Omega}$$

$$\varphi(x,t)=0 \qquad for \quad x\in\partial\Omega=\Gamma(t)$$

The Level Set Method

- Let us consider in the most general case the following form of curve propagation: $C(p,t) = F(\mathcal{K})\mathcal{N}$
- Addressing the problem in a higher dimension…
- The level set method represents the curve in the form of an implicit surface: $\varphi(x,y,t) : \mathcal{R}^2 \times [0,T] \rightarrow \mathcal{R}$
- That is derived from the initial contour according to the following condition:

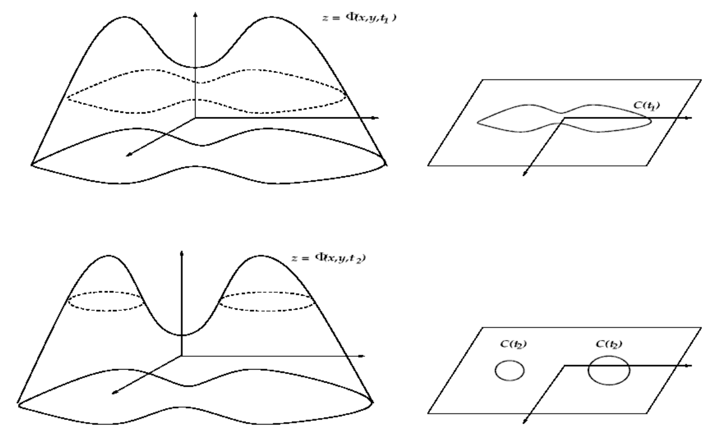
$$C(p,0) = \{(x,y) : \varphi(x,y,0) = 0\}$$



$\{x|\varphi(x,t) = 0\}$ defines $\Gamma(t)$.

The Level Set Method

- Construction of the implicit function



$$C(p,0) = \{(x,y) : \varphi(x,y,0) = 0\}$$

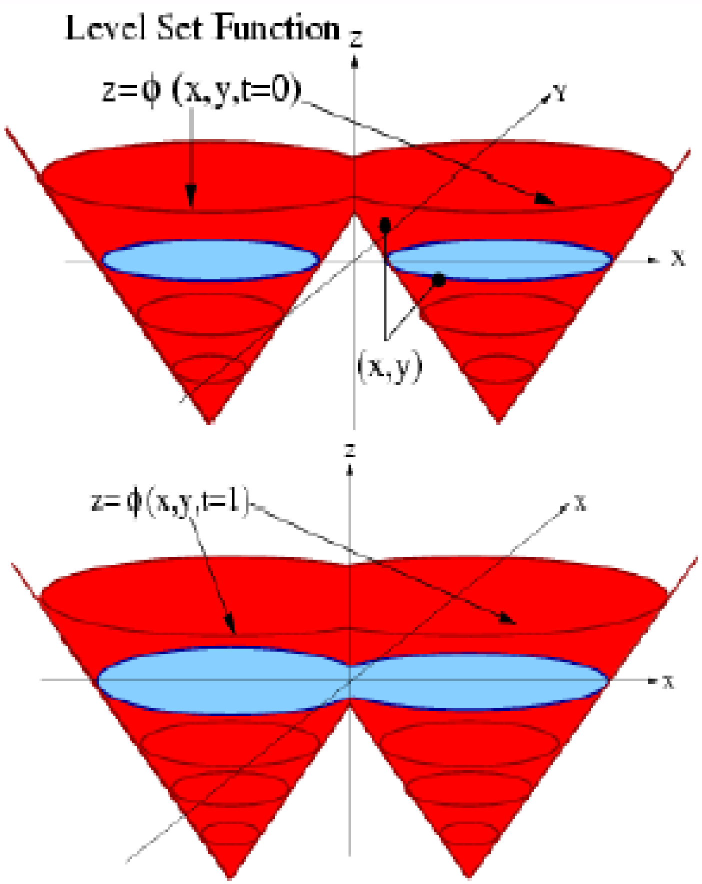
$$C(p,t) = \{(x,y) : \varphi(x,y,t) = 0\}\,,\; C(t) = \varphi^{-1}(0)$$

- And taking the derivative with respect to time (using the chain rule)

$$\varphi(C(t),t) = 0 \quad \Rightarrow \quad \frac{\partial \varphi}{\partial C} \cdot \underbrace{\frac{\partial C}{\partial t}}_{FN} \overset{(1)}{+} \underbrace{\frac{\partial \varphi}{\partial t}}_{\varphi_t} = 0$$

What is Level Set

- Adding an extra dimension to the problem



The level set function:

$$z = \Phi(x,y,t)$$

Contour at time *t:*

$$0 = \Phi(x,y,t)$$

The level set PDE:

$$\Phi_t + F|\nabla\Phi| = 0$$

$$\text{given } \Phi(x,y,t=0)$$

Level Set 一水平集

- Level set 的运动可以表示为

$$\frac{\partial \varphi}{\partial t} + v \cdot \nabla \varphi = 0$$

- 假设 $v_N = v \cdot \frac{\nabla \varphi}{|\nabla \varphi|}$ 则上式成为

$$\frac{\partial \varphi}{\partial t} + v_N |\nabla \varphi| = 0$$

Level Set —水平集

- 小结
  - Level set 方法实际上就是求解一个随时间变化的偏微分方程

$$\frac{\partial \varphi}{\partial t} + v_N |\nabla \varphi| = 0$$

  - 其中 v_N 表示可以是任何关于时间，位置，几何等量的函数。

  - 有时提到 Level set 方法是指其数值解方法
  - 应用Level set方法需要解决两个问题
    - 如何列出有意义的方程求解实际问题
    - 如何能快速、稳定地求出方程的数值解

Level Set—水平集

- 通常，演化速度F含有三种成份：

$$\frac{\partial \mathbf{C}}{\partial t} = g(\alpha + \beta k)\mathbf{n} \qquad \alpha, \beta \in \mathbf{R}, \beta \geq 0$$

1. 与曲率相关的所谓扩散项起到保持曲线的光滑性的作用
2. 对流项为曲线演化提供动力支持
3. 速度衰减因子使速度在边缘轮廓处停止

Level set 的数值解法

- 几种差分方法的稳定性分析
  - 假设求解一阶一维方程 $u_t + au_x = 0,\ u(x,0) = u_0(x)$
  - 向前差分格式 $$u_j^{n+1} = u_j^n - ar(u_j^n - u_{j-1}^n)$$
  - 向后差分格式 $$u_j^{n+1} = u_j^n - ar(u_{j+1}^n - u_j^n)$$
  - 中心差分格式 $$u_j^{n+1} = u_j^n - ar(u_{j+1}^n - u_{j-1}^n)/2$$

Level set 的数值解法

- 向前差分格式的稳定性条件 $0 \le ar \le 1$
- 向后差分格式的稳定性条件 $-1 \le ar \le 0$
- 中心差分格式的稳定性条件 恒不稳定

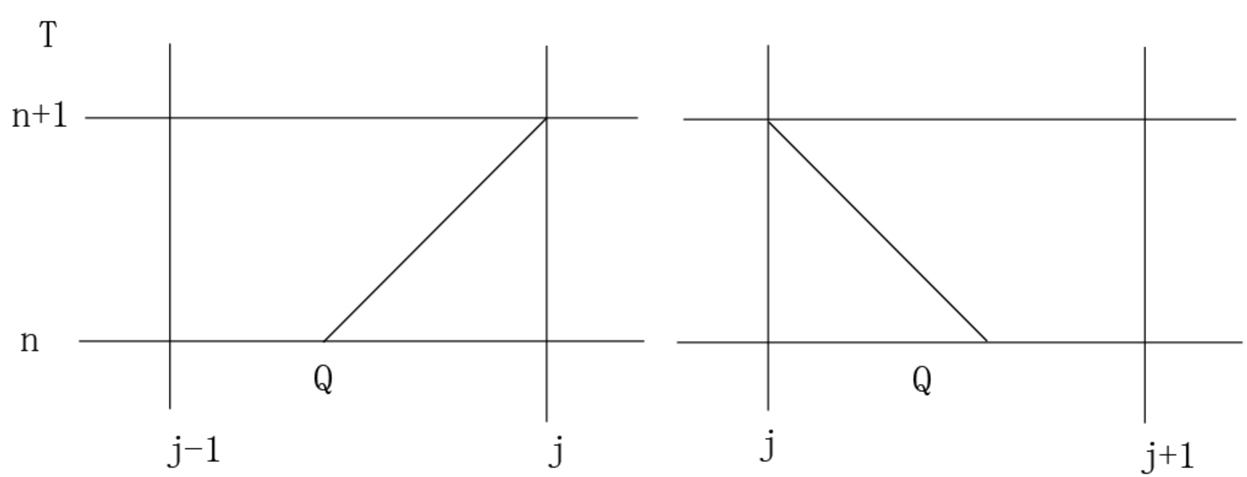
（稳定性条件的分析方法（如Von Neumann方法）参见数值分析的教材）

Level set 的数值解法—CIR格式

- 所以选择差分格式要根据a的符号来判断

解为
$$u_j^{n+1}=\begin{cases}u_j^n-ar(u_j^n-u_{j-1}^n),\ a\geq 0\\ u_j^n-ar(u_{j+1}^n-u_j^n),\ a<0\end{cases}$$

或
$$\frac{u_j^{n+1}-u_j^n}{\tau}+a^+\frac{u_j^n-u_{j-1}^n}{h}-a^-\frac{u_{j+1}^n-u_j^n}{h}=0$$



Level set 的数值解法—CIR格式

- 其中

$$a^{\pm} = \frac{1}{2}(|a| \pm a)$$

上式称为Courant-Isaacson-Rees格式（CIR格式）

在求解其它类型的偏微分方程时也要用到类似的格式，不过分析其稳定性要复杂很多。

Level set 的数值解法—LF格式

- 如果将中心差分格式的 u_j^n 替换为 $\frac{1}{2}(u_{j+1}^n + u_{j-1}^n)$ 则得到Lax-Friedrichs格式

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{ar}{2}(u_{j+1}^n - u_{j-1}^n)$$

其稳定性条件为 $|ar| \leq 1$

Level set 的数值解法

- Upwind 差分法
- Hamilton-Jacobi ENO
- Hamilton-Jacobi WENO
- TVD Runge-Kutta

Upwind 差分法

- 假设 $t^n = n\Delta t$

$$\frac{\partial \varphi}{\partial t} + v \cdot \nabla \varphi = 0 \quad \Rightarrow \quad \frac{\varphi^{n+1} - \varphi^n}{\Delta t} + v^n \cdot \nabla \varphi^n = 0$$

根据CIR格式，先考虑一维的情况，当 $v^n > 0$ 时，曲线从左往右移动，所以要用到 φ_i^n 左边的值，即用向后差分来估计 $\nabla \varphi^n$. 同理，当 $v^n < 0$ 时，用向前差分。

Upwind 差分法

- 算法的精度

$$O(\Delta x)$$

- 算法的稳定条件

$$\Delta t \max\{\frac{|v_x|}{\Delta x}+\frac{|v_y|}{\Delta y}+\frac{|v_z|}{\Delta z}\} \in (0,1)$$

可见*upwind*差分法虽然简单，但是精度不高，计算速度慢

Hamilton-Jacobi ENO 方法

- ENO: Essentially Nonoscillatory（不波动，不摇动）
- 基本思想：用尽量光滑的多项式插值 φ 然后再求解 φ_x。用HJ ENO方法可以更精确地估计 φ_x^+ 或者 φ_x^-。
- 定义算子

$$D_i^0 = \varphi_i$$

$$D_{i+1/2}^1 \varphi = \frac{D_{i+1}^0 \varphi - D_i^0 \varphi}{\Delta x}$$

$$D_i^2 \varphi = \frac{D_{i+1/2}^1 \varphi - D_{i-1/2}^1 \varphi}{2\Delta x}$$

$$\cdots$$

Hamilton-Jacobi ENO 方法

- 上面的分步差分可以用来重建如下形式的插值多项式

$$\varphi(x)=Q_0(x)+Q_1(x)+Q_2(x)+Q_3(x)$$

然后通过对$\varphi(x)$的求导可以得到 φ_x^- 或 φ_x^+ 的估计值。

$$\varphi_x(x_i)=Q_1'(x_i)+Q_2'(x_i)+Q_3'(x_i)$$

- 例如，为了估计φ_x^-，应该从$k=i$-1开始，为了估计φ_x^+，应该从$k=i$开始。

然后定义 $Q_1(x)=(D^1_{k+1/2}\varphi)(x-x_i)$

Hamilton-Jacobi ENO 方法

即　$Q_1'(x) = D^1_{k+1/2}\varphi$

得到 φ_x 的一阶精度估计（其形式是和upwind方法得到的形式是一样的）

- 同样可以得到二阶和更高阶精度的估计。

二阶精度估计

$$Q_2'(x_i) = D^2_{k+1/2}\varphi(2(i-k)-1)\Delta x$$

三阶精度估计$k^*=k-1$

$$Q_3'(x_i) = D^3_{k+1/2}\varphi(3(i-k^*)^2 - 6(i-k^*)^2 + 2)(\Delta x)^2$$

Hamilton-Jacobi WENO 方法

- WENO: Weighted ENO
- 当计算 $(\varphi_x^-)_i$时，三阶精度的HJ ENO算法需要知道 $\{\varphi_{i-3},\varphi_{i-2},\varphi_{i-1},\varphi_i,\varphi_{i+1},\varphi_{i+2}\}$ 的值，共有3种HJ ENO估计$(\varphi_x^-)_i$的方法。定义

$$v_1 = D^-\varphi_{i-2}$$

$$v_2 = D^-\varphi_{i-1}$$

$$v_3 = D^-\varphi_i$$

$$v_4 = D^-\varphi_{i+1}$$

$$v_5 = D^-\varphi_{i+2}$$

则三种估计为

$$\varphi_x^1 = \frac{v_1}{3} - \frac{7v_2}{6} + \frac{11v_3}{6}$$

$$\varphi_x^2 = -\frac{v_2}{3} + \frac{5v_3}{6} + \frac{v_4}{3}$$

$$\varphi_x^3 = \frac{v_3}{3} + \frac{5v_4}{6} - \frac{v_5}{6}$$

Hamilton-Jacobi WENO 方法

- HJ ENO方法的目的就是从上面3个估计中选出一个最光滑的多项式逼近。
- 后来有人提出这种HJ ENO方法可以进一步通过将3个式子加权来提高精度。
- WENO的形式可以写成

$$\varphi_x = w_1\varphi_x^1 + w_2\varphi_x^2 + w_3\varphi_x^3,$$

$$w_1 + w_2 + w_3 = 1, \quad 0 \le w_i \le 1$$

$$w_1 = 0.1, w_2 = 0.6, w_3 = 0.3$$

可以证明对光滑区域可以达到5阶精度！

TVD Runge-Kutta 方法

- 前面提到的HJ ENO 和 HJ WENO可以达到5阶精度，向前Euler算法（Upwind算法）只能达到1阶精度。
- 用TVD RK方法可以达到更高阶的精度
- TVD: total variation diminishing
- 一阶TVD RK就是向前Euler算法。
- 二阶TVD RK和二阶RK算法相似。

TVD Runge-Kutta 方法

- 三阶TVD RK算法

$$\frac{\varphi^{n+1}-\varphi^{n}}{\Delta t}+v^{n}\cdot\nabla\varphi^{n}=0$$

$$\frac{\varphi^{n+2}-\varphi^{n+1}}{\Delta t}+v^{n+1}\cdot\nabla\varphi^{n+1}=0$$

$$\varphi^{n+1/2}=\frac{3}{4}\varphi^{n+1}+\frac{1}{4}\varphi^{n+2}$$

$$\frac{\varphi^{n+3/2}-\varphi^{n+1/2}}{\Delta t}+v^{n+1/2}\cdot\nabla\varphi^{n+1}=0$$

$$\varphi^{n+1}=\frac{1}{3}\varphi^{n}+\frac{2}{3}\varphi^{n+3/2}$$

TVD Runge-Kutta 方法

- 虽然4阶以上的TVD RK方法存在，但在实际应用中在时间上的精度的提高并不会对结果有太大的改进，而且从计算复杂性上考虑会得不偿失。

求解Level set的数值方法小结

- Level set 的一般形式可以表达为

$$\frac{\partial \varphi}{\partial t} + v \cdot \varphi = 0$$

$$v = a(x) \cdot \nabla \varphi - \mu(x) \nabla \cdot \left(\frac{\nabla \varphi}{|\nabla \varphi|}\right)$$

其中 $a(x) \cdot \nabla \varphi$ 称为对流项，$\nabla \cdot \left(\frac{\nabla \varphi}{|\nabla \varphi|}\right)$ 为曲率。

求解Level set的数值方法小结

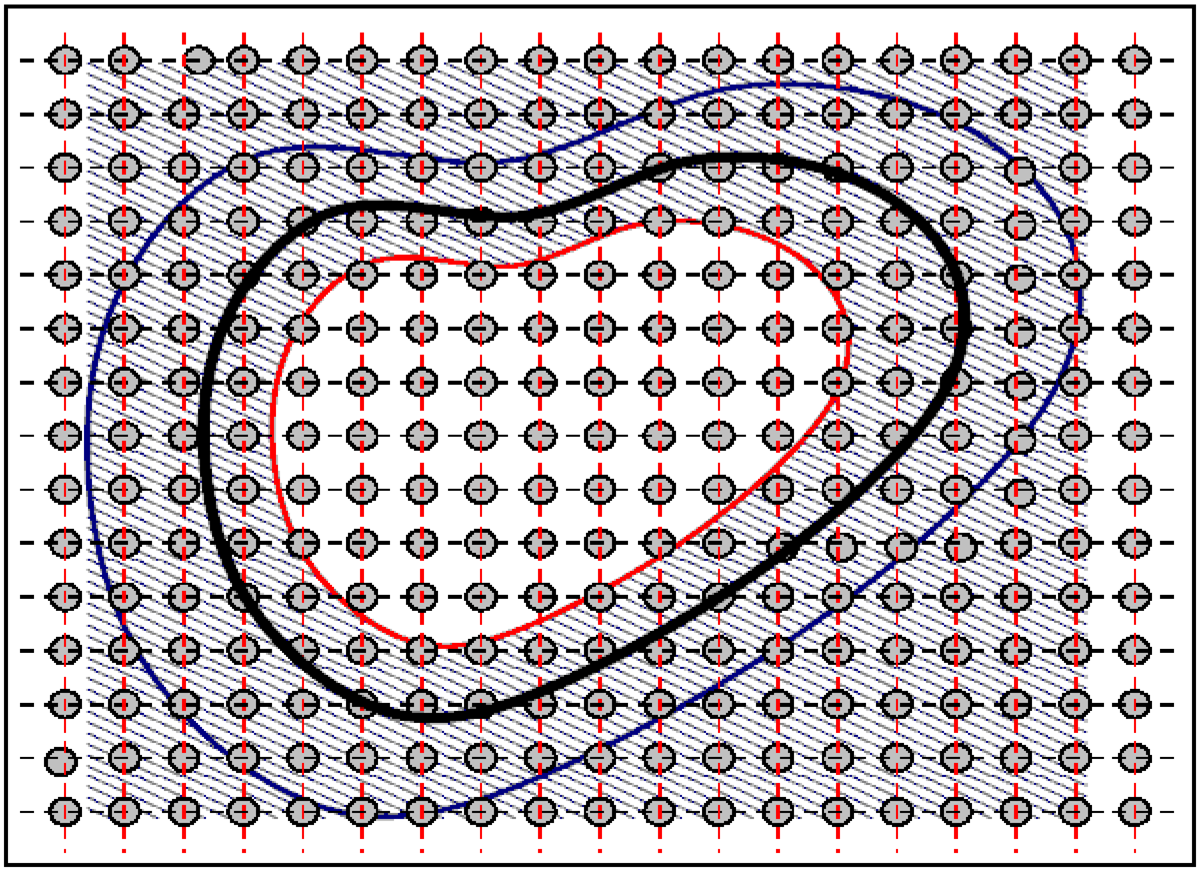
- 求解该方程一般分为3步
  - 用ENO,WENO或upwind方法求解对流项。
  - 用中心差分的方法估算曲率。
  - 用TVD RK方法来求解。

$$\frac{\partial \varphi}{\partial t} + v \cdot \varphi = 0$$

$$v = \boxed{a(x) \cdot \nabla \varphi} - \mu(x)\boxed{\nabla \cdot (\frac{\nabla \varphi}{|\nabla \varphi|})}$$

对流项　　　　曲率

更高级的解法？！？



Outward Band

$\Phi(s) = +d$

Front Position

$\Phi(s) = 0$

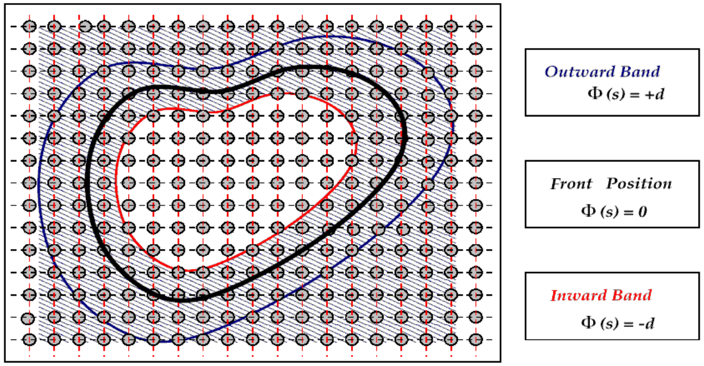
Inward Band

$\Phi(s) = -d$

From theory to Practice (Narrow Band)

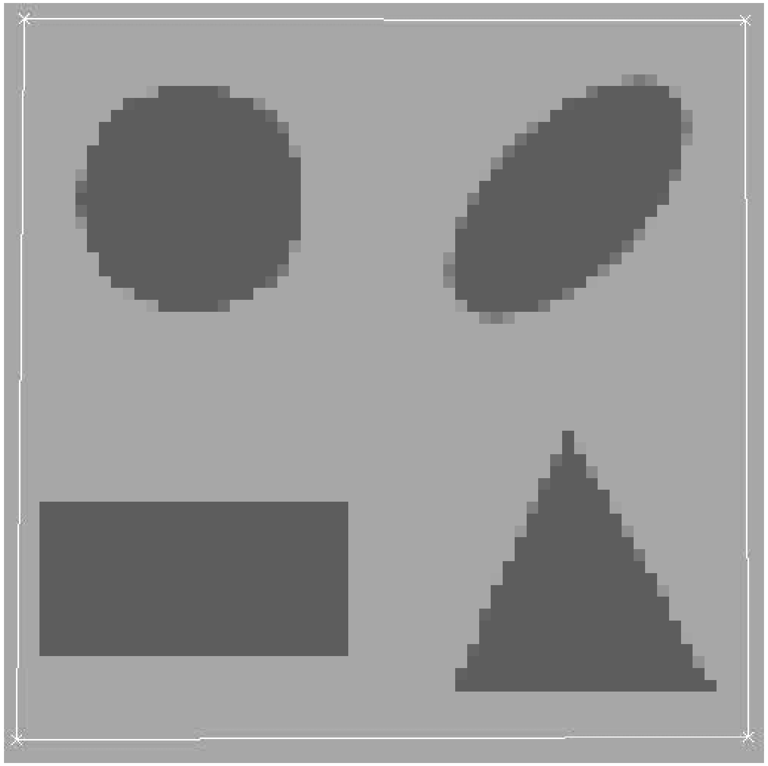
[Chop:93, Adalsteinsson-Sethian:95]

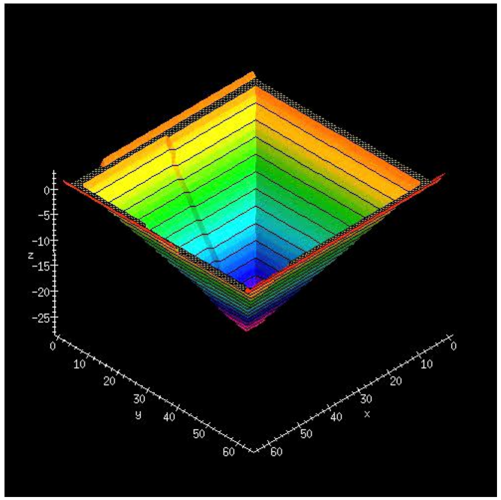
- Central idea: we are interested on the motion of the zero-level set and not for the motion of each iso-phote of the surface
  - Extract the latest position
  - Define a band within a certain distance
  - Update the level set function
  - Check new position with respect the limits of the band
  - Update the position of the band regularly, and re-initialize the implicit function

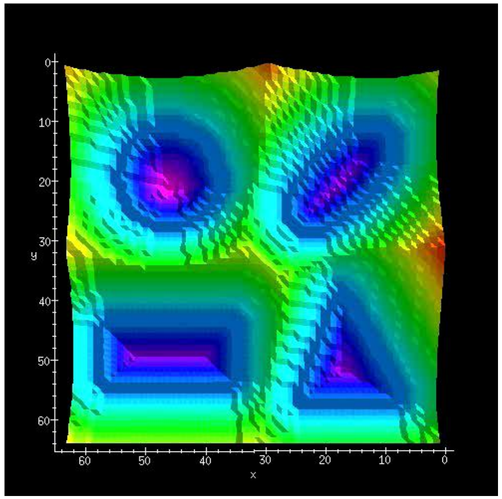


- Significant decrease on the computational complexity, in particular when implemented efficiently and can account for any type of motion flows

Narrow Band (the basic derivation)







Results are courtesy: R. Deriche

Handling the Distance Function

- The distance function has to be frequently re-initialized…
  - Extraction of the curve position & re-initialization:
    - Using the marching cubes one can recover the current position of the curve, set it to zero and then re-initialize the implicit function: the Borgefors approach, the Fast Marching method, explicit estimation of the distance for all image pixels…
  - Preserving the curve position and refinement of the existing function (Susman-smereka-osher:94)

$$\frac{d}{d\tau}\phi_m = \operatorname{sgn}\left(\phi_m^0\right)\left(1 - |\nabla\phi_m|\right)$$

  - Modification on the level set flow such that the distance transform property is preserved (gomes-faugeras:00)
    - Extend the speed of the zero level set to all iso-photes, rather complicated approach with limited added value?

From theory to Practice (Fast Marching)

[Tsitsiklis:93,Sethian:95]

- Central idea: "move" the curve one pixel in a progressive manner according to the speed function while preserving the nature of the implicit function

- Consider the stationary equation $F\ |\nabla T| = 1$.

- Such an equation can be recovered for all $[\frac{\partial C}{\partial t} = F\mathcal{N}]$ flows where the speed function has one sign (either positive or negative), propagation takes place at one direction

- If T(x,y) is the time when the implicit function reaches (x,y):

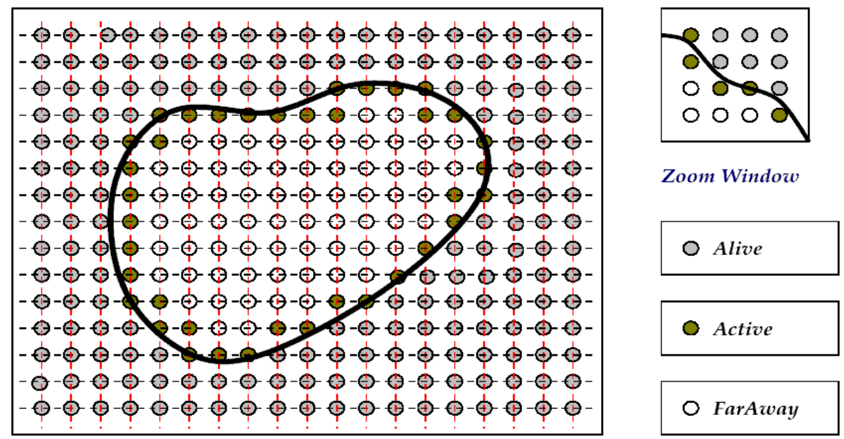
$$T(C(p,t)) \stackrel{\triangle}{=} t \quad \Rightarrow \quad \nabla T \cdot C_t = 1$$
$$\Rightarrow \quad \nabla T \cdot \left(F \frac{\nabla T}{|\nabla T|}\right) = 1$$
$$\Rightarrow \quad F\ |\nabla T| = 1$$

Fast Marching (continued)

- Consider the stationary equation $F\ |\nabla T| = 1$ · in its discrete form:

$$\frac{1}{F^2_{\{i,j\}}} = \max\left(D^{-x}_{\{i,j\}}T,0\right)^2 + \min\left(D^{+x}_{\{i,j\}}T,0\right)^2 + \max\left(D^{-y}_{\{i,j\}}T,0\right)^2 + \min\left(D^{+y}_{\{i,j\}}T,0\right)^2$$

- And using the assumption that the surface propagates in one direction, the solution can be obtained by outwards propagation from the smallest T value…



  - active pixels, the curve has already reached them
  - alive pixels, the curve could reach them at the next stage
  - far away pixels, the curve cannot reach them at this stage

Fast Marching (continued)

❑ INITIAL STEP

- Initialize $[T=0]$ for the all pixels of the front (active), their first order neighbors alive and the rest far away
- For the first order neighbors, estimate the arrival time according to: $\left[T_{\{i,j\}} = \frac{1}{F_{\{i,j\}}}\right]$
- While for the rest the crossing time is set to infinity $[T_{\{i,j\}} = \infty]$

❑ PROPAGATION STEP

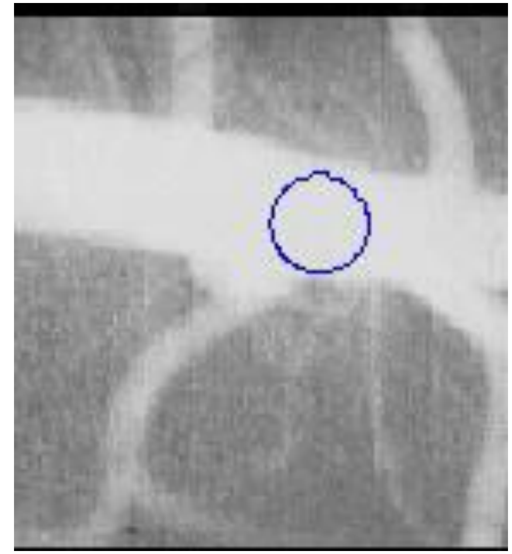
- Select the pixel with the lowest arrival time from the alive ones
- Change his label from alive to active and for his first order neighbors:
  - If they are alive, update their T value according to

$$\frac{1}{F^2_{\{i,j\}}} = \max\left(D^{-x}_{\{i,j\}}T, 0\right)^2 + \min\left(D^{+x}_{\{i,j\}}T, 0\right)^2 + \max\left(D^{-y}_{\{i,j\}}T, 0\right)^2 + \min\left(D^{+y}_{\{i,j\}}T, 0\right)^2$$

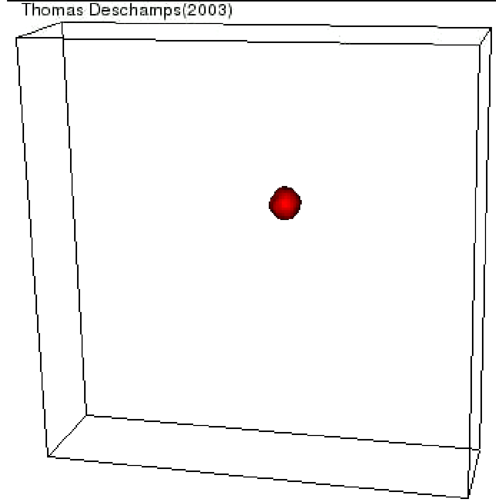
  - If they are far away, estimate the arrival time according to: $\left[T_{\{i,j\}} = \frac{1}{F_{\{i,j\}}}\right]$

Fast Marching Pros/Cons, Some Results

- Fast approach for a level set implementation
- Very efficient technique for re-setting the embedding function to be distance transform
- Single directional flows, great importance on initial placement of the contours
- Absence of curvature related terms or terms that depend on the geometric properties of the curve…
- Results are courtesy: J. Sethian, R. Malladi, T. Deschamps, L. Cohen



Thomas Deschamps(2003)



Level set 与变分方程的关系

- 例(Total Variation模型)：

$$F(u) = \int |\nabla u| dx + \lambda \int |f - u|^2 dx$$

用Euler-Lagrange方程化为偏微分方程为

$$u = f + \frac{1}{2\lambda} \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right)$$

问题的求解可以化为Level set演化的形式，即

$$u_t = -(u - f) + \frac{1}{2\lambda} \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right)$$

Level-set 实战

$$\frac{\partial \varphi}{\partial t} + v \cdot \nabla \varphi = 0$$

Level-set 实战

图像轮廓提取

边界检测和轮廓线提取

- 隐式动态轮廓模型
  - 用隐式模型可以跟踪拓扑变化的轮廓
  - 隐式轮廓线的微分方程表达为

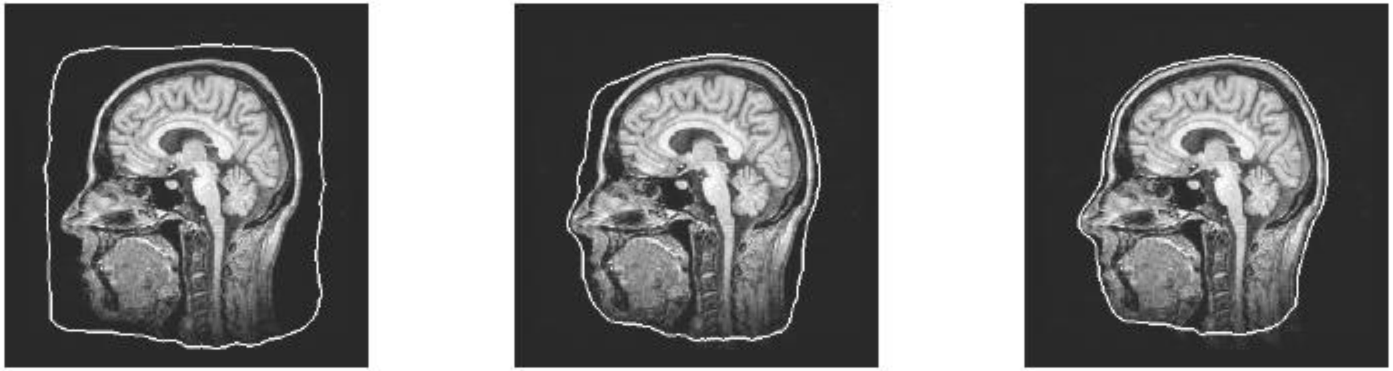
$$\frac{\partial u}{\partial t} = g(x)|\nabla u|(\nabla \cdot (\frac{\nabla u}{|\nabla u|}) + k)$$

$$u(x,0) = u_0(x)$$

  - 注意轮廓不但被梯度驱动，而且被曲率驱动

边界检测和轮廓线提取—隐式动态轮廓模型

- 试验结果



不用边界表达的动态轮廓线算法

- 经典的图像分割算法（如snakes）都需要将图像的边界特殊表示，然后计算其长度等进行优化，而实际上边界的长度可以用数学表达式表示
- 定义Heaviside函数

$$H(x) = \begin{cases} 1, \text{ if } x \geq 0 \\ 0, \text{ if } x < 0 \end{cases}$$

不用边界表达的动态轮廓线算法

其导数为 $\delta(x) = \frac{d}{dx}H(x)$ (in the sense of distributions),
则边界的长度可以表示为

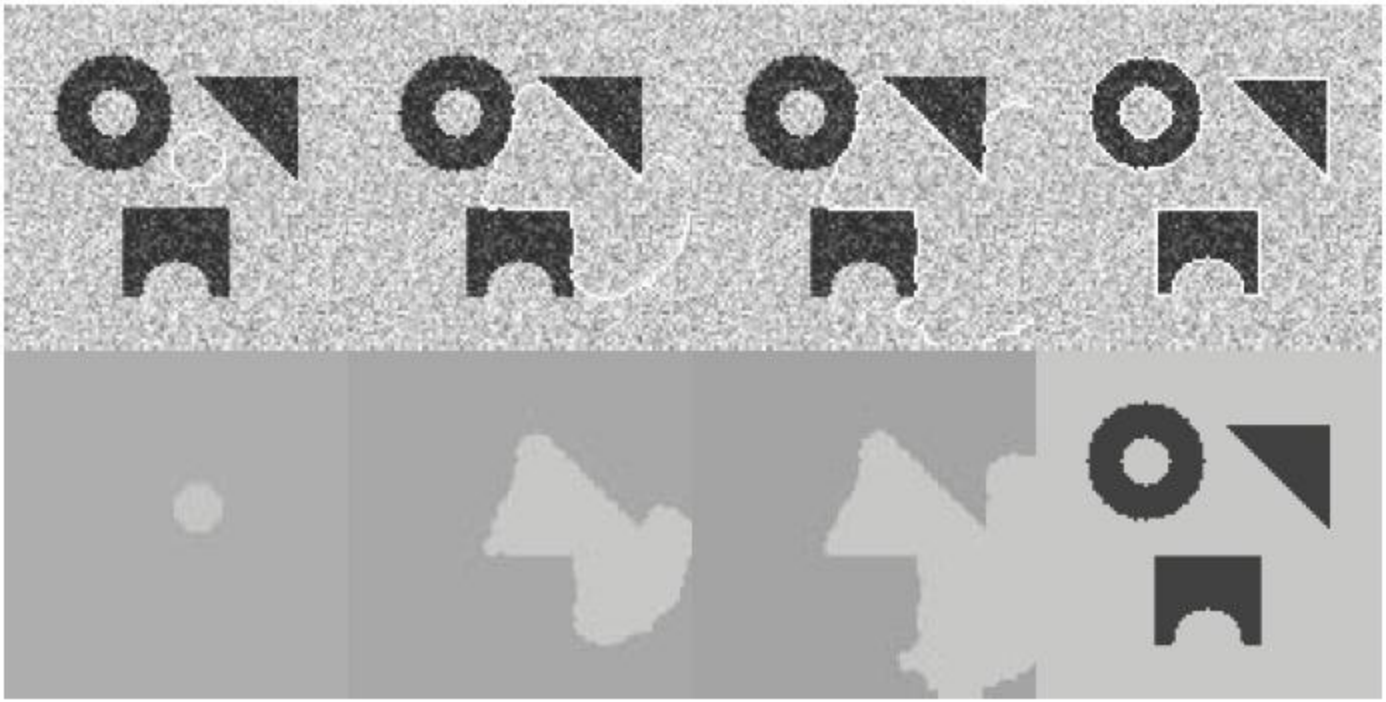
$$\begin{cases} \text{length}\{\phi = 0\} = \int_\Omega |\nabla H(\phi)| = \int_\Omega \delta(\phi)|\nabla\phi|, \\ \text{area}\{\phi \geq 0\} \quad = \int_\Omega H(\phi)dxdy, \end{cases}$$

- 这样无边界表达的优化方程为

$$\begin{aligned} F(\phi, c_1, c_2) = {} & \mu \int_\Omega \delta(\phi)|\nabla\phi| + \nu \int_\Omega H(\phi)dxdy \\ & + \lambda_1 \int_\Omega |u_0 - c_1|^2 H(\phi)dxdy + \lambda_2 \int_\Omega |u_0 - c_2|^2(1 - H(\phi))dxdy. \end{aligned}$$

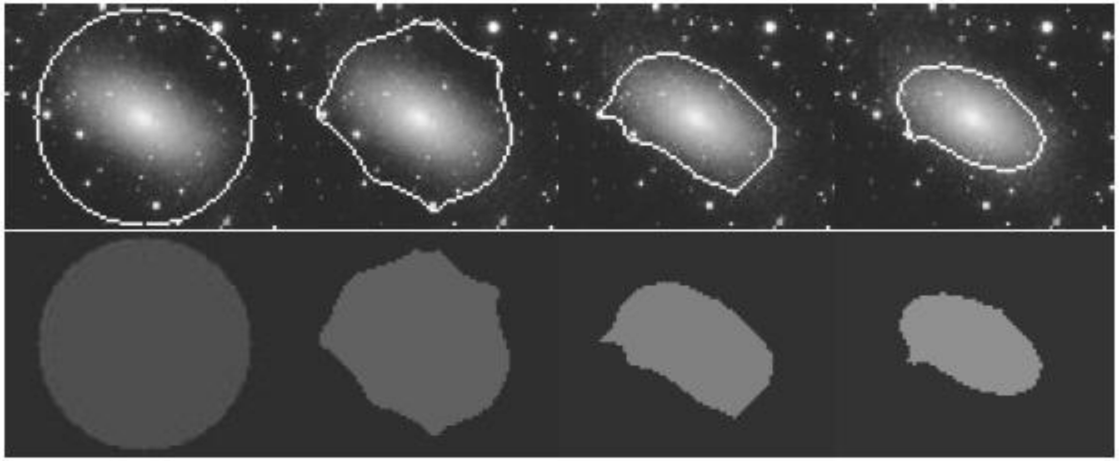
不用边界表达的动态轮廓线算法

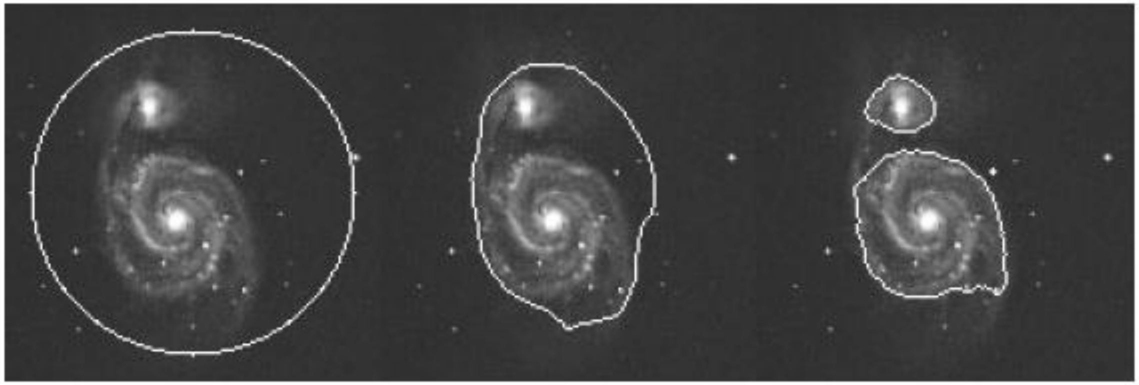
- 试验结果



不用边界表达的动态轮廓线算法

- 试验结果



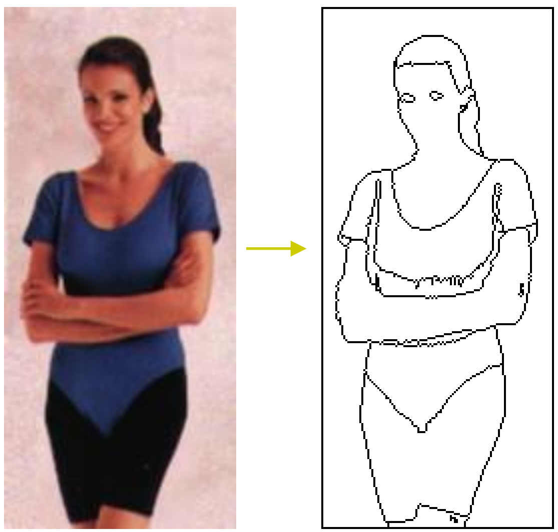


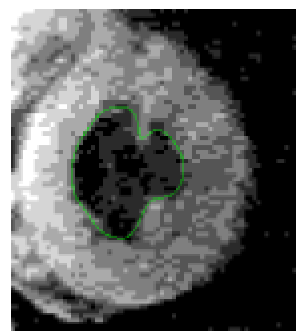
Level-set 实战

图像分割

What is image segmentation?

Definition: Separate the original image into regions that are meaningful for a specific task. (shape recovery)





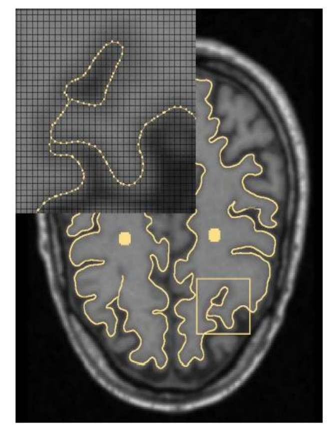


Image Segmentation

- Region based
  - Thresholding
  - Region grow
  - Region splitting and merging
  - classification
- Edge based
  - Border tracing
  - Graph searching
  - Dynamic programming (DP)
  - Hough transforms
- Deformable models
  - Snake
  - Level Set

Why deformable models in image segmentation?

- used a very simple form
- a set of control points
- moving control points
- form closed loops

基于Fast Marching技术的图像分割

- 算法框架

```
InitTValueMap()
InitTrialLists()
while (ExistTrialPixels()) {
        pxl = FindLeastTValue()
        MarkPixelAlive(pxl)
        UpdateLabelMap(pxl)
        AddNeighboursToTrialLists(pxl)
        UpdateNeighbourTValues(pxl)
}
```

Mumford-Shah 图像分割

- Mumford-Shah模型

$$E(f,C)=\alpha\iint_{\Omega}\left|\nabla f\right|^{2}dA+\beta\iint_{\Omega\backslash C}(f-g)^{2}dA+\gamma\oint_{C}ds$$

Mumford-Shah 图像分割

- 实验结果

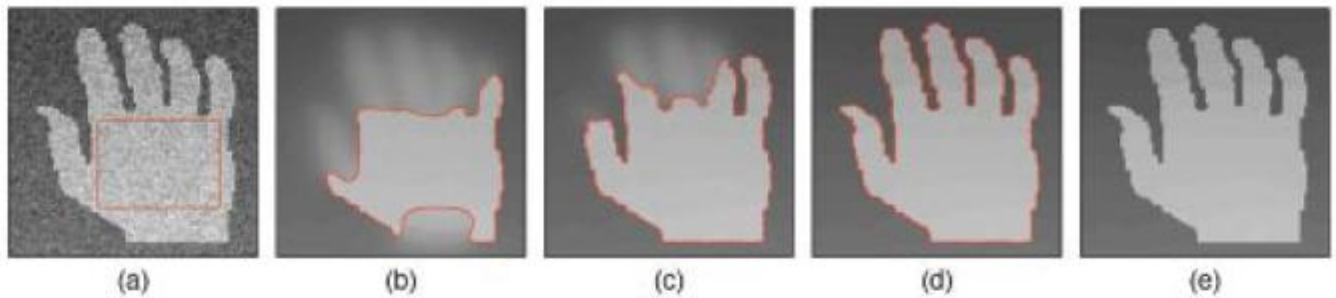


Fig. 2. Outward flow from inside.

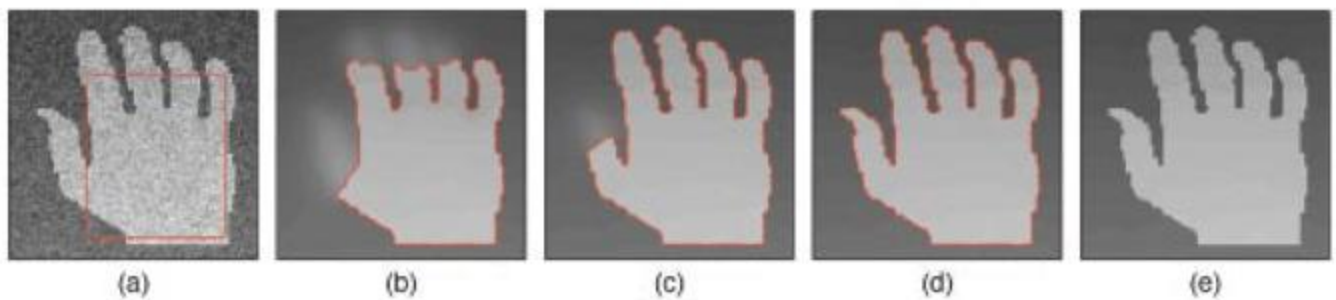
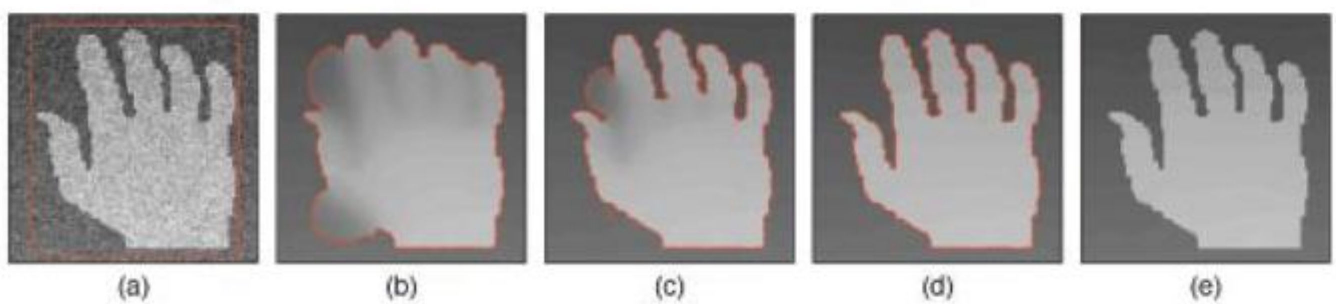


Fig. 3. Bidirectional flow.



Mumford-Shah 图像分割

- 实验结果

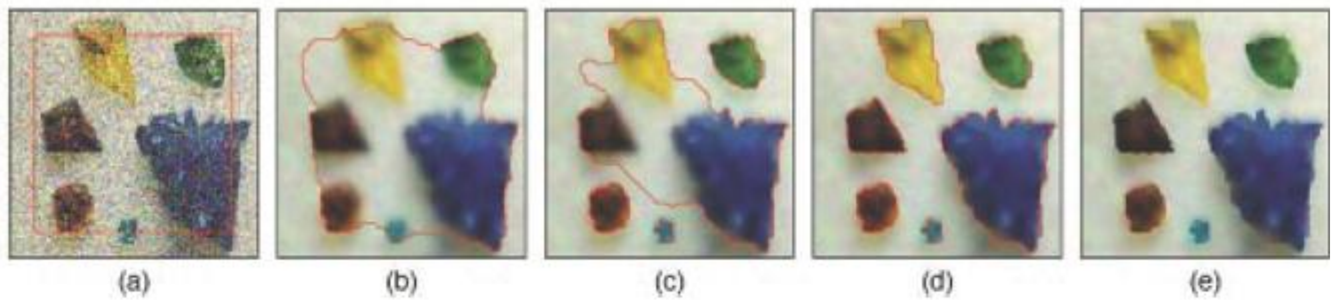
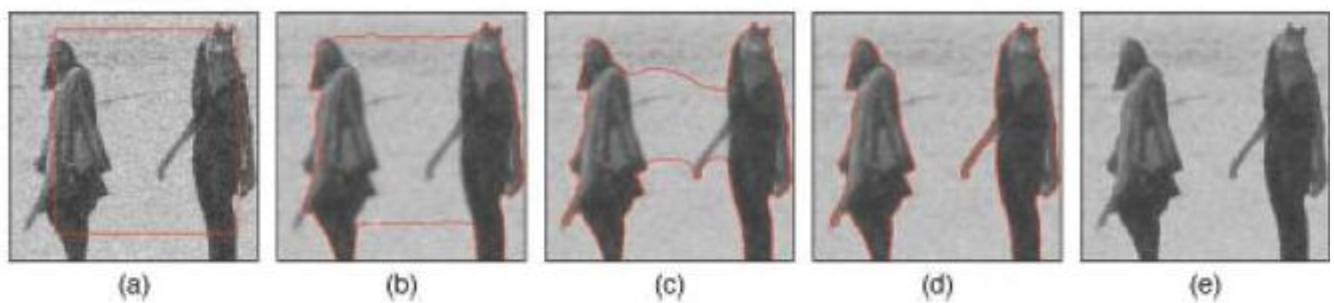
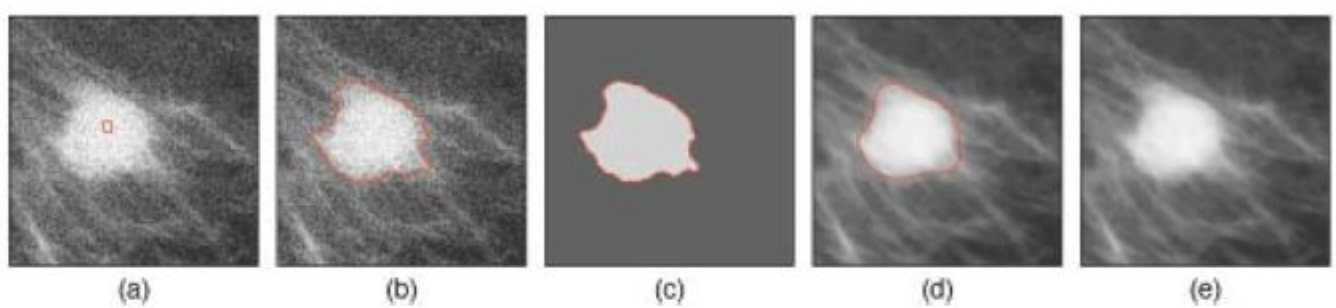


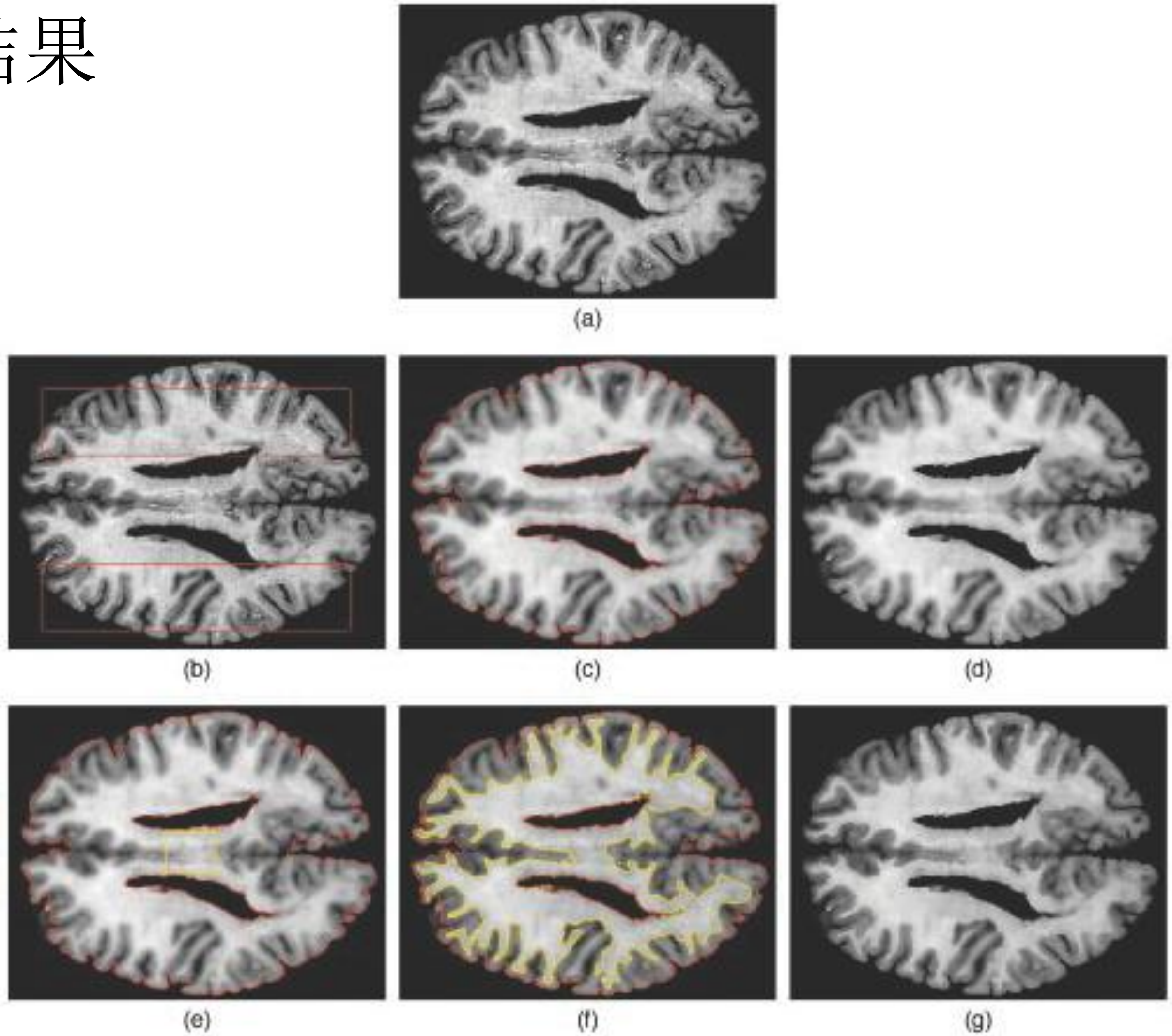
Fig. 7. Segmentation and smoothing of a color image with six distinct foreground regions.





Mumford-Shah 图像分割

- 试验结果



(a) (b) (c) (d) (e) (f) (g)

Level-set 实战

医学图像处理

基于知识的医学图像分割

- 基于知识的图像分割使得一些先验的模型可以被嵌入到分割的过程中, 提高精确性.
- 用level set方法考虑的图像特征:灰度分布(距离函数),局部曲率,全局形状信息(统计信息)...
- 加入先验信息的方法:

  样本训练(利用距离场)

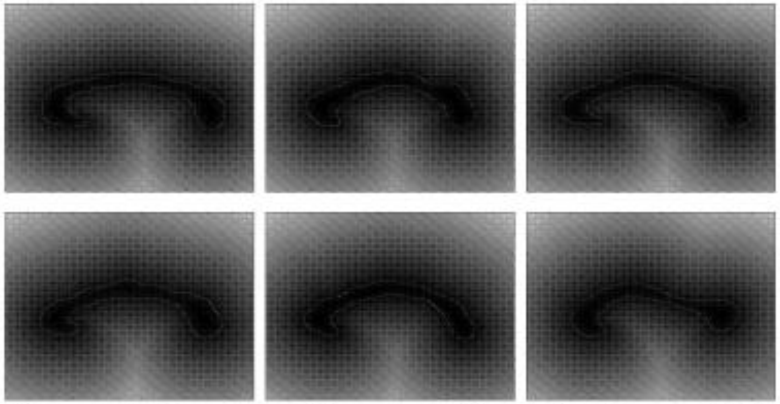


Figure 2. Corpus callosum outlines for 6 out of 51 patients in the training set embedded as the zero level set of a higher dimensional signed distance surface.

基于知识的医学图像分割

- 样本训练: 假设用距离函数表示的*n*个样本为

$$\mathcal{T} = \{u_1, u_2, \ldots, u_n\}.$$

则平均表面可以表示为

$$\mu = \frac{1}{n} \sum u_i.$$

- 然后可以将这些样本信息用主成分分析的方法表示为一个*k*(<*n*)个参数的方程, 最后得到一个对形状的Gauss分布表达式

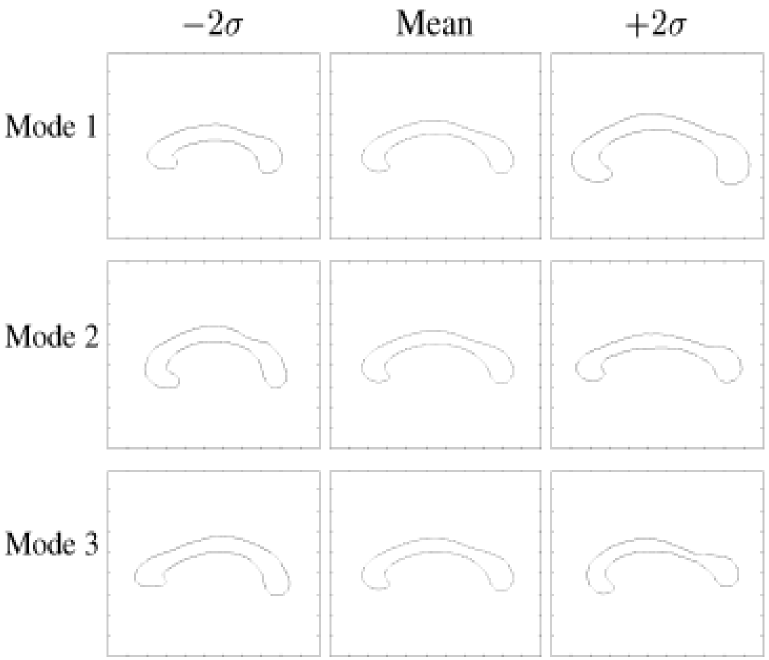


Figure 3. The three primary modes of variance of the corpus callosum training dataset.

基于知识的医学图像分割

- 学习得到形状的Gauss分布形式为

$$P(\alpha) = \frac{1}{\sqrt{(2\pi)^k |\Sigma_k|}} \exp\left(-\frac{1}{2}\alpha^{\top}\Sigma_k^{-1}\alpha\right)$$

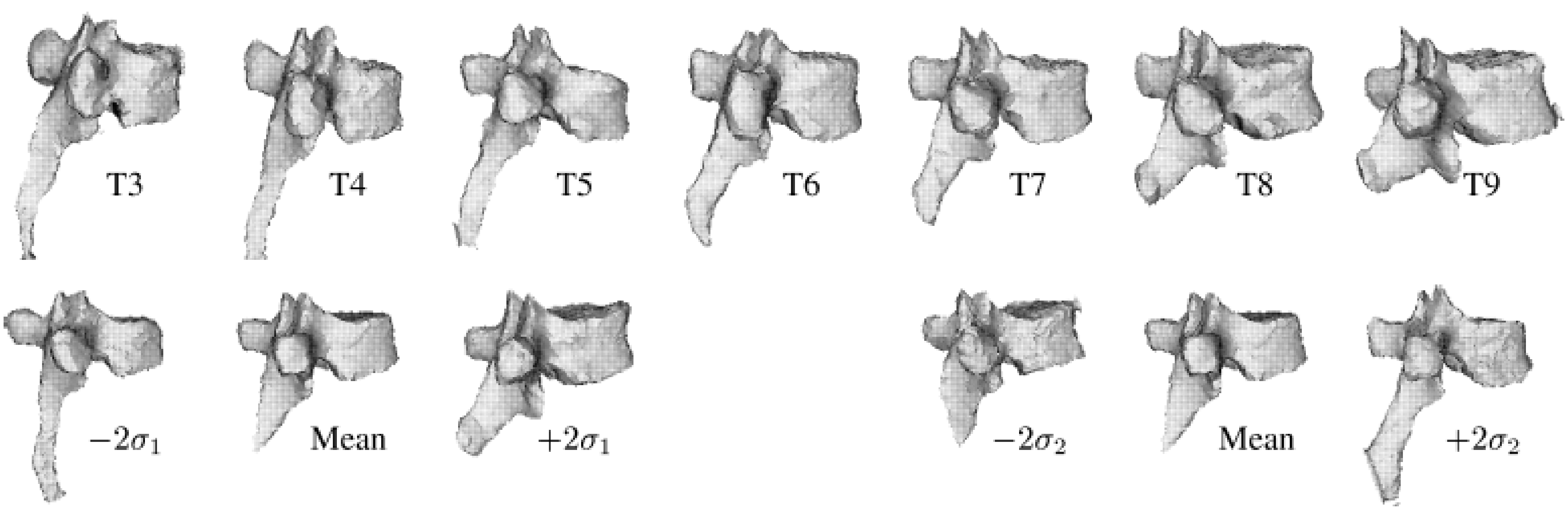


Figure 4. Top: Three-dimensional models of seven thoracic vertebrae (T3-T9) used as training data. Bottom left and right: Extracted zero level set of first and second largest mode of variation respectively.

医学图像分析—
基于知识的医学图像分割

- 估计物体的形状(shape)和朝向(pose)
  - 形状：α
  - 朝向：p （一组参数，如旋转角度、平移距离等）
  - 转变为最大后验概率估计(MAP)问题：

$$\langle \alpha_{\text{MAP}}, p_{\text{MAP}} \rangle = \underset{\alpha, p}{\operatorname{argmax}} P(\alpha, p \mid u, \nabla I)$$

  - 用Bayes定理可以得到

$$\begin{aligned} P(\alpha, p \mid u, \nabla I) &= \frac{P(u, \nabla I \mid \alpha, p) P(\alpha, p)}{P(u, \nabla I)} \\ &= \frac{P(u \mid \alpha, p) P(\nabla I \mid \alpha, p, u) P(\alpha) P(p)}{P(u, \nabla I)} \end{aligned}$$

医学图像分析—
基于知识的医学图像分割

- 参数假设估计
  - 内部项:假设边界的演化始终都是在物体内部的

$$P(u \mid \alpha, p) = \exp\left(-V_{outside}\right)$$

  - 梯度项:假设梯度绝对值和估计图像*u**满足Gauss分布

$$P(\nabla I \mid u^*, u) = \exp\left(-\left|\, h(u^*) - |\nabla I|\, \right|^2\right)$$

  - 朝向估计:假设均匀分布

$$P(p) = \mathcal{U}(-\infty, \infty)$$

  - 形状参数:前面已推导

$$P(\alpha) = \frac{1}{\sqrt{(2\pi)^k |\Sigma_k|}} \exp\left(-\frac{1}{2}\alpha^\top \Sigma_k^{-1} \alpha\right)$$

基于知识的医学图像分割

- 曲线演化--同时考虑图像梯度-曲率信息和先验信息

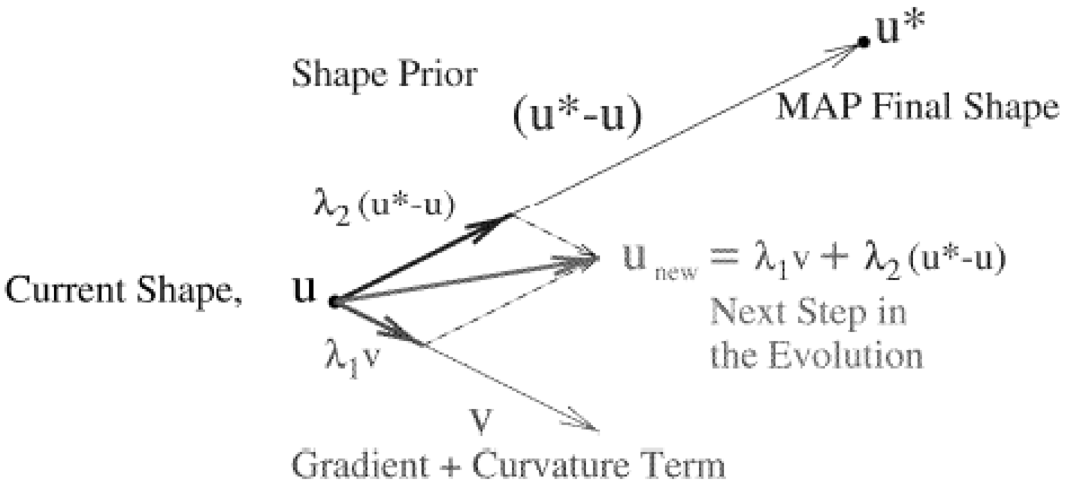


Figure 8. Illustration of the various terms in the evolution of the surface, u. The surface u^*, is the maximum *a posteriori* final shape. To update u, we combine the standard gradient and curvature update term, v, and the direction of the MAP final shape, $u^* - u$.

基于知识的医学图像分割

- Level Set的曲线演化
  - 原始的无先验知识的演化(只根据曲率-梯度信息)

$$u(t+1) = u(t) + \lambda_1 \left(\mathrm{g} \left(c + \kappa \right) \left| \nabla u(t) \right| + \nabla u(t) \cdot \nabla g \right)$$

  - 假设在时刻*t*用前面的方法估计出了一个最大可能的分割表面*u**, 则为了使演化朝着估计的方向前进

$$u(t+1) = u(t) + \lambda_2 \ \left(u^*(t) - u(t) \right)$$

基于知识的医学图像分割

- 最后得到整个的演化公式,其中 λ_1 和 λ_2 控制先验知识和无先验知识项(梯度-曲率)之间的平衡

$$u(t+1) = u(t) + \lambda_1 \left(g\,(c+\kappa)\,|\nabla u(t)| + \nabla u(t) \cdot \nabla g \right) + \lambda_2 \left(u^*(t) - u(t) \right)$$

基于知识的医学图像分割

- 实验结果

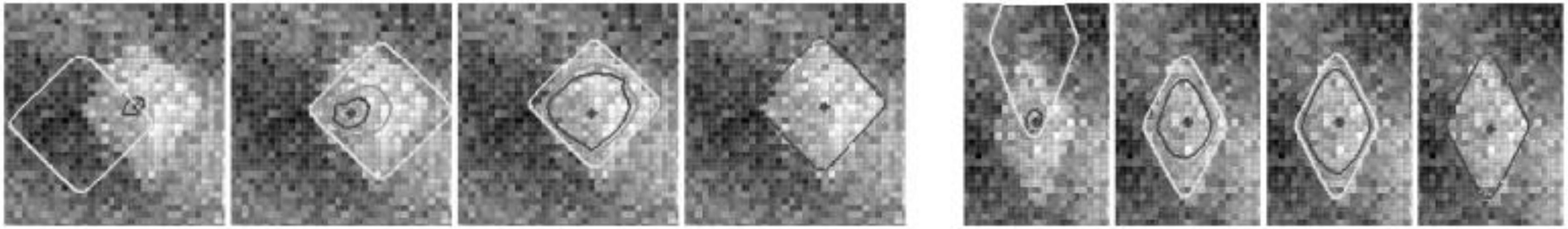


Figure 9. Several time steps in the curve evolution process of segmenting two rhombi. The training set for the rhombus consisted of rhombi of various sizes and aspect ratios. The red curve is the zero level set of the evolving surface. The green curve is the next step in the curve evolution. The yellow curve is the MAP estimate of the position and shape of the final curve.

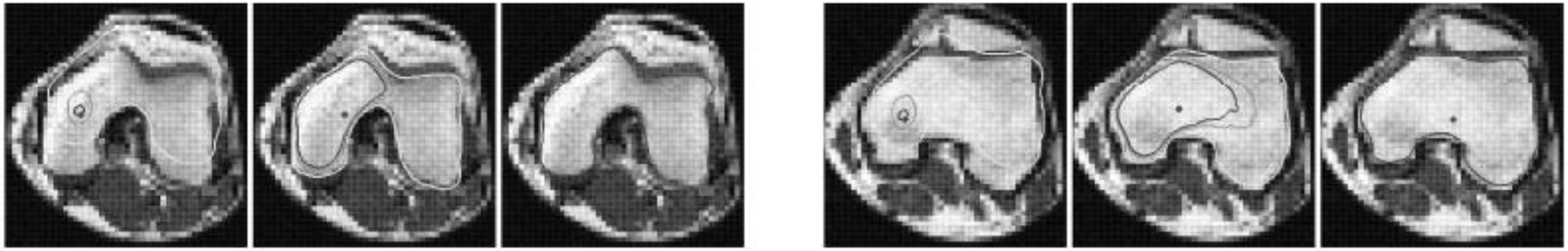


Figure 10. Initial, middle, and final steps in the evolution process of segmenting two slices of the femur. The training set consisted of 18 slices of the same femur, leaving out the slice being segmented and its neighbors.

基于知识的医学图像分割

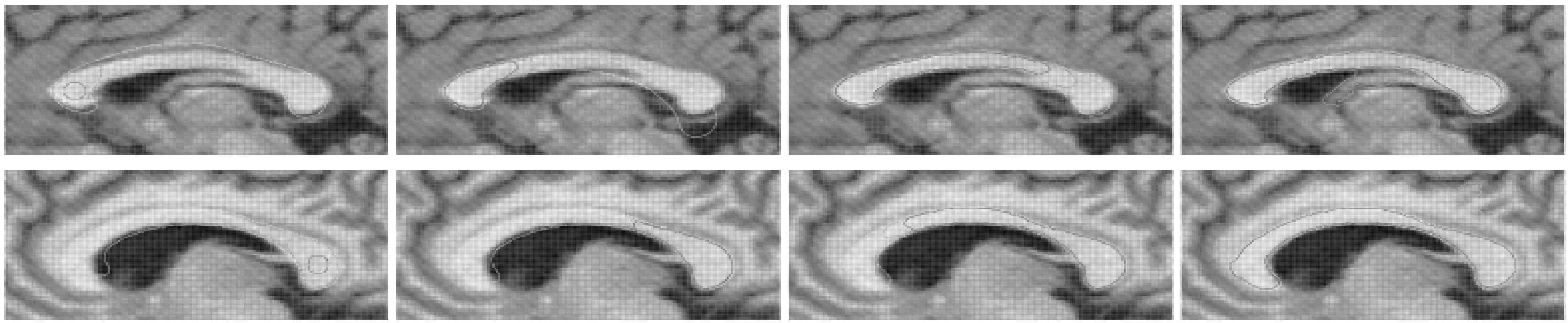


Figure 11. Four steps in the segmentation of two different corpora callosa. The last image in each case shows the final segmentation in red. The cyan contour is the standard evolution *without* the shape influence.

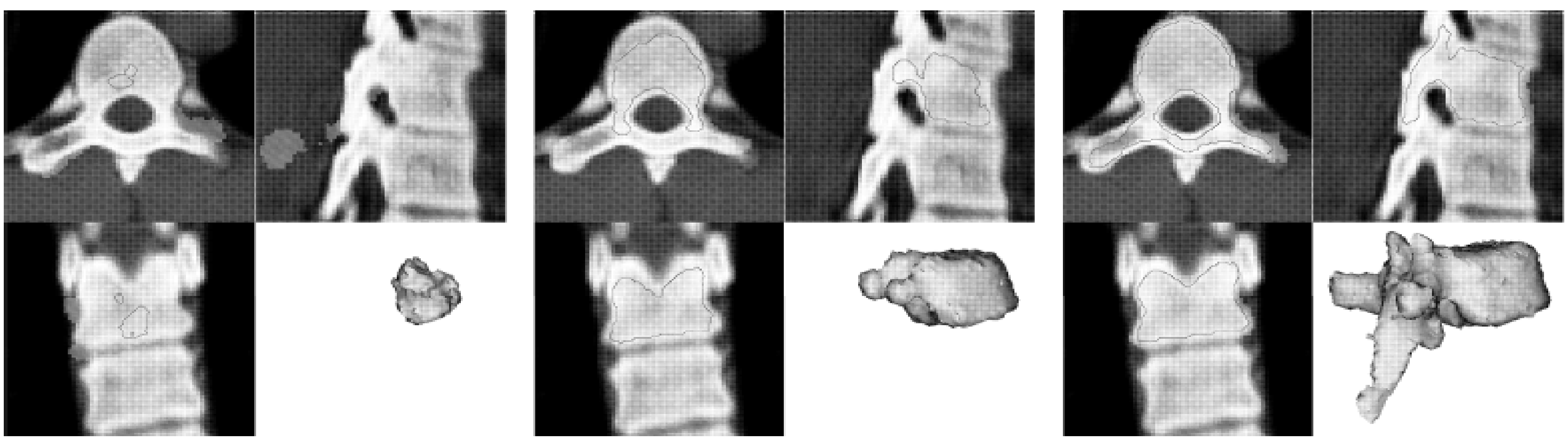


Figure 12. Early, middle, and final steps in the segmentation of the vertebra T7. Three orthogonal slices and the 3D reconstruction are shown for each step. The red contour is a slice through the evolving surface. The yellow overlay is a slice through the inside of the MAP final surface.

Level-set 实战

图像修复

图像恢复和重建—图像填补

- 图像填补(inpainting)
  - 假设原始图像为 $I_0(i,j):[0,M]\times[0,N]\rightarrow \mathbb{R}$,
    图像填补算法将恢复一系列图像
    $$I(i,j,n):[0,M]\times[0,N]\times\mathbb{N}\rightarrow\mathbb{R}$$
    使得 $lim_{n\rightarrow\infty}I(i,j,n)=I_R(i,j)$
  - 即表达为 $I^{n+1}(i,j)=I^n(i,j)+\Delta t I_t^n(i,j),\forall(i,j)\in\Omega$
    其中$I_t^n(i,j)$是由一些规则定义的

图像恢复和重建—图像填补

- 演化规则的定义：
  - 假设图像是光滑的
  - 演化应该保持边界
  - …

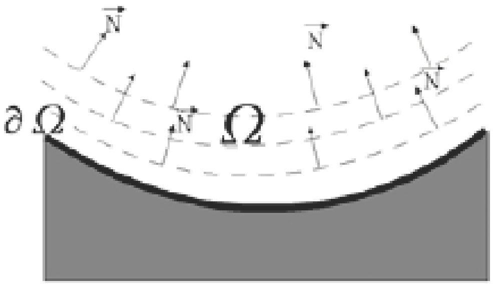


Figure 1: Propagation direction as the normal to the signed distance to the boundary of the region to be inpainted.

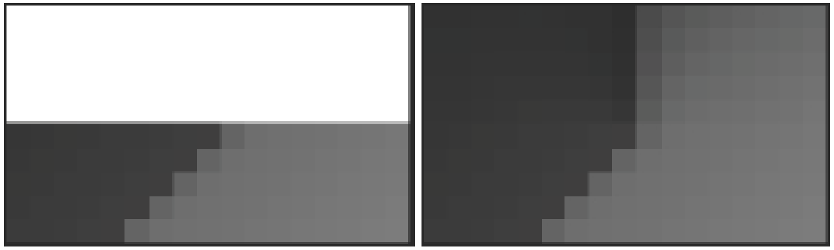


Figure 2: Unsuccessful choice of the information propagation direction. Left: detail of the original image, region to be inpainted is in white. Right: restoration.

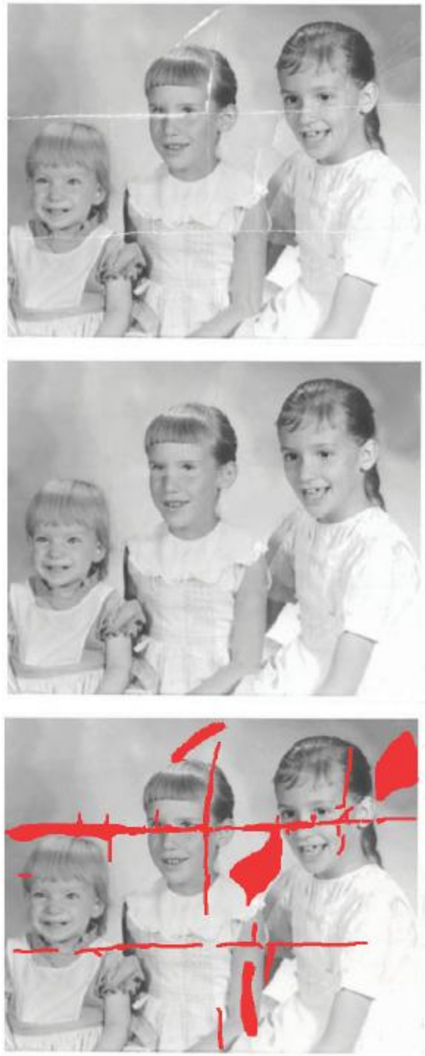
图像恢复和重建—图像填补

- 演化的数学表达

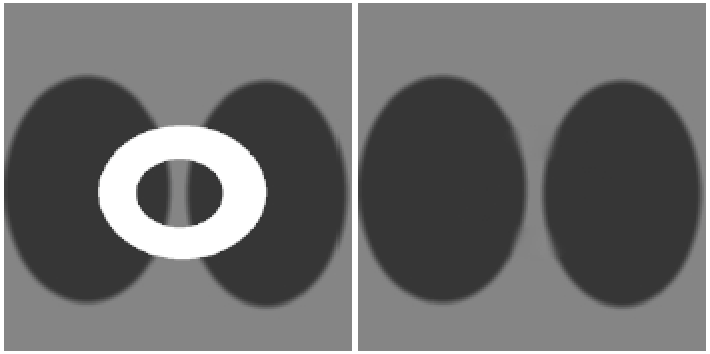
$$\frac{\partial I}{\partial t}(x,y,t) = g_\epsilon(x,y)\kappa(x,y,t)\left|\nabla I(x,y,t)\right|, \forall(x,y) \in \Omega^\epsilon$$

- 试验结果

图像恢复和重建—图像填补



图像恢复和重建—图像填补



图像恢复和重建—图像填补

- 其他利用level set方法的图像填补算法
  - TV inpainting

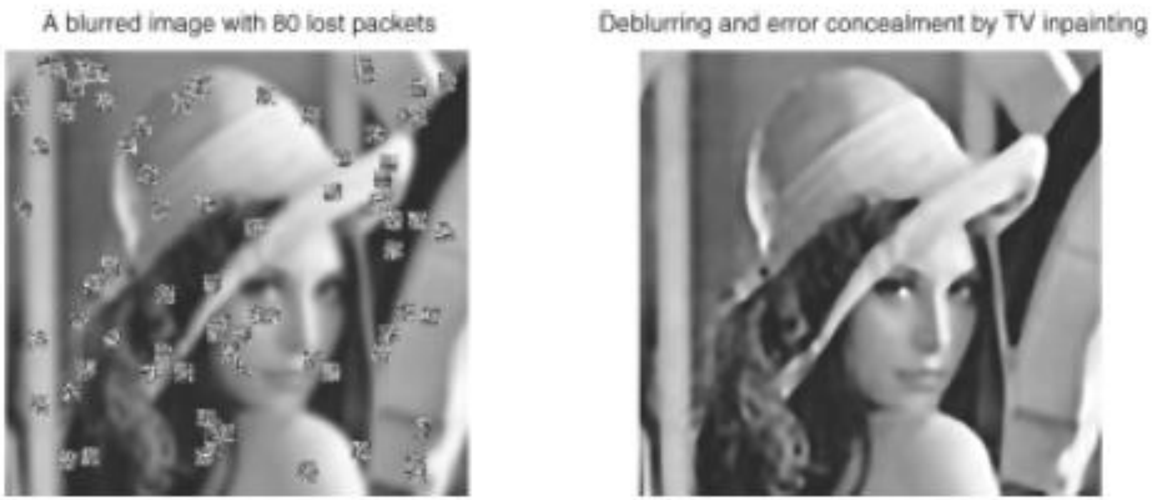


Figure 1. TV inpainting for the error concealment of a blurry image.

图像恢复和重建—图像填补

- Mumford-Shah inpainting

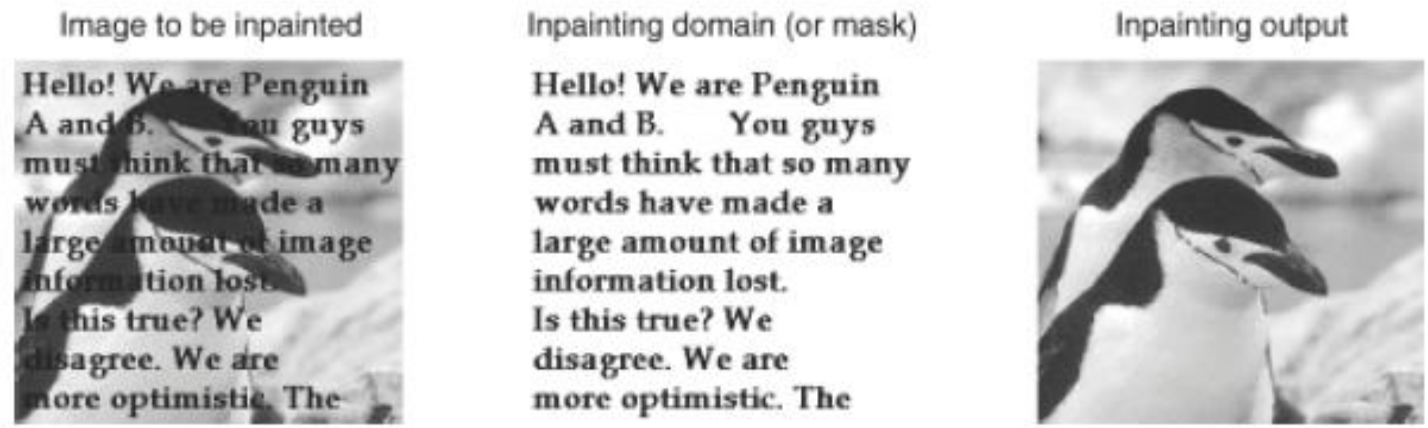
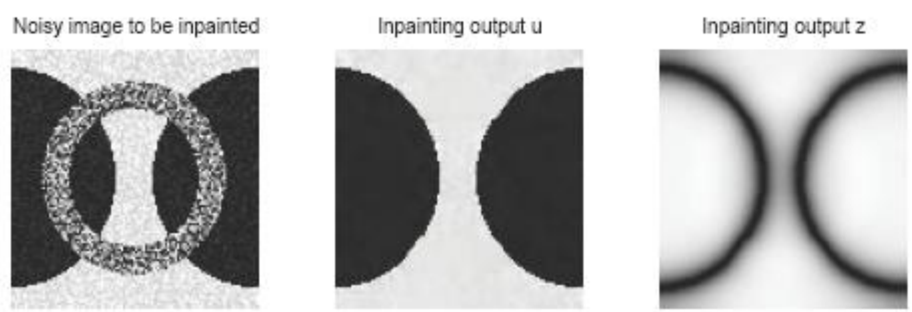


Figure 2. Mumford-Shah inpainting for text removal.



图像恢复和重建—图像填补

- Mumford-Shah-Euler inpainting
  - 可以插值保持形状的弧度

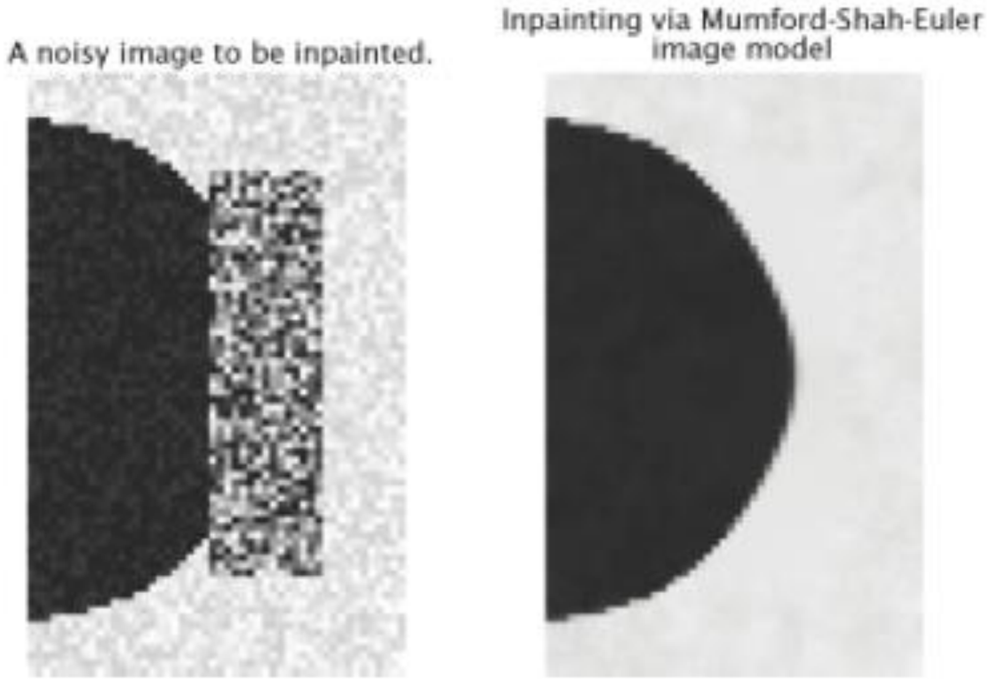


Figure 3. Smooth inpainting by the Mumford-Shah-Euler model.

图像恢复和重建—图像去噪

- TV 模型

$$\inf_u \left\{ E(u) = \int_\Omega |\nabla u| + \lambda \|f - u\|_* \right\}.$$

Level set 演化形式

$$u_t = -\frac{1}{2\lambda} \triangle \left[\operatorname{div}\left(\frac{\nabla u}{|\nabla u|} \right) \right] - (K^* K u - K^* f), \quad u(0, \cdot, \cdot) = f(\cdot, \cdot).$$

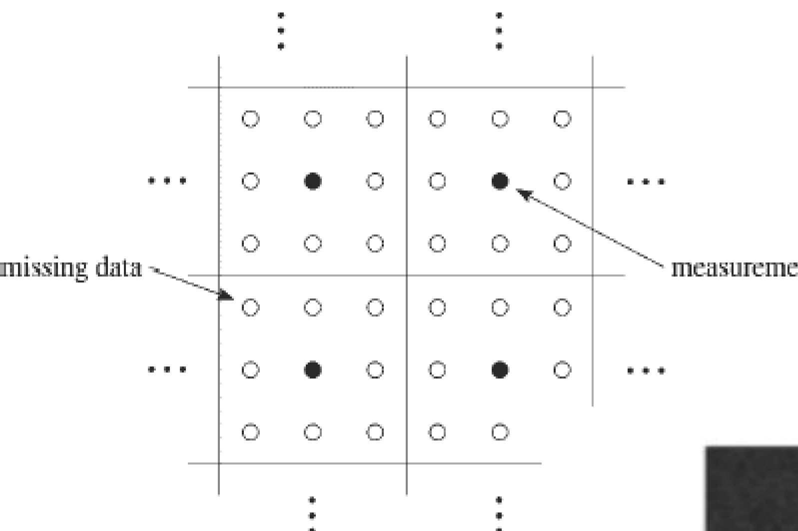
图像恢复和重建—图像去噪

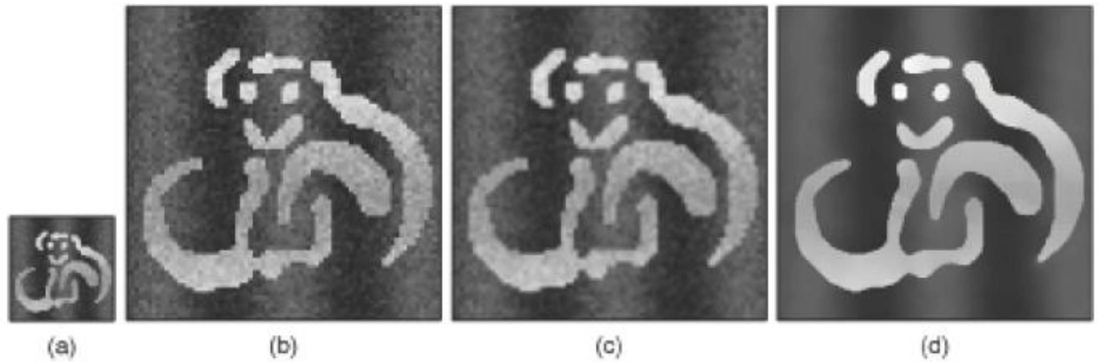
- 试验结果

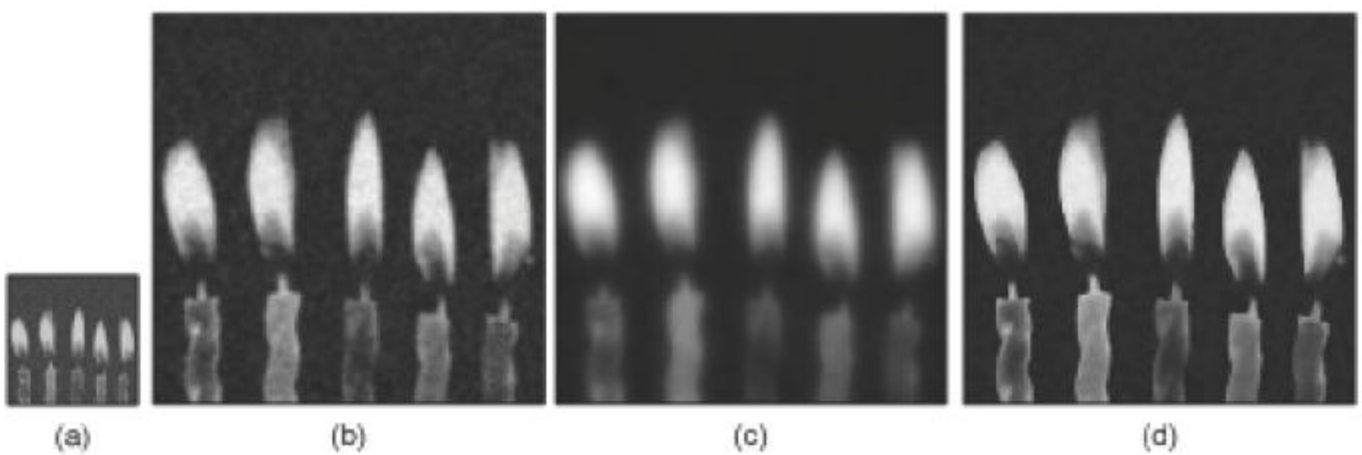
  （不同权值）

图像恢复和重建—图像放大

- 用Mumford-Shah模型的图像放大

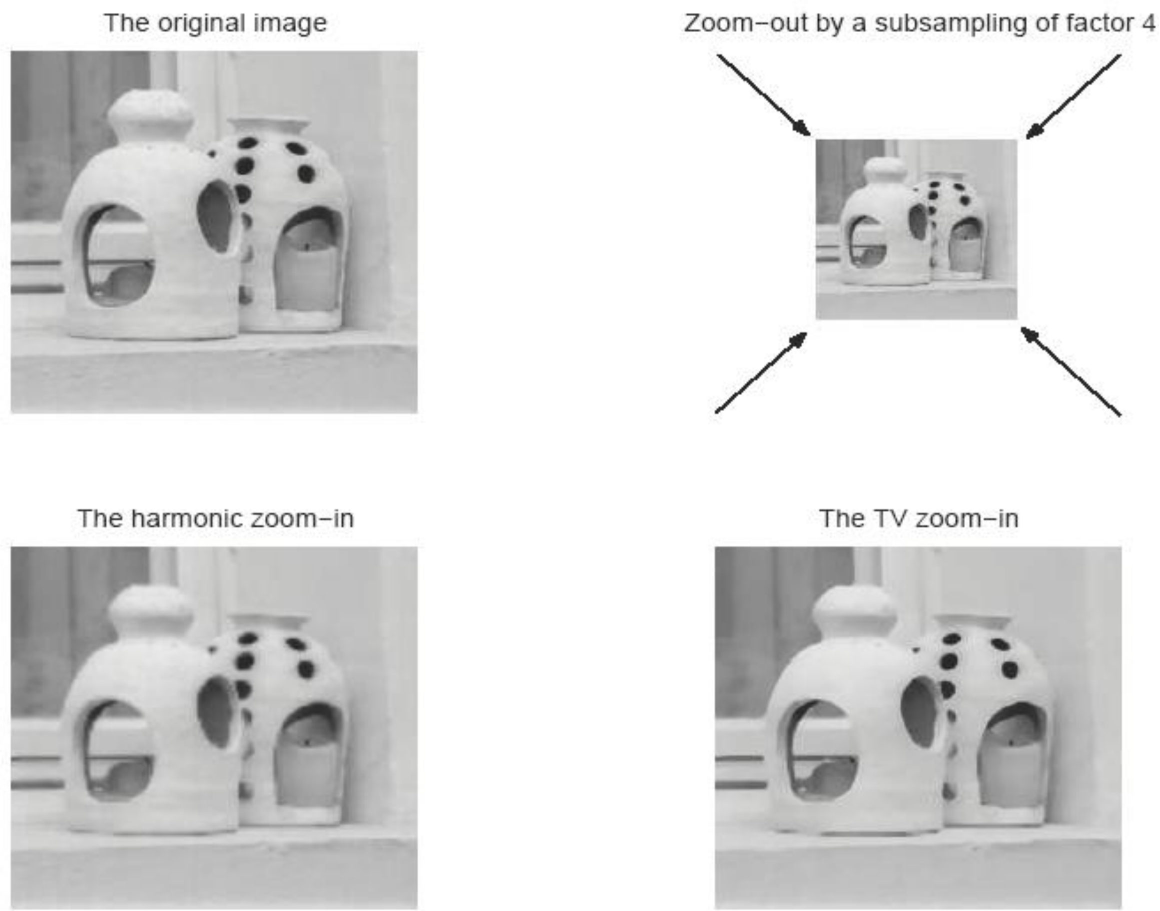






图像恢复和重建—图像放大

- TV 模型的图像放大



Level-set 实战

立体视觉

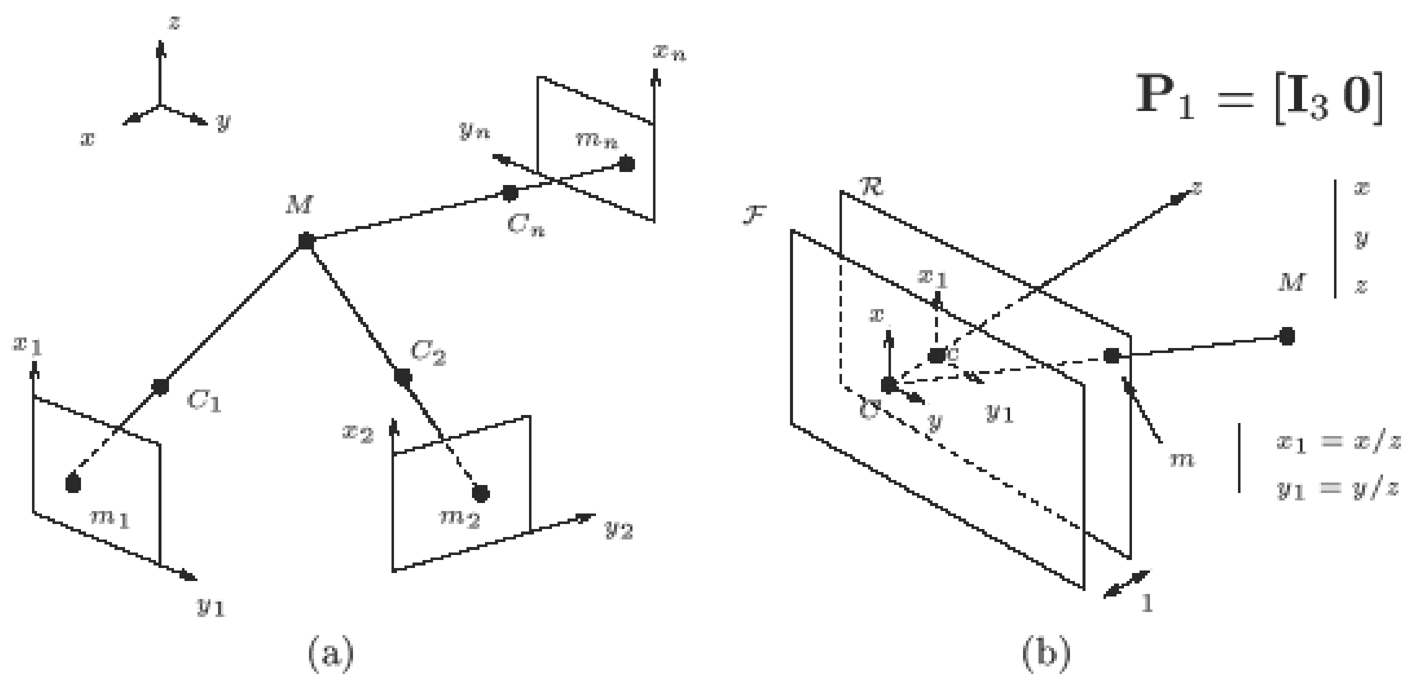
基于变分方法立体视觉

- 问题：假设相机的内参和外参都已知，求解图像中的对应点。即图像的**匹配**或**对应**。
  - 图像的匹配一般是通过建立一个变分方程并通过求解其局部最优解来解决的。
  - 一旦对应点找到，就通过可以**重建**三维图像。
- 用变分方法可以不用分开匹配和重建过程

基于变分方法立体视觉

- 给出三维中的点M，其在屏幕上的坐标可以通过和一个3×4的变换矩阵P表达，即

$$\mathbf{P}_2 = [\mathbf{R}^T \ -\mathbf{R}^T\mathbf{t}]$$



$$\mathbf{P}_1 = [\mathbf{I}_3 \ \mathbf{0}]$$

基于变分方法立体视觉

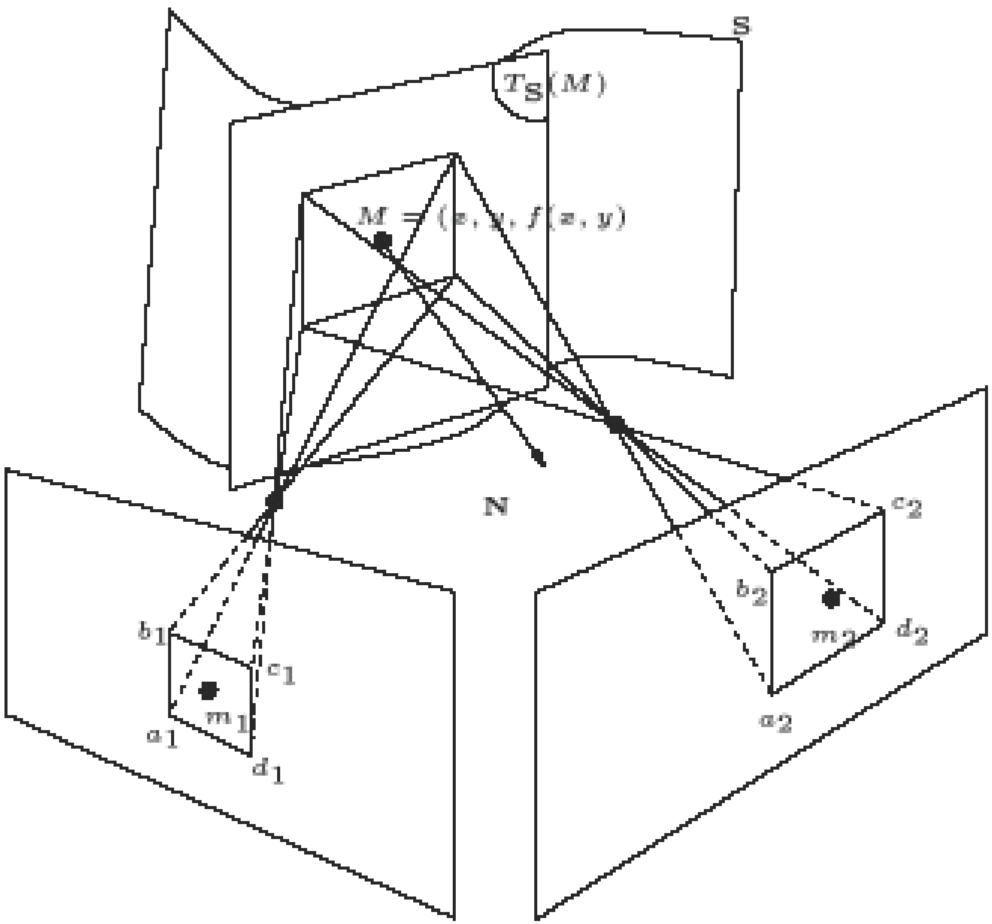
- 一个简单的匹配模型
  - 假设空间物体表面可以通过光滑函数$[x, y, f(x,y)]^T$来表示，而每一点在两幅图的位置为 $m_1(x,y,f), m_2(x,y,f)$ 并且假设物体上的同一点在两幅图上呈现的亮度相同，即 $I_1(m_1) = I_2(m_2)$
  - 则问题化为求解如下的方程

$$C_1(f) = \int\int (I_1(m_1(x,y)) - I_2(m_2(x,y))^2 dxdy$$

  - 相应的Euler-Lagrange方程为

$$(I_1 - I_2)(\nabla I_1 \cdot \frac{\partial \mathbf{m}_1}{\partial f} - \nabla I_2 \cdot \frac{\partial \mathbf{m}_2}{\partial f}) = 0$$

基于变分方法立体视觉



基于变分方法立体视觉

- 实验结果

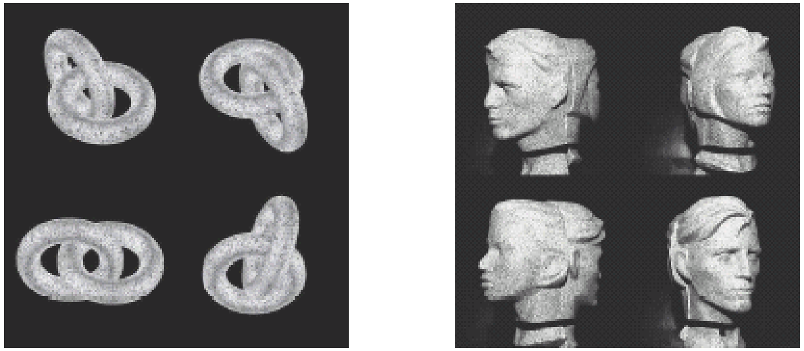


Fig. 4. Multicamera images of 3D objets. On the left hand side, two imbricated synthetic toruses (24 images). On the right hand side, real images: two heads stuck together (18 images)

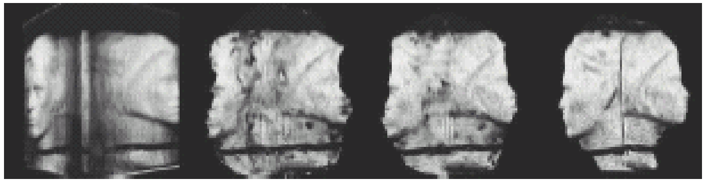


Fig. 6. Recovering process for the two heads

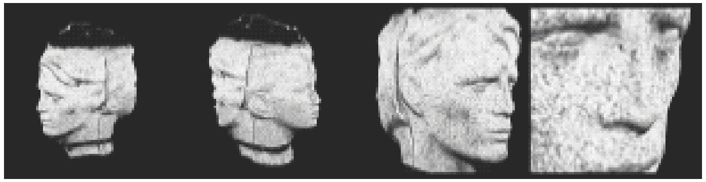


Fig. 7. Some views of the recovered object.

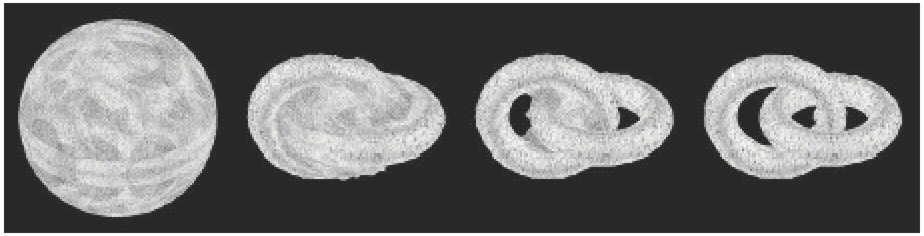


Fig. 5. Recovering process for the two toruses

基于距离函数的散乱点的重建

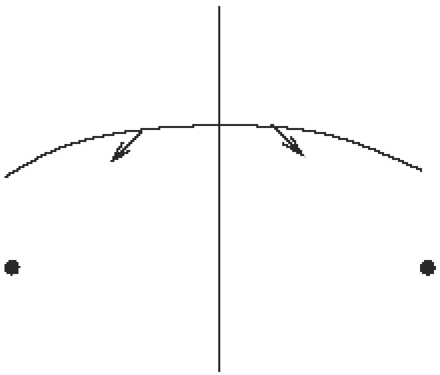
- 基本思想：先用一个大的初始曲面将散乱点包围住，然后用level set演化的方式使曲面逼近到散乱点，同时满足一定的内部约束和外部约束。
- 能量的约束利用距离函数，即

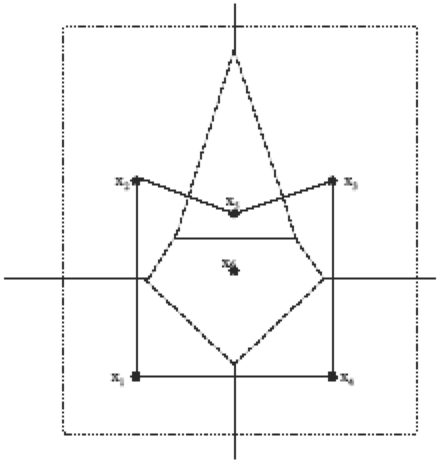
$$E(\Gamma) = \left[\int_{\Gamma} d^p(\boldsymbol{x}) ds \right]^{\frac{1}{p}}, \quad 1 \le p \le \infty,$$

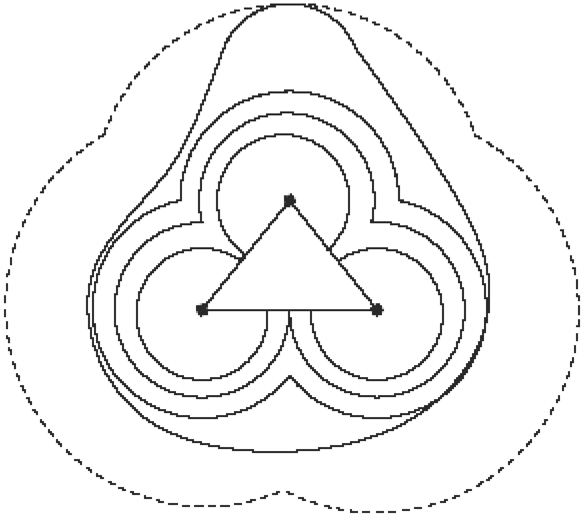
基于距离函数的散乱点的重建

- Level set演化方程

$$\frac{\partial \phi}{\partial t} = |\nabla \phi| \nabla \cdot \left[d \frac{\nabla \phi}{|\nabla \phi|} \right] = |\nabla \phi| \left[\nabla d \cdot \frac{\nabla \phi}{|\nabla \phi|} + d \nabla \cdot \frac{\nabla \phi}{|\nabla \phi|} \right]$$

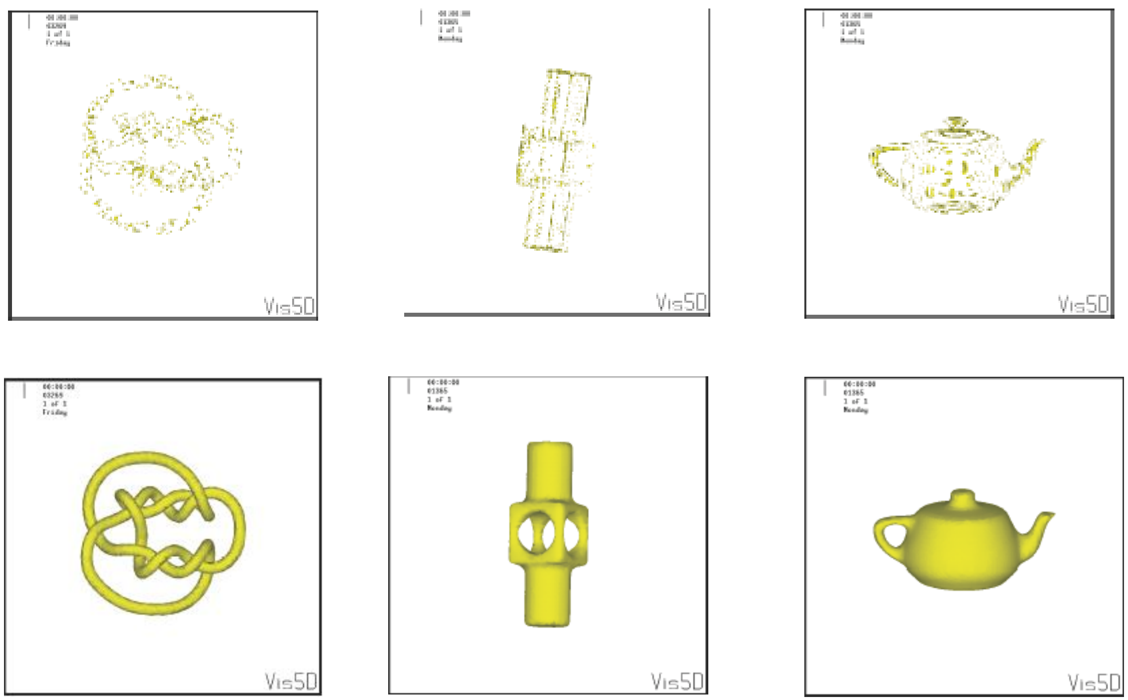

(a)

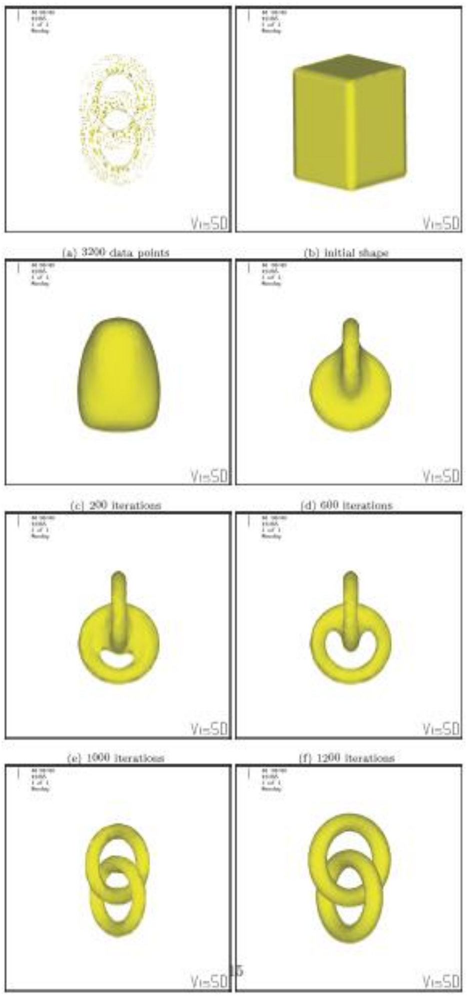

(b)



基于距离函数的散乱点的重建

- 试验结果





基于距离函数的散乱点的重建

- 试验结果

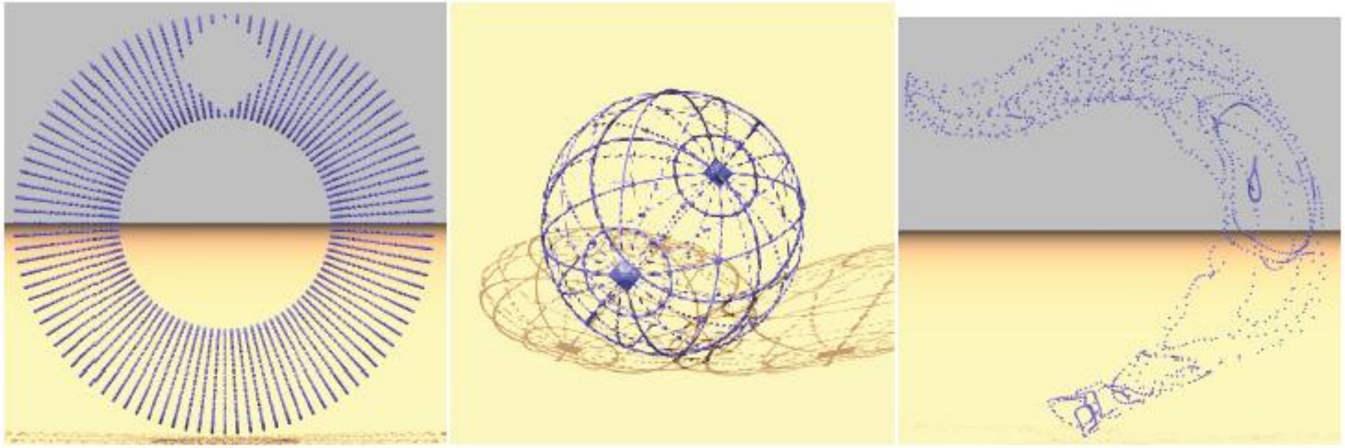
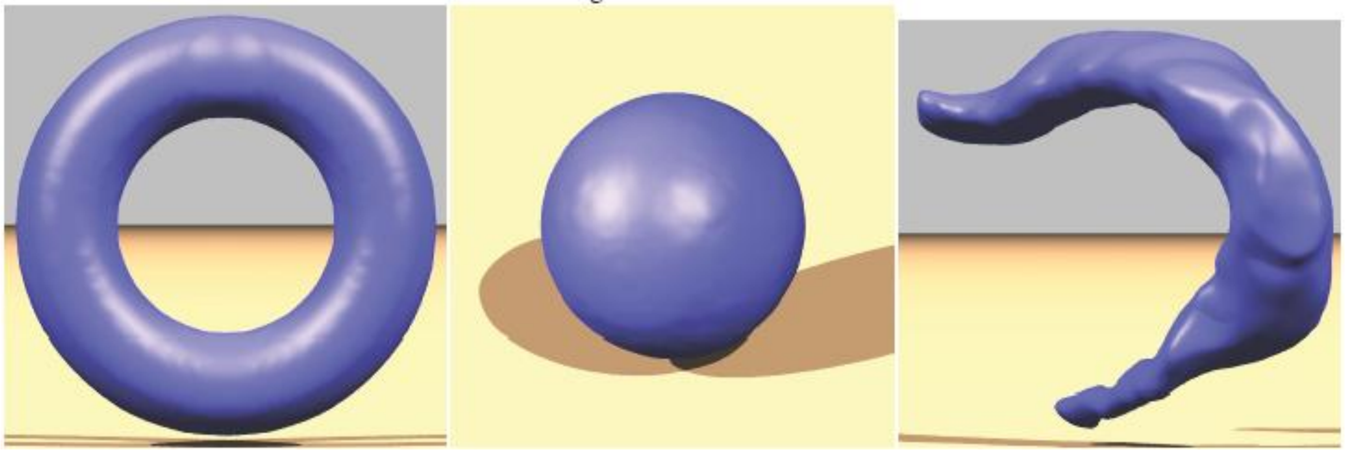
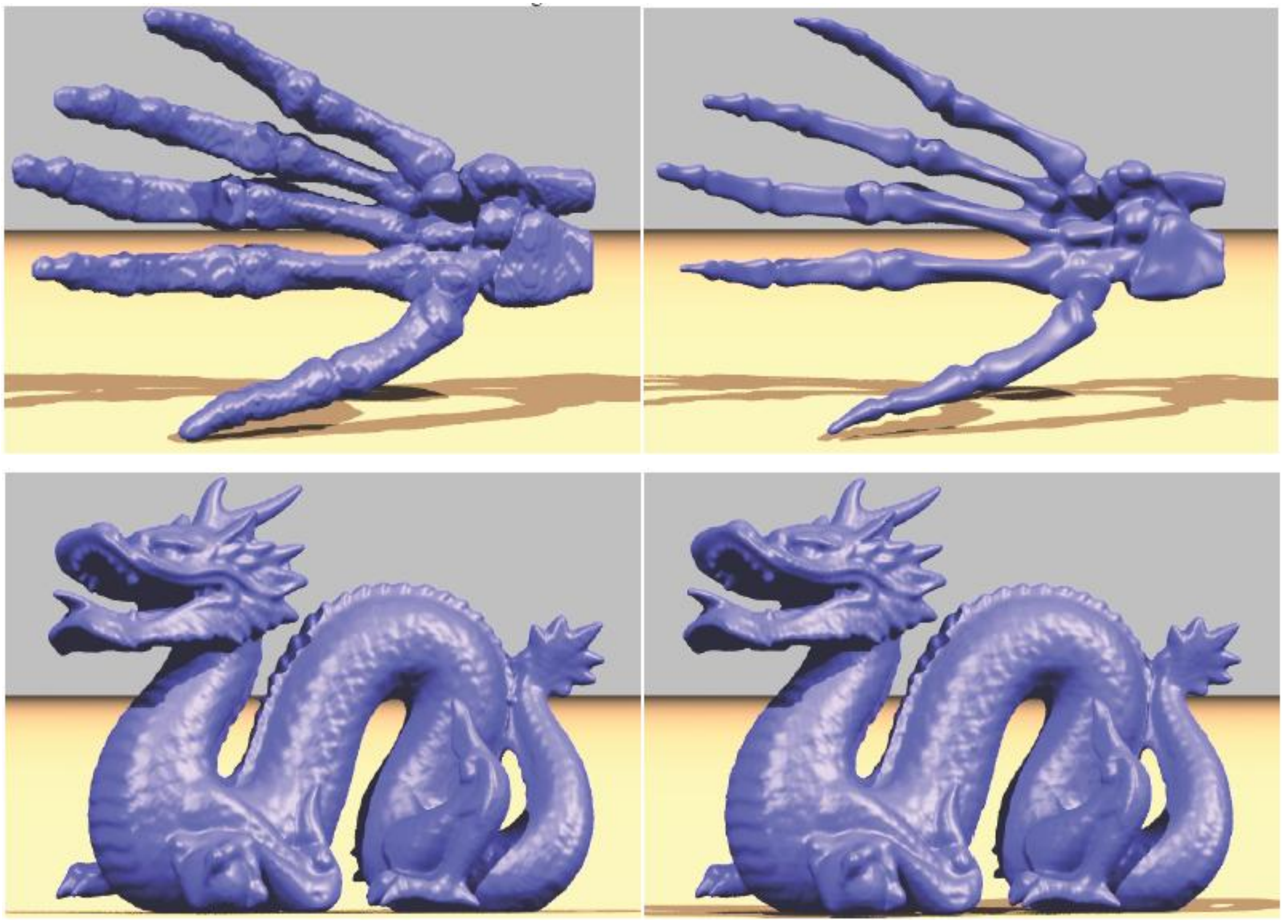


Figure 4: initial data



基于距离函数的散乱点的重建

- 试验结果



基于距离函数的散乱点的重建

- 试验结果

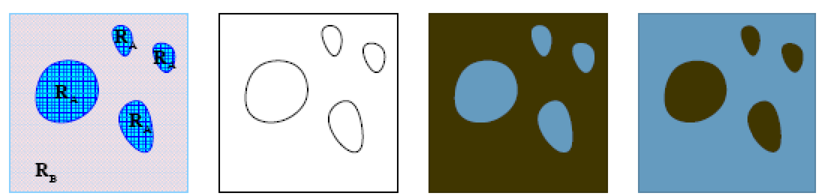
不同分辨率

Level-set 实战

运动分析

运动分析
— Geodesic Active Region

- 地测线动态区域(Geodesic Active Region)
- 定义和假设
  - 只有两个区域需要区分:前景和背景
  - I为输入图像由(h_A,h_B)组成
  - $\mathcal{P}(\mathcal{R}) = \{\mathcal{R}_A, \mathcal{R}_B\}$ 是对图像的一个分割
  - $\partial\mathcal{P}(\mathcal{R}) = \{\partial\mathcal{R}_A, \partial\mathcal{R}_B\}$ 是两个区域的公共边界



(a) (b) (c) (d)

FIG. 1. Geodesic Active Region Model: (a) the input, (b) the boundary-based information, (c) the region-based information corresponding to hypothesis h_A, [the information is proportional to the frame intensities] (d) the region-based information corresponding to hypothesis h_B [the information is proportional to the frame intensities].

运动分析
— Geodesic Active Region

- 边界表示(Geodesic Active Contour)

$$E(\partial\mathcal{R}) = \sum_{X\in\{A,B\}} \int_0^1 \underbrace{g\left(\underbrace{p_{b,X}(I(\partial\mathcal{R}_X(c_x)))}_{X\ boundary\ probability}\right)}_{X\ boundary\ attraction} \underbrace{\left|\partial\dot{\mathcal{R}}_X(c_x)\right|}_{X\ regularity} dc_x.$$

运动分析

— Geodesic Active Region

- 区域表示
  - Bayes形式

$$p_S(\mathcal{P}(\mathcal{R})|I) = \frac{p(I|\mathcal{P}(\mathcal{R}))}{p(I)} p(\mathcal{P}(\mathcal{R}))$$

- $p(I|\mathcal{P}(\mathcal{R}))$ 是给定分割 $\mathcal{P}(\mathcal{R})$ 的后验概率.
- $p(\mathcal{P}(\mathcal{R}))$ 是 $\mathcal{P}(\mathcal{R})$ 在所有可能的分割中的概率.
- $p(I)$ 是图像*I*出现的概率

运动分析
— Geodesic Active Region

- 假设前面的Bayes形式不受 $p(\mathcal{P}(\mathcal{R}))$, $p(I)$ 的影响,则有

$$p_S(\mathcal{P}(\mathcal{R})|I) = p(I|\{\mathcal{R}_A, \mathcal{R}_B\})$$

- 进一步假设区域分割的独立性,则有

$$p_S(\mathcal{P}(\mathcal{R})|I) = p([I|\mathcal{R}_A] \cap [I|\mathcal{R}_B]) = p(I|\mathcal{R}_A)\ p(I|\mathcal{R}_B)$$

- 假设象素的独立性

$$p(I|\mathcal{R}_X) = \prod_{s \in \mathcal{R}_X} p_X(I(s)) \quad X \in \{A, B\}.$$

- 综上得到

$$p_S(\mathcal{P}(\mathcal{R})|I) = \prod_{s \in \mathcal{R}_A} p_A(I(s)) \prod_{s \in \mathcal{R}_B} p_B(I(s)).$$

运动分析
— Geodesic Active Region

- 最后得到区域上的表示形式为

$$E(\partial\mathcal{P}(\mathcal{R})) = \underbrace{-\iint_{\mathcal{R}_A} \log\left[\underbrace{p_A(I(x,y))}_{h_A\ probability}\right] dxdy}_{\mathcal{R}_A\ fitting\ measurement} \underbrace{-\iint_{\mathcal{R}_B} \log\left[\underbrace{p_B(I(x,y))}_{h_B\ probability}\right] dxdy}_{\mathcal{R}_B\ fitting\ measurement}.$$

运动分析
— Geodesic Active Region

- Geodesic Active Region的表达式

$$E(\partial\mathcal{R}) = -\sum_{X\in\{A,B\}} \underbrace{\alpha \overbrace{\iint_{\mathcal{R}_X} \log\left[\underbrace{p_{r,X}(I(x,y))}_{h_X\ probability}\right] dxdy}^{\mathcal{R}_X\ fitting\ measurement}}_{X\ Region\ Term}$$

$$\sum_{X\in\{A,B\}} \underbrace{(1-\alpha)\int_0^1 \overbrace{g\left(\underbrace{p_{b,X}(I(\partial\mathcal{R}_X(c_x)))}_{X\ boundary\ probability}\right)}^{X\ boundary\ attraction} \overbrace{\left|\partial\dot{\mathcal{R}}_X(c_x)\right|}^{X\ regularity} dc_x}_{X\ Boundary\ Term}.$$

运动分析
— Geodesic Active Region

- Geodesic Active Region的level set 形式
  - 定义

$$\delta_\alpha(\phi) = \begin{cases} 0, & |\phi| > \alpha, \\ \frac{1}{2\alpha}\left(1 + \cos\left(\frac{\pi\phi}{\alpha}\right)\right), & |\phi| < \alpha, \end{cases}$$

$$\mathscr{H}_\alpha(\phi) = \begin{cases} 1, & \phi > \alpha, \\ 0, & \phi < -\alpha, \\ \frac{1}{2}\left(1 + \frac{\phi}{\alpha} + \frac{1}{\pi}\sin\left(\frac{\pi\phi}{\alpha}\right)\right), & |\phi| < \alpha \end{cases}$$

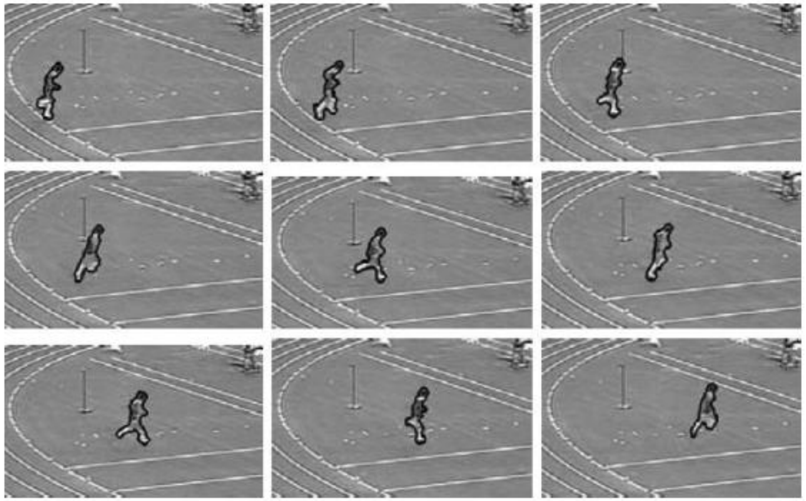
  - 边界和区域表示的Level set 形式

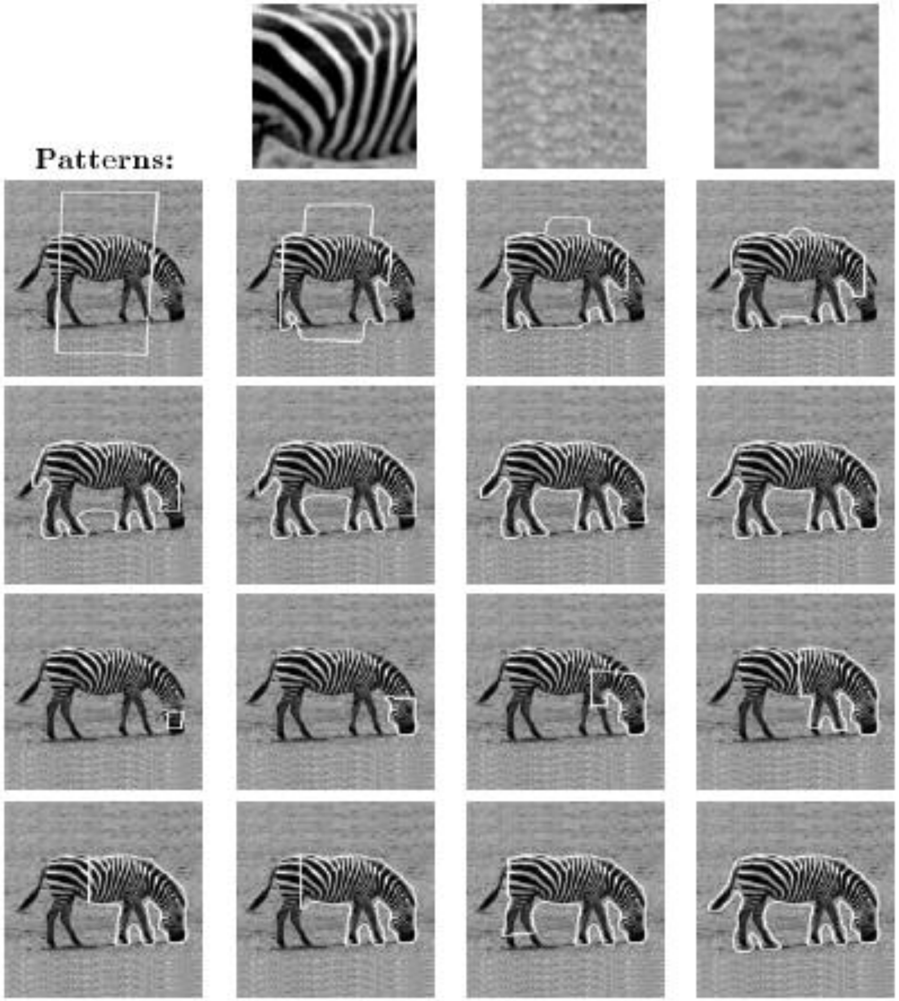
$$E_{\text{geodesic}}(\phi) = \iint_\Omega \delta_\alpha(\phi) b(;)|\nabla\phi|\, \mathrm{d}\Omega,$$

$$E_{\text{regional}}(\phi) = \underbrace{\iint_\Omega \mathscr{H}_\alpha(\phi) r_O(;)\, \mathrm{d}\Omega}_{\text{class } A} + \underbrace{\iint_\Omega (1 - \mathscr{H}_\alpha(\phi)) r_B(;)\, \mathrm{d}\Omega}_{\text{class } B}$$

运动分析
— Geodesic Active Region

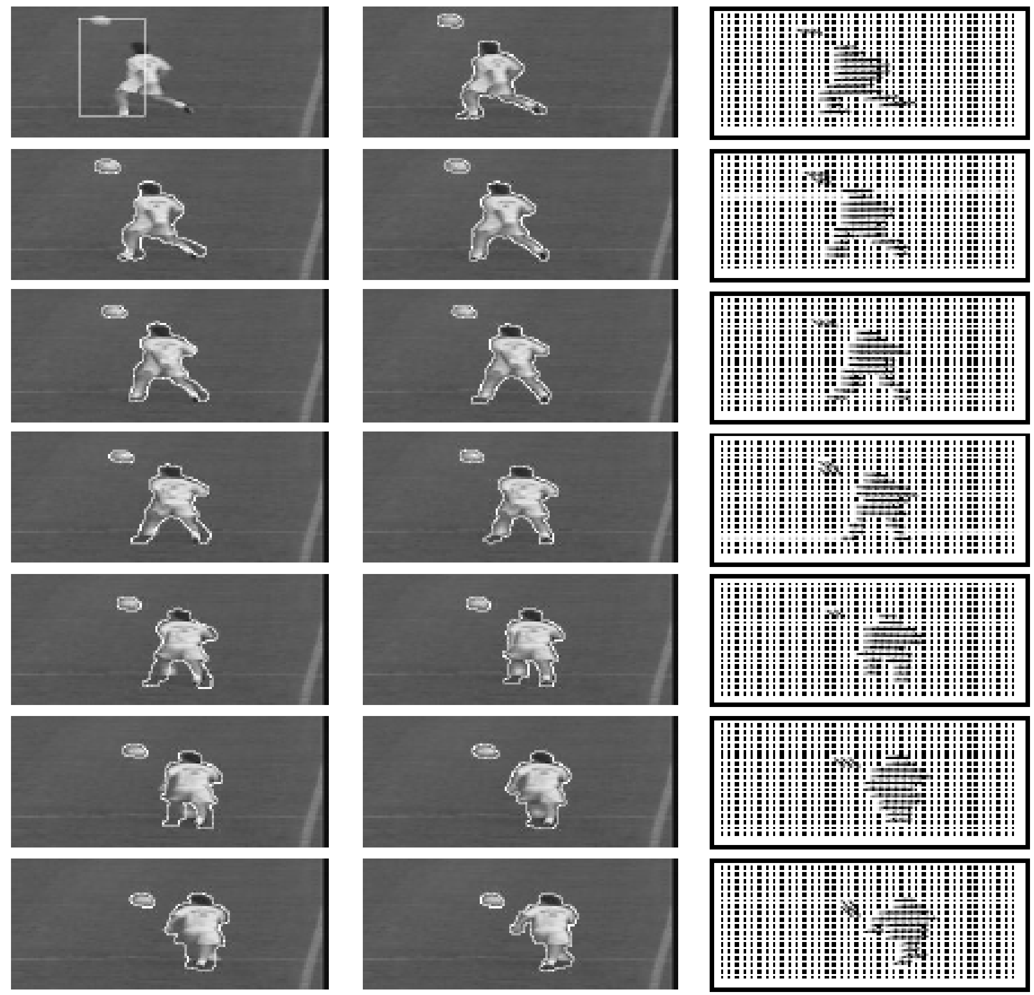
- GAR在视频跟踪中的试验结果





运动分析
— Geodesic Active Region

- 试验结果



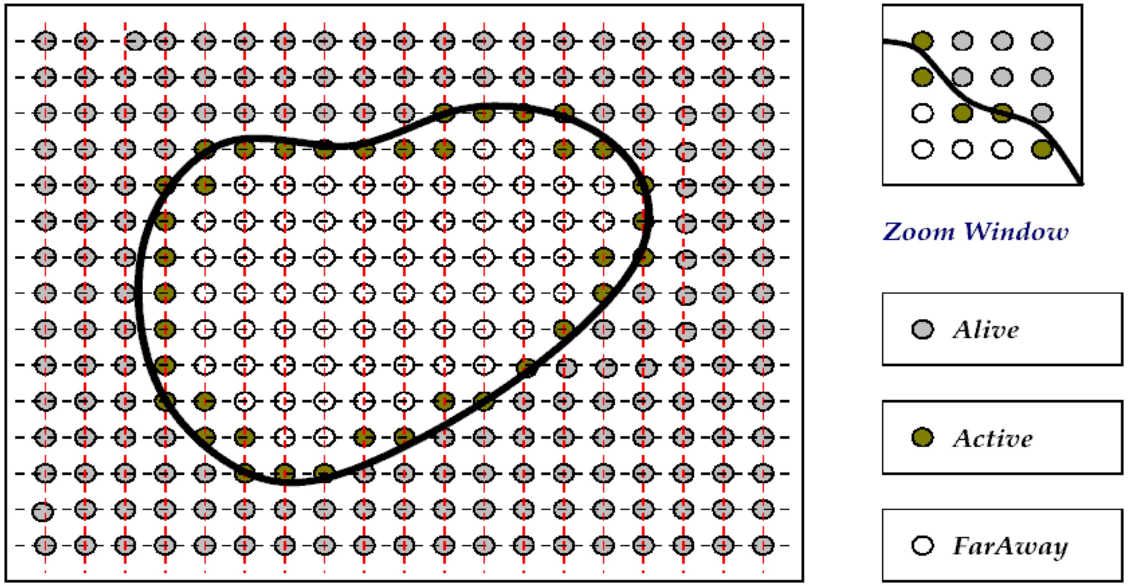
PDE方法的未来

PDE是吞噬计算能力的霸王龙

- 数值天气预报
- 主要采用各种流体模拟的
- 偏微分方程

  - 3公里分辨率的气象预报计算需要
  - >10万亿次/Sec

PDE离散求解的并行性



$$\frac{\partial \varphi}{\partial t} + v \cdot \nabla \varphi = 0$$

运算设备的发展

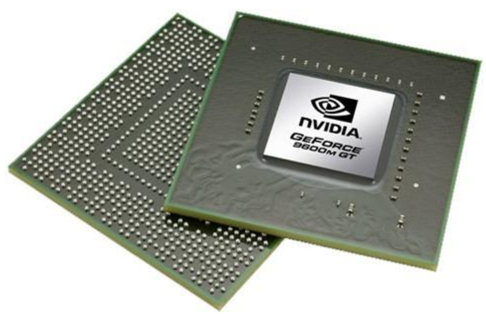
- 大型集群 >100万亿次计算能力

运算设备的发展

- 个人计算平台 >1万亿次

CPU → GPU → GPGPU

Intel Multi Core



nVidia kernel

Tesla C1060

量变引起质变

- 研究对象

- 计算方法

参考资源

- Level set 的Matlab 工具箱
  http://www.cs.ubc.ca/~mitchell/ToolboxLS/
  用matlab实现了三种数值算法

- 有好的C++语言实现的Level Set工具箱吗？

参考文献

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- [2] James A. Sethian. *Level Set Methods and Fast Marching Methods*. Cambridge University Press (1999).
- [3] J.A. Sethian. Level Set Methods: Evolving interfaces in geometry, fluid mechanics, computer vision, and materials science. Cambridge University Press(1996).