Poisson Equation

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致谢

- 感谢沈向洋博士提供的部分课件内容

Outline

- Partial differential equation
- From variational methods to PDE
- Poisson Equation
- Application
  - Poisson Image Editing
  - Poisson/Laplacian Mesh Editing
- How to solve?
- More …

Siméon Denis Poisson

- His teachers: *Laplace*, *Lagrange*, …
- Poisson’s terms:
  - Poisson's equation
  - Poisson's integral
  - Poisson distribution
  - Poisson brackets
  - Poisson's ratio
  - Poisson's constant

1781-1840, France

“Life is good for only two things: to study mathematics and to teach it.”

Background:

- Partial Differential Equations (PDE)

$$E(f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}) = 0$$

- The PDE's which occur in physics are mostly second order and linear:

$$A \cdot f_{xx} + 2B \cdot f_{xy} + C \cdot f_{yy} + D \cdot f_x + E \cdot f_y + F \cdot f + G = 0$$

$$A \cdot f_{xx} + 2B \cdot f_{xy} + C \cdot f_{yy} + D \cdot f_x + E \cdot f_y + F \cdot f + G = 0$$

$A \cdot C < B^2$:
- Hyberbolic
  - wave equation: $$\Delta f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$$

$A \cdot C = B^2$:
- Parobolic
  - heat equation: $$\frac{\partial f}{\partial t} = k \cdot \Delta f$$

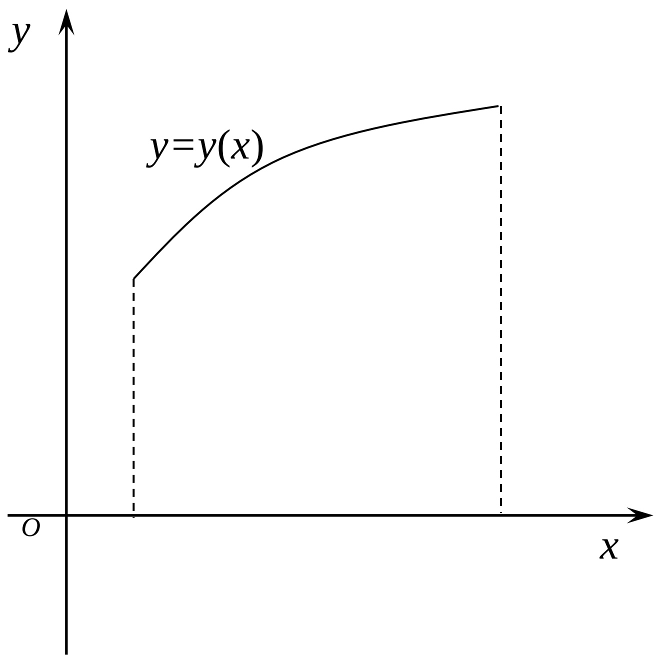
$A \cdot C > B^2$:
- Elliptic
  - Laplace equation: $$\Delta f = 0$$
  - Poisson equation: $$\Delta f = -\rho$$

变分命题与一般极值问题

- 历史上有很多有名的极值问题，其求解方法可统称为变分法
  - 两点间的最短连线问题
  - 最速下降线问题
  - 短程线问题
  - …

两点间的最短连线问题

- 为什么“任意两点间的最短连线是连接两端的直线”？



两点间的最短连线问题

- 为什么“任意两点间的最短连线是连接两端的直线”？

- 问题的假设：
  - 二维平面空间，一点是坐标原点(0,0)，一点在(a,b)
  - 两点间的连接曲线是 $y=y(x)$
  - 曲线的弧长微元是 $\mathrm{d}s^2=\mathrm{d}x^2+\mathrm{d}y^2$ 或 $\mathrm{d}s=\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$
  - 曲线的总弧长是

$$\underline{s}=\int_0^a (1+y'^2)^{1/2}\,\mathrm{d}x$$

s是标量，是$y'(x)$的一个广义函数，称为泛函，可记为$s(y')$

两点间的最短连线问题

- 问题的数学描述：找出具有曲线$y(x)$使得

$$\min_{y'} \int_0^a (1+y'^2)^{1/2}\,\mathrm{d}x$$

- 同时必须满足端点约束条件 (constraint condition)

$$\begin{cases} y(0)=0 & x=0 \\ y(a)=b & x=a \end{cases}$$

变分命题 (II)

- 第一类变分问题：
  - 被积函数包括一阶导数的变分问题
  - 满足端点约束条件
  - 在所有的足够光滑函数$y(x)$中，求使以下泛函为极值

$$\Pi(y)=\int_{\alpha}^{\beta}F(x,y,y')\mathrm{d}x$$

- 第二类变分问题：
  - 两个待定函数：$y(x)$，$z(x)$
  - 满足约束条件：$\varphi(x, y, z) = 0$
  - 满足端点约束条件
  - 在所有的足够光滑函数$y(x)$，$z(x)$中，求使以下泛函为极值

$$\Pi(y,z)=\int_{\alpha}^{\beta}F(x,y,y',z,z')\mathrm{d}x$$

变分法中的符号

- 给定**函数**$y(x)$
  - 宗量：x
  - 函数：$y(x)$
  - 宗量的增量：Δx
  - 函数的增量：
    - $\Delta y = y(x+\Delta x) - y(x)$
  - 当两点无限接近：
    - $\Delta x \to dx$，$\Delta y \to dy$
  - 略去高阶微量：
    - $dy = y'(x)dx$
  - 当在x处取得函数极值
    - $dy = 0$

- 给定**泛函**$\Pi(y)$
  - 宗量：y
  - 泛函：$\Pi(y)$
  - 函数的变分：δy
  - 泛函的变分：
    - $\delta\Pi = \Pi(y+\delta y) - \Pi(y)$
  - 在计算$\delta\Pi$时可以展开$\Pi(y+\delta y)$中的被积函数只保留线性项
  - 当在y处取得泛函极值
    - $\delta\Pi = 0$

函数$y(x)$在定义域内与$y(x)+\delta y(x)$处处无限接近

第一类变分问题

- 设函数$y(x)$是下式的极值解

$$\Pi(y)=\int_{\alpha}^{\beta}F(x,y,y')\mathrm{d}x$$

  - 且满足端点条件 $y(\alpha)=\bar{y}_1, y(\beta)=\bar{y}_2$

- 设其邻近的函数$y(x)+\delta y(x)$也满足端点条件
  - 因此端点变分满足

$$\delta y(\alpha)=\delta y(\beta)=0$$

- 泛函的变分为

$$\begin{aligned}\delta\Pi &=\Pi(y+\delta y)-\Pi(y)\\ &=\int_{\alpha}^{\beta}\{F(x,y+\delta y,y'+\delta y')-F(x,y,y')\}\mathrm{d}x\end{aligned}$$

第一类变分问题

- 根据微量计算规则，设$y(x)$和$y(x)+\delta y(x)$是有一阶接近的曲线

$$F(x,y+\delta y,y'+\delta y')=F(x,y,y')+\left[\frac{\partial}{\partial y}F(x,y,y')\right]\delta y+\left[\frac{\partial}{\partial y'}F(x,y,y')\right]\delta y'$$

- 引入简写符号

$$F=F(x,y,y'),F_y=\frac{\partial}{\partial y}F(x,y,y'),F_{y'}=\frac{\partial}{\partial y'}F(x,y,y')$$

- 可得

$$\delta F=F_y\delta y+F_{y'}\delta y'$$

第一类变分问题

- 泛函的变分为：

$$\boxed{\delta\Pi=\int_{\alpha}^{\beta}\delta F\mathrm{d}x=\int_{\alpha}^{\beta}\left[F_{y}\delta y+F_{y'}\delta y'\right]\mathrm{d}x}$$

- 下面将证明函数$y(x)$的导数的变分等于函数$y(x)$的变分的导数，亦即导数和变分两种运算可以互换运算顺序：

$$\boxed{\delta y'=(\delta y)'}$$

变分问题的欧拉方程

- 由预备定理可知

$$\boxed{F_y - \frac{\mathrm{d}}{\mathrm{d}x} F_{y'} = 0, \alpha \le x \le \beta}$$

- 如果展开 $\dfrac{\mathrm{d}F_{y'}}{\mathrm{d}x}$

$$\boxed{F_y - \frac{\partial^2 F}{\partial x \partial y} - \frac{\partial^2 F}{\partial y \partial y'} y' - \frac{\partial^2 F}{\partial y' \partial x} y'' = 0}$$

- 其中$F(x,y,y')$必须具有二阶偏导数，$y(x)$也必须具有二阶偏导数。

由此把变分问题转化为微分方程求解

变分法求解(1)
最短连线问题

- 其变分极值问题为

$$\delta\Pi = \int_0^{\alpha} \left[1+(y'+\delta y')^2\right]^{1/2} \mathrm{d}x - \int_0^{\alpha} \left[1+y'^2\right]^{1/2} \mathrm{d}x$$

- 略去$\delta y'$的高次微量得

$$\delta\Pi = \int_0^{\alpha} \frac{y'+\delta y'}{\left[1+y'^2\right]^{1/2}} \mathrm{d}x = 0$$

- 分部积分，并利用$\delta y(0)=0$，$\delta y(a)=0$，得

$$\delta\Pi = -\int_0^{\alpha} \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y'}{\left(1+y'^2\right)^{1/2}}\right]\delta y \mathrm{d}x = 0$$

变分法求解(1)
最短连线问题

- 由变分法预备定理，给出以下微分方程

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y'}{\left(1+y'^2\right)^{1/2}}\right]=0$$

- 积分得

$$\frac{y'}{\left(1+y'^2\right)^{1/2}}\equiv const \quad \Longrightarrow \quad y'\equiv const$$

- 由端点约束条件即得 $y=\dfrac{b}{a}x$

泛函变分问题的一般求解步骤

1. 从物理上建立泛函及其条件
2. 通过泛函变分，利用变分法基本预备定理求得欧拉方程
3. 在边界条件下求解欧拉方程，即微分方程求解

变分法与欧拉方程

- 变分法与欧拉方程代表同一物理问题
- 欧拉方程求解和从变分法求数值近似解（如有限元，利兹法，伽辽金法等），其效果一样

- 欧拉方程求解很困难，但从泛函求近似解通常很方便，因而变分法一直被广为重视。
- 但并不是所有的微分方程都能找到相对应的泛函问题。

Poisson Equation

$$\Delta f = -\rho$$

$$\Delta \equiv \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}$$

$$\rho = \rho(x, y)$$

平衡膜—泊松方程

Variational interpretation

$$f^* = \arg\min_f \iint_\Omega \underbrace{\left\| \nabla f - \mathbf{v} \right\|^2}_{F} \quad \text{s.t } f^*|_{\partial\Omega} = f|_{\partial\Omega}$$

Euler Equation: $F_f - \frac{\partial}{\partial x} F_{f_x} - \frac{\partial}{\partial y} F_{f_y} = 0$

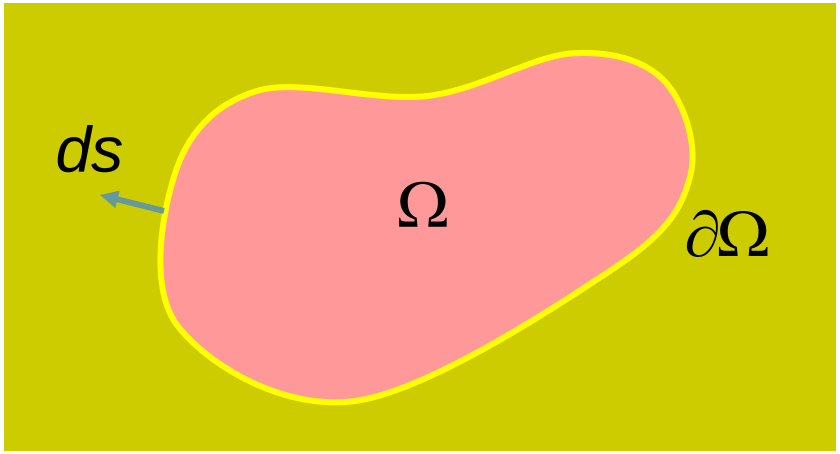
$$\boxed{\Delta f = div(\mathbf{v}) \ \text{ s.t } f^*|_{\partial\Omega} = f|_{\partial\Omega}}$$

$\mathbf{v}$ is a guidance field, needs not to be a gradient field.

Boundary conditions

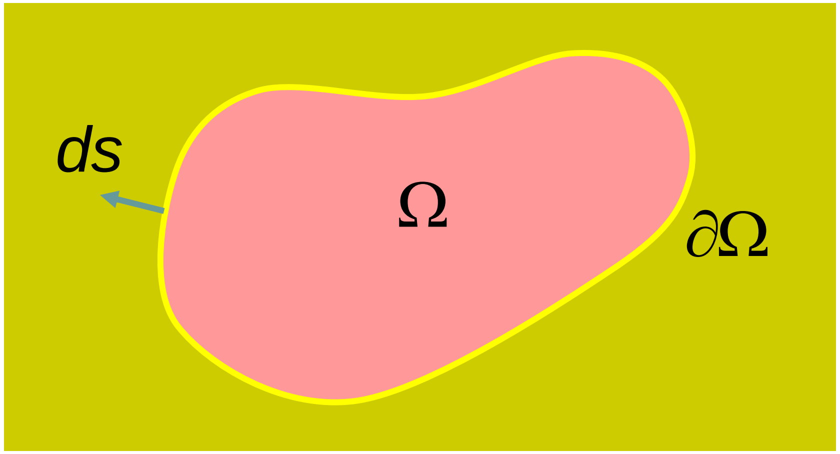
- *Dirichlet* boundary conditions: $f\,|_{\partial\Omega}$

- *Neumann* boundary conditions: $\frac{\partial f}{\partial \mathbf{s}}\,|_{\partial\Omega}$

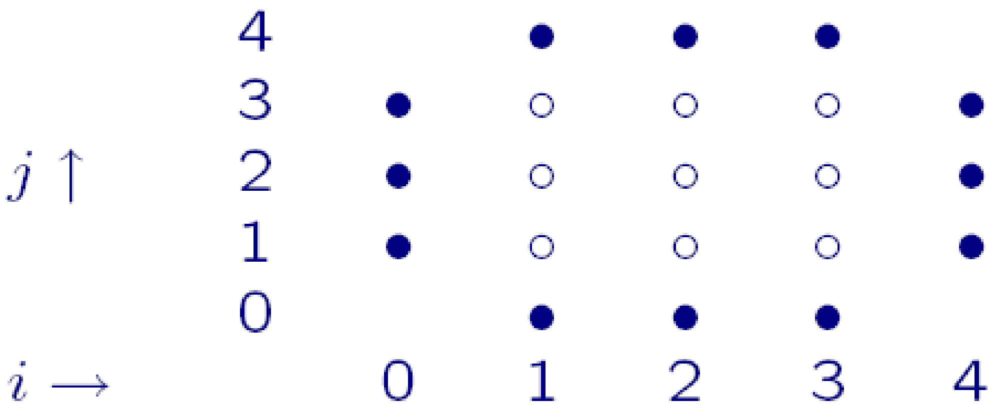


Existence of solution

The solution of an Poisson Equation is uniquely determined in Ω , if *Dirichlet* boundary conditions or *Neumann* boundary conditions are specified on $\partial\Omega$



Discrete Poisson Equation



$$\Delta f = -\rho$$

$$\frac{f_{i+1,j} + f_{i-1,j} - 2f_{i,j}}{h^2} + \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{h^2} = \rho_{i,j}$$

Matrix Nature of Poisson Equation

$$\begin{bmatrix} -4 & 1 & & & & & & & \\ 1 & -4 & 1 & & & & & & \\ & 1 & -4 & 1 & & & & & \\ & & 1 & -4 & 1 & & & & \\ & & & 1 & -4 & 1 & & & \\ & & & & 1 & -4 & 1 & & \\ & & & & & 1 & -4 & 1 & \\ & & & & & & 1 & -4 & 1 \\ & & & & & & & 1 & -4 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{12} \\ f_{22} \\ f_{32} \\ f_{13} \\ f_{23} \\ f_{33} \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{12} \\ b_{22} \\ b_{32} \\ b_{13} \\ b_{23} \\ b_{33} \end{bmatrix}$$

Poisson Equation Solver

- Direct method
- Iterative methods
  - Jacobi, Gauss-Seidel, SOR
- Multigrid method

Physical Origins of Poisson Equation

- Electrostatic potential

$$\Delta\Phi = -\frac{\rho(\mathbf{x})}{\varepsilon_0}$$

- Gravitational potential

$$\Delta\Phi = -4\pi G\rho(\mathbf{x})$$

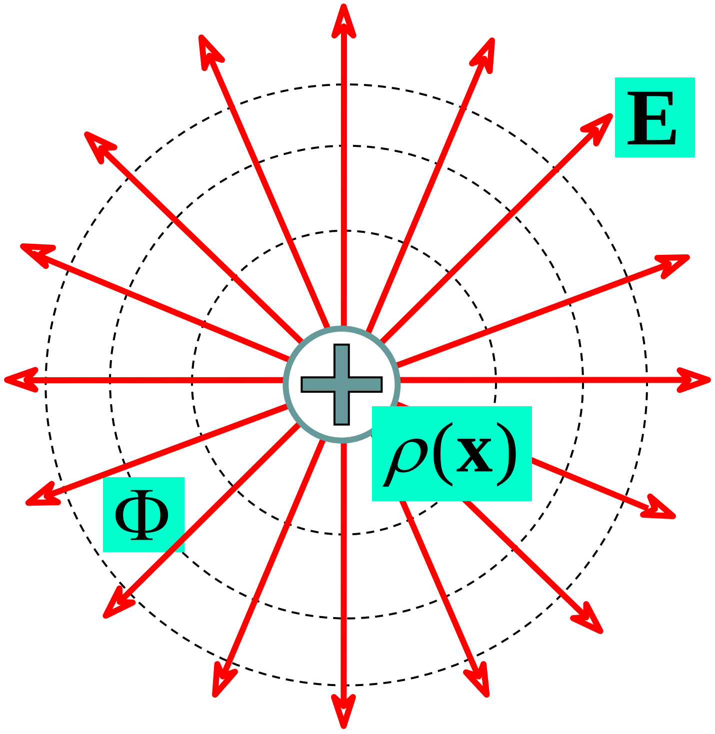
Electrostatic potential

$$\mathbf{F} = \frac{q_1 q_2 \mathbf{r}}{4\pi\varepsilon_0 r^3}$$

$\rho(\mathbf{x})$ Charge Density

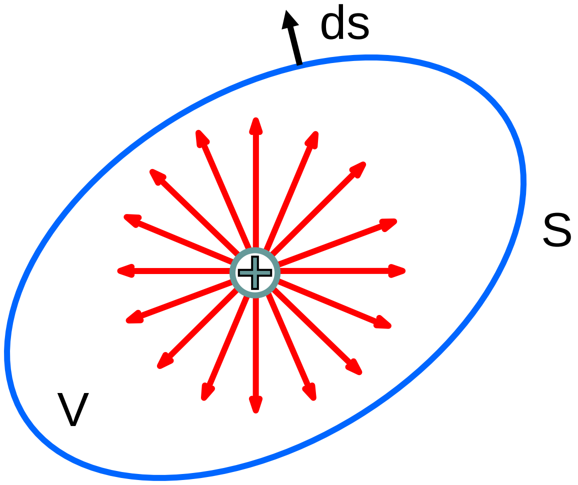
Φ Electric Potential

$\mathbf{E}$ Electric Field



$$\mathbf{E} = -\nabla\Phi$$

Derivations



Gauss's Law:

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \int_V \boxed{\frac{\rho(\mathbf{x})}{\varepsilon_0}} dv$$

Gauss's theorem:

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \int_V \boxed{\nabla \cdot \mathbf{E}} dv$$

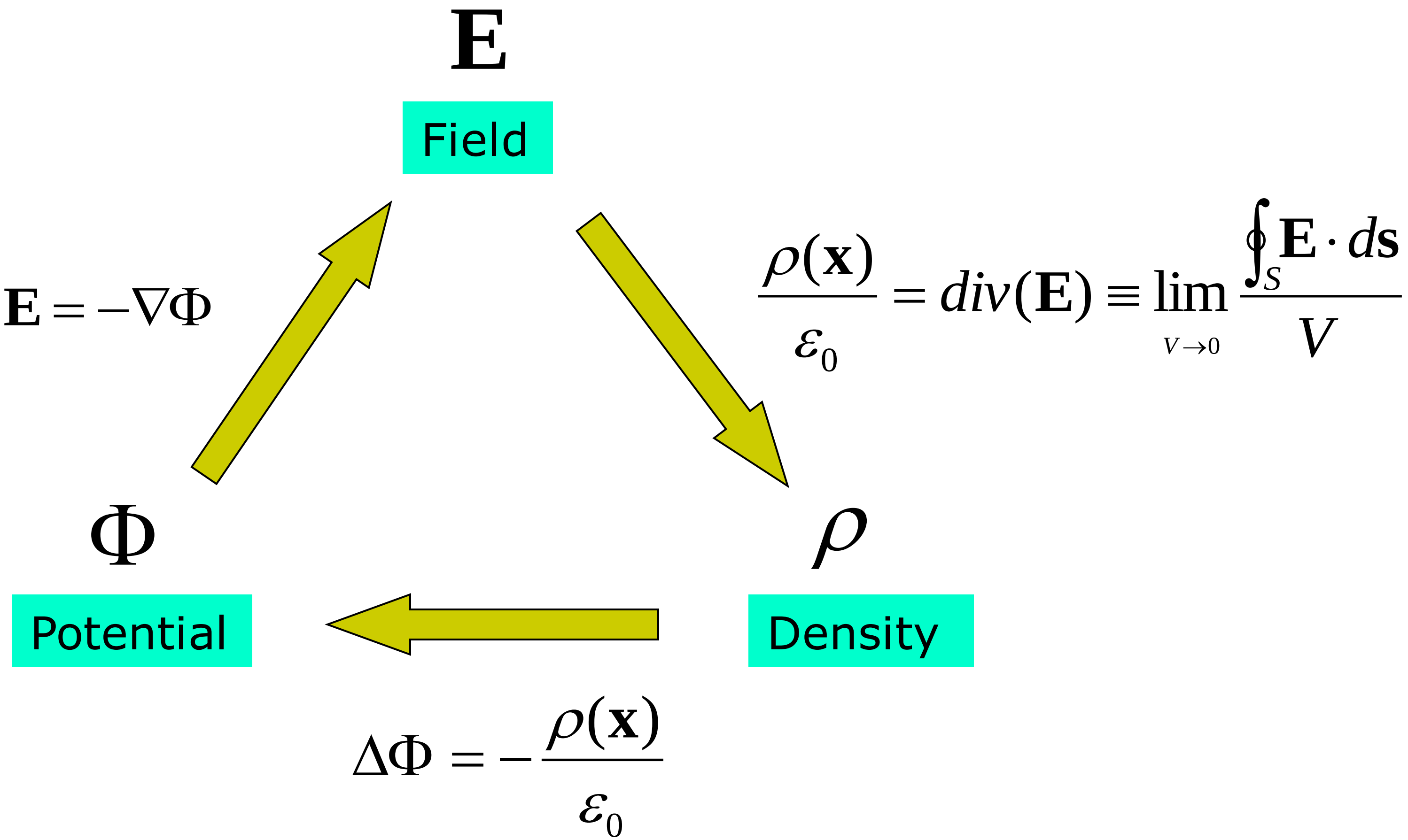
$$\boxed{\nabla \cdot \mathbf{E}} = \boxed{\frac{\rho(\mathbf{x})}{\varepsilon_0}}$$

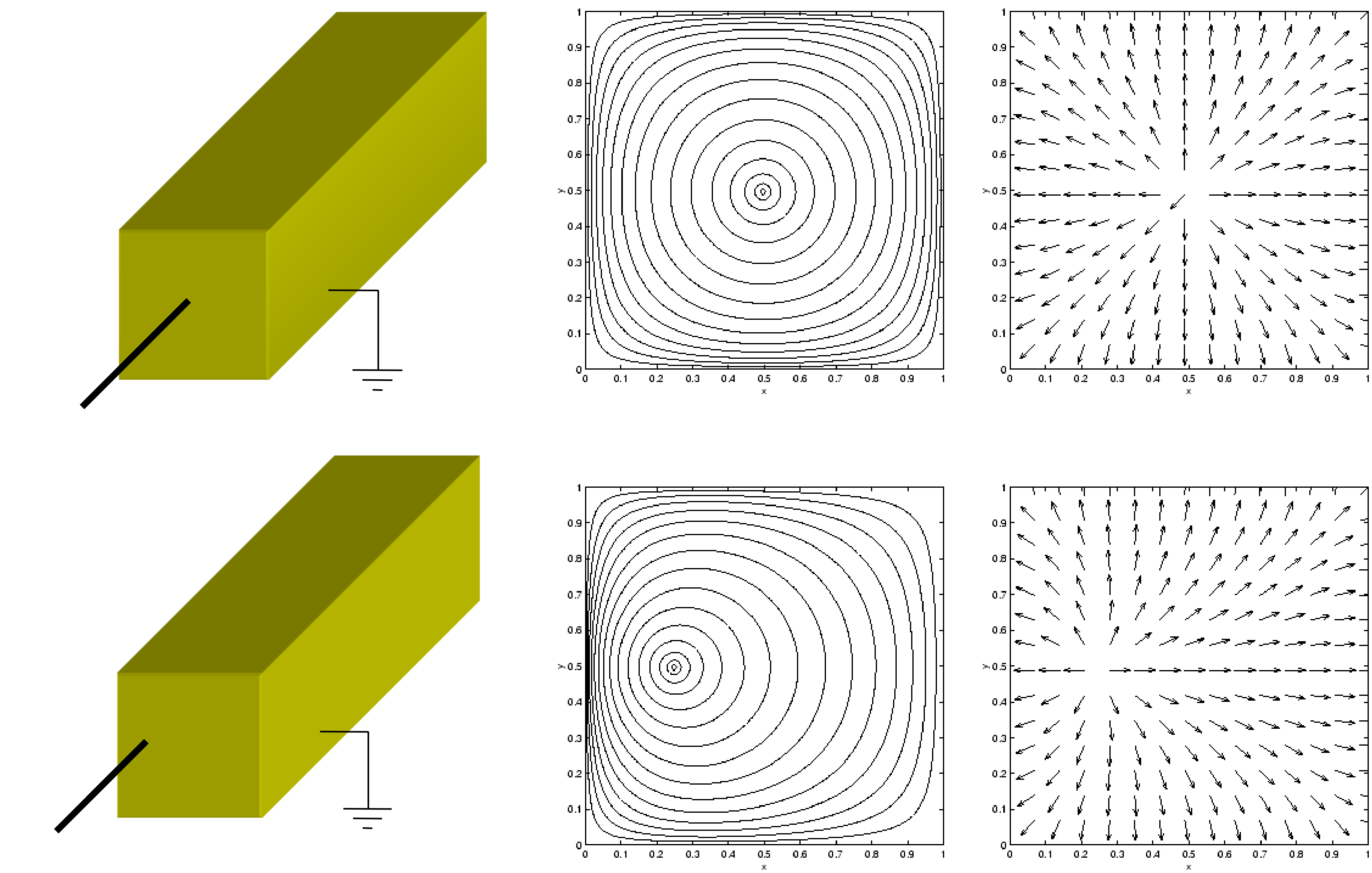
$$\mathbf{E} = -\nabla\Phi$$

$$\boxed{\Delta\Phi = -\frac{\rho(\mathbf{x})}{\varepsilon_0}}$$

Poisson Equation

Relationships





$$\rho(\mathbf{x}) = \delta(x_0, y_0)$$

Φ

$\mathbf{E}$

Gravitational potential

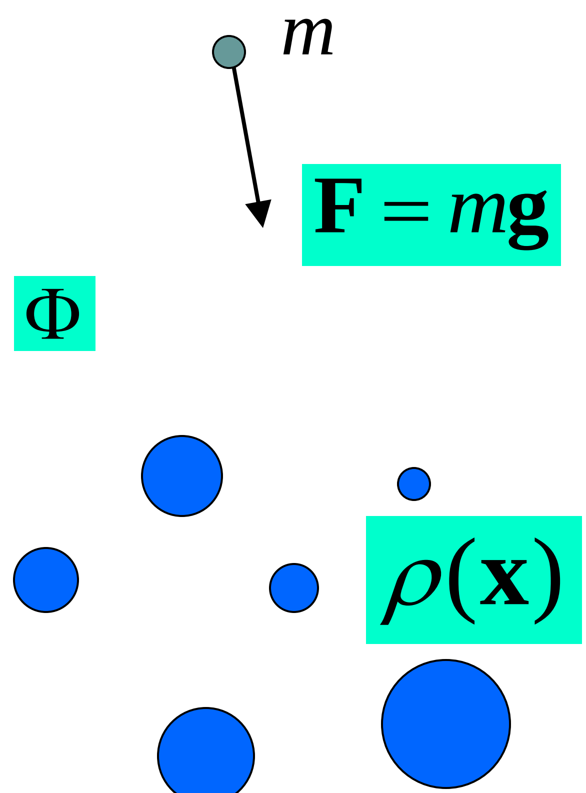
$$\mathbf{F} = \frac{mMG\mathbf{r}}{r^3}$$

$\rho(\mathbf{x})$ Mass Density

Φ Gravitational Potential

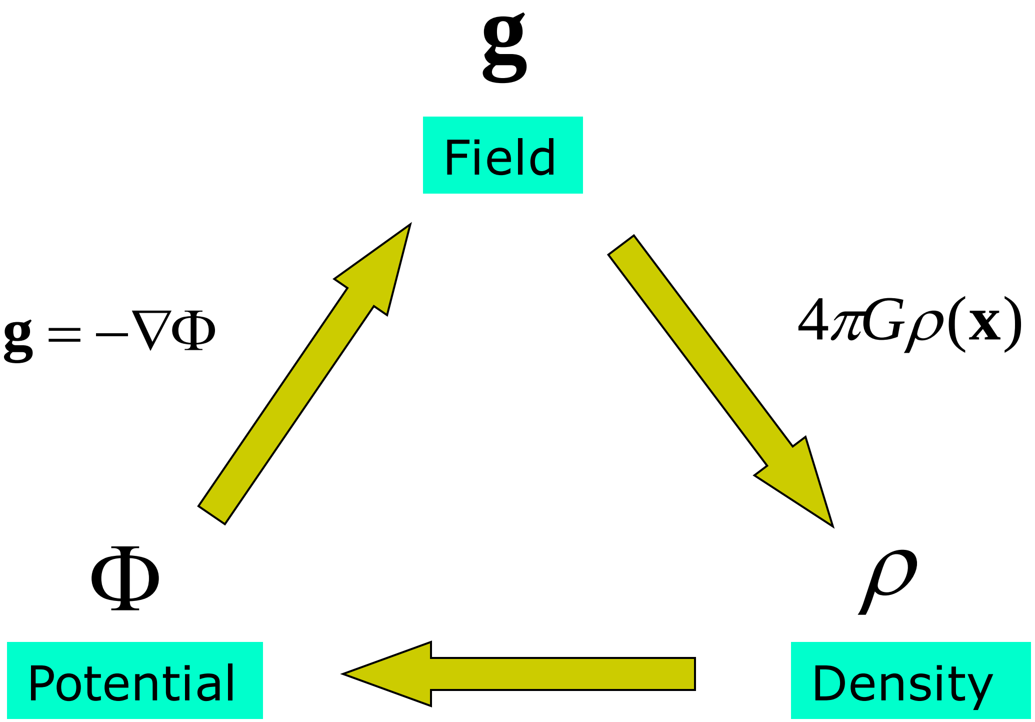
$\mathbf{g}$ Force Field

(acceleration)



$$\mathbf{g} = -\nabla\Phi$$

Relationships

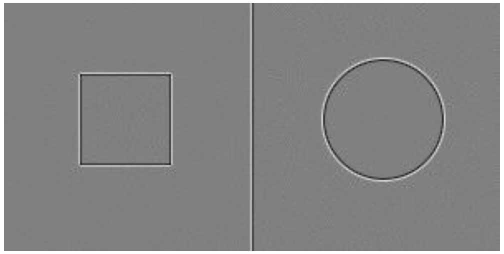


$$\mathbf{g} = -\nabla\Phi$$

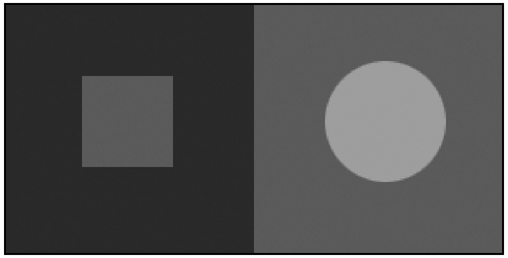
$$4\pi G\rho(\mathbf{x}) = div(\mathbf{g}) \equiv \lim_{V\to 0} \frac{\oint_S \mathbf{g}\cdot d\mathbf{s}}{V}$$

$$\boxed{\Delta\Phi = -4\pi G\rho(\mathbf{x})}$$

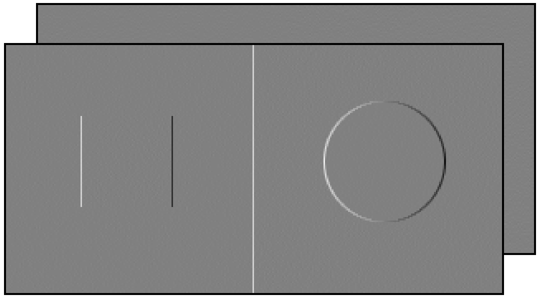
Analogy for Image



$\rho(\mathbf{x})$ Image Density



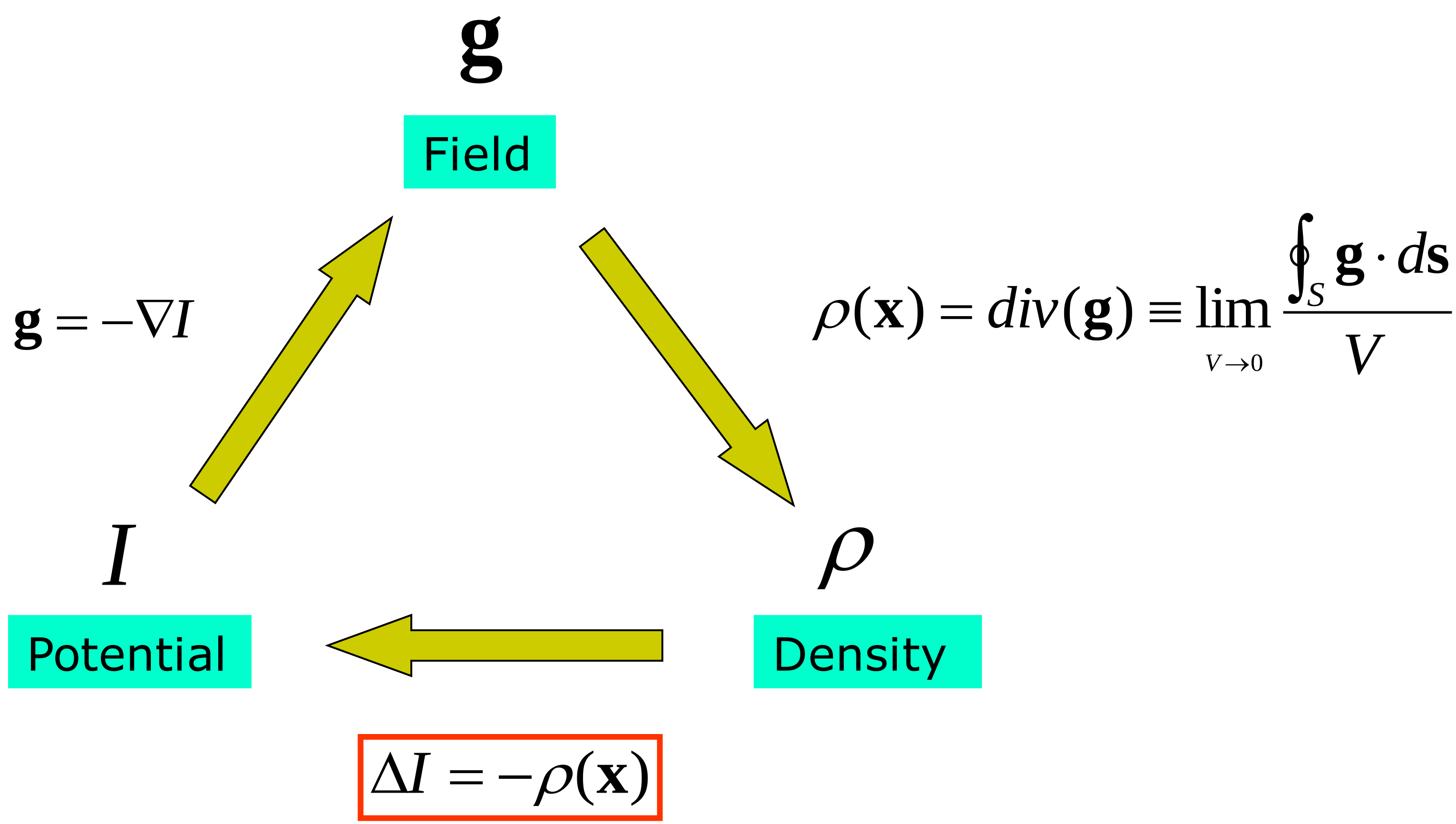
I Image (Potential)

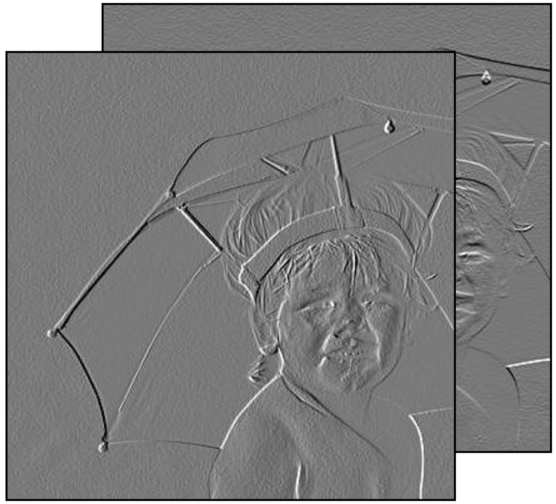


$\mathbf{g}$ Image Gradient

$$\mathbf{g} = -\nabla I$$

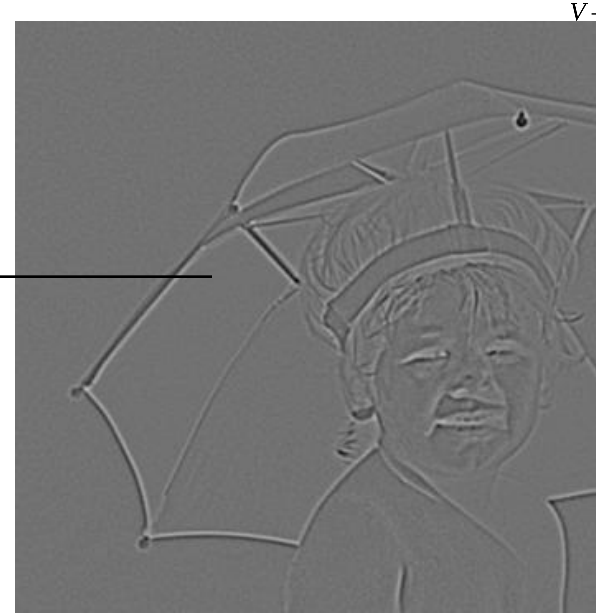
Relationships



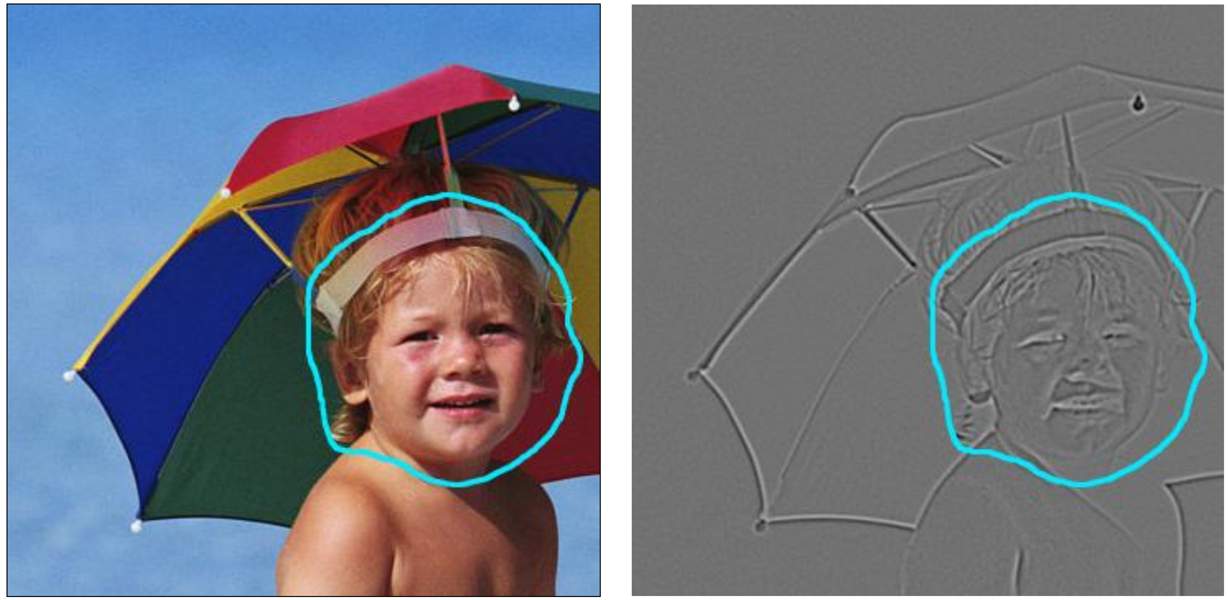


$$\mathbf{g} = -\nabla I$$

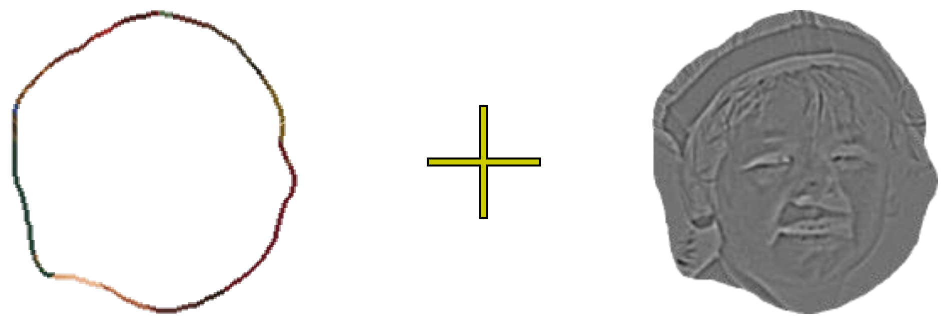
$$\rho(\mathbf{x}) = div(\mathbf{g}) \equiv \lim_{V \to 0} \frac{\oint_S \mathbf{g} \cdot d\mathbf{s}}{V}$$



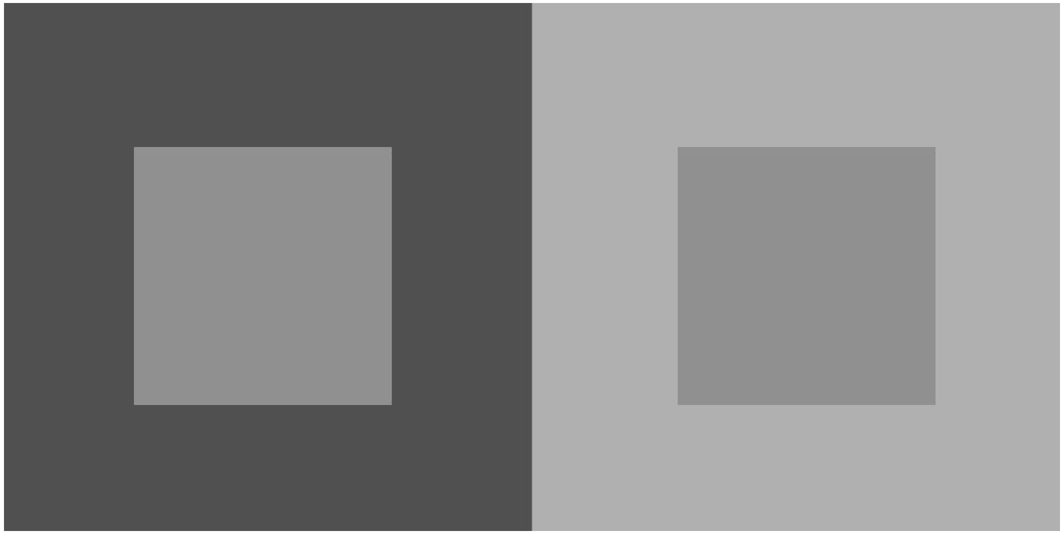
$$\boxed{\Delta I = -\rho(\mathbf{x})}$$



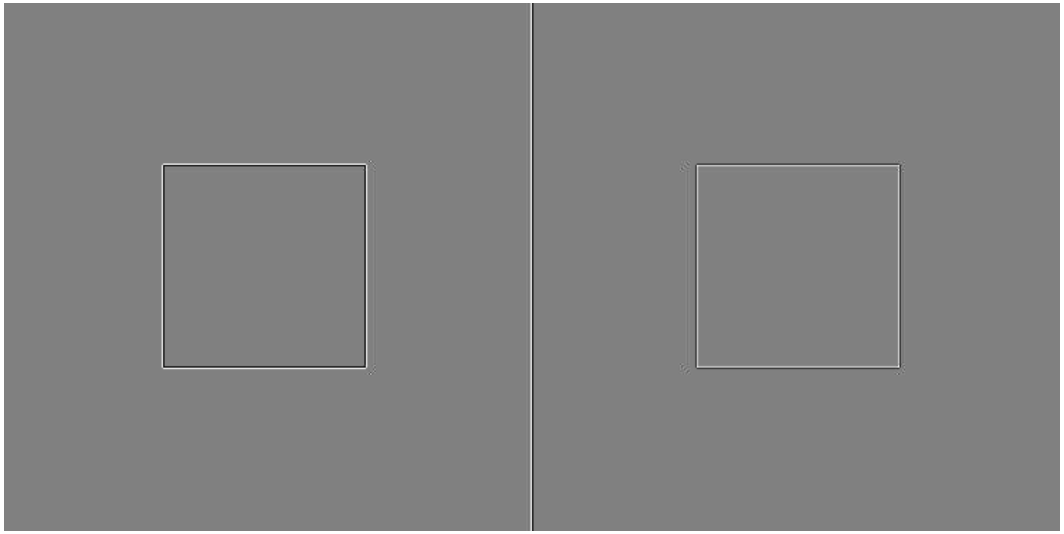
$$\boxed{\Delta I = -\rho(\mathbf{x})}$$

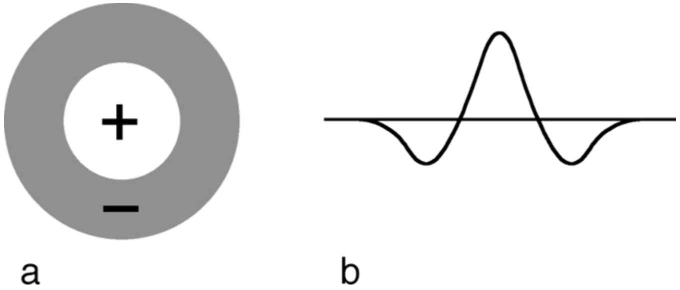


simultaneous contrast effect



Why Laplace operator?





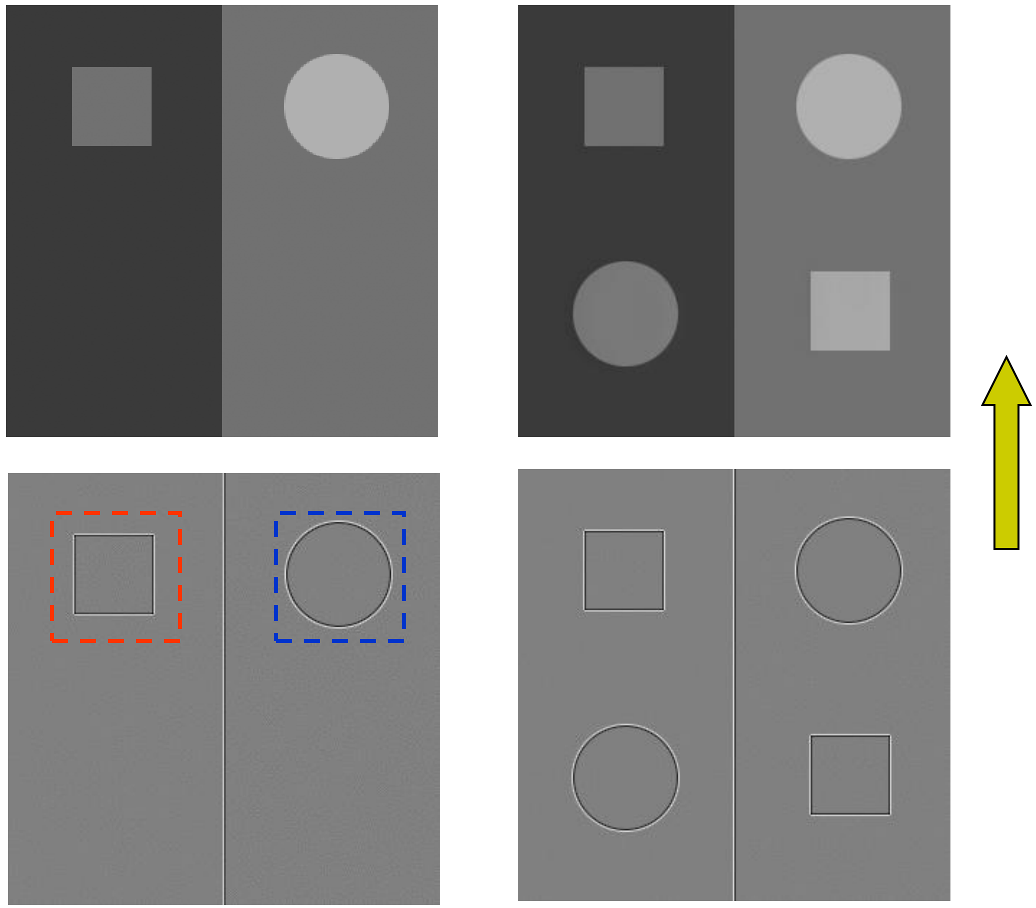
Motivation for Image Editing

I

Potential

ρ

Density



$$\boxed{\Delta I = -\rho(\mathbf{x})}$$

Poisson Image Editing

P. Pérez, M. Gangnet, and A. Blake

Poisson Image Editing. *SIGGRAPH 2003*

Seamless Cloning

- Precise selection: tedious and unsatisfactory
- Alpha-Matting: powerful but involved
- Seamless cloning: loose selection but no seams?

Cloning by solving Poisson Equation

$$\Delta I = div(\nabla I_A) \qquad \text{s.t.} \quad I|_{\partial\Omega} = I_B|_{\partial\Omega}$$

I_A $\qquad$ I_B

Why we do analogy for image?

- Easier in the image gradient domain
  - *Local* editing → *global* effects
  - *Seamless* - cloning, editing, tilling

Variational interpretation

$$I^* = \arg\min_f \iint_\Omega \underbrace{\left\| \nabla I - \nabla I_A \right\|^2}_{F} \quad \text{s.t } I^* |_{\partial\Omega} = I_B |_{\partial\Omega}$$

Euler Equation: $F_f - \frac{\partial}{\partial x} F_{f_x} - \frac{\partial}{\partial y} F_{f_y} = 0$

$$\boxed{\Delta I = div(\nabla I_A) \quad \text{s.t.} \quad I |_{\partial\Omega} = I_B |_{\partial\Omega}}$$

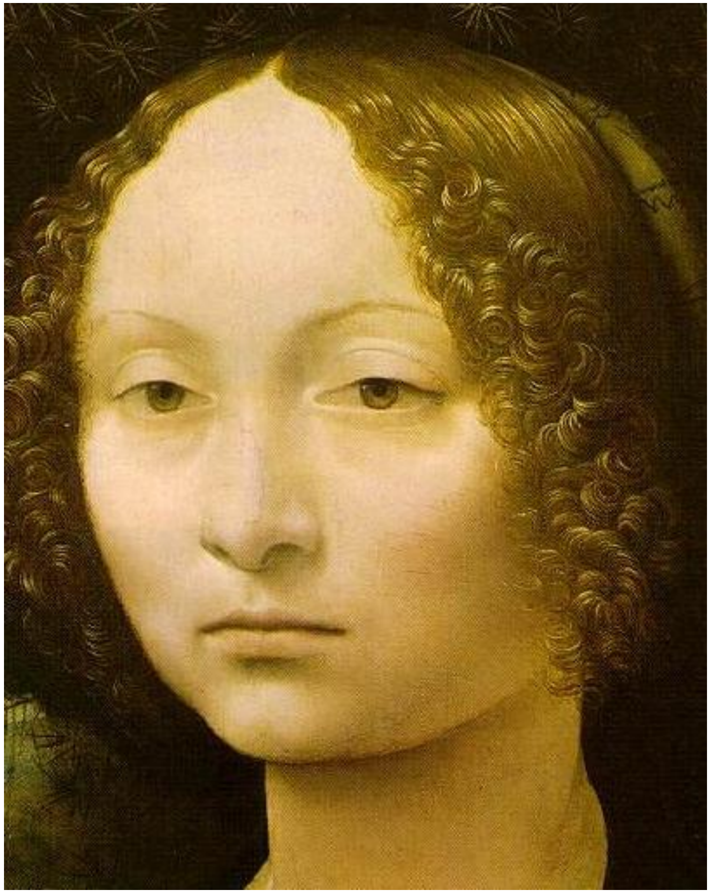
∇I_A is a gradient field to be cloned.

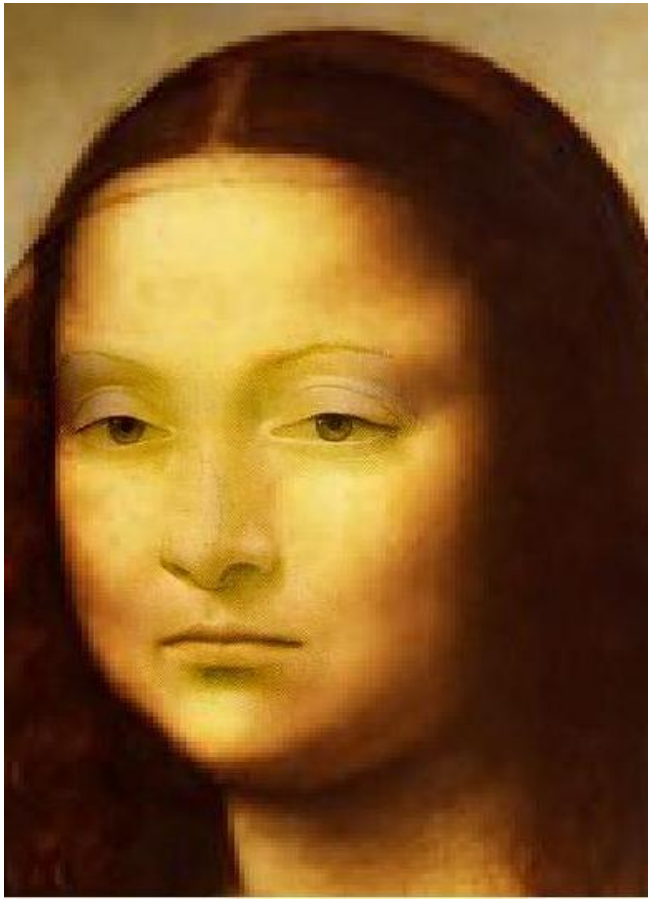
conceal

compose

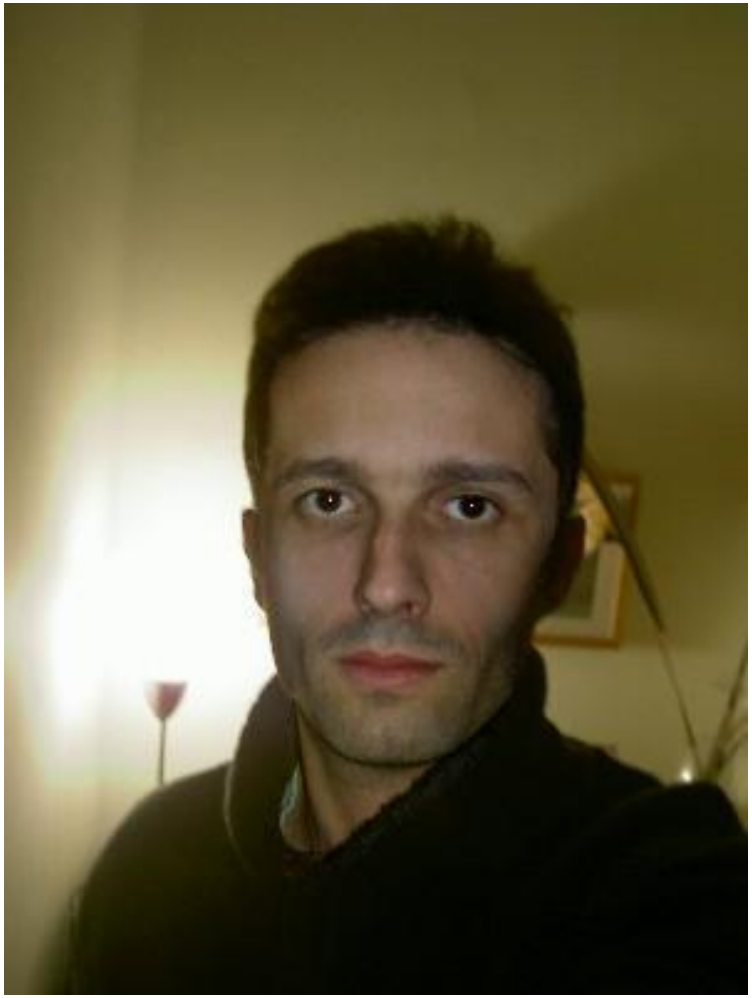
change texture

change features





mix lights



Seamless Editing

I_A

I

$$\Delta I = div(T(\nabla I_A)) \qquad \text{s.t.}\, I|_{\partial\Omega} = I_A|_{\partial\Omega}$$

change colors

change colors

Seamless Tiling

Single image

Seamless Tiling

Multiple images tiled at random

Poisson Matting

J. Sun, J. Jia, C. K. Tang, H. Y. Shum

Poisson Matting. *SIGGRAPH 2004.*

Image Composition

- Composition Equation

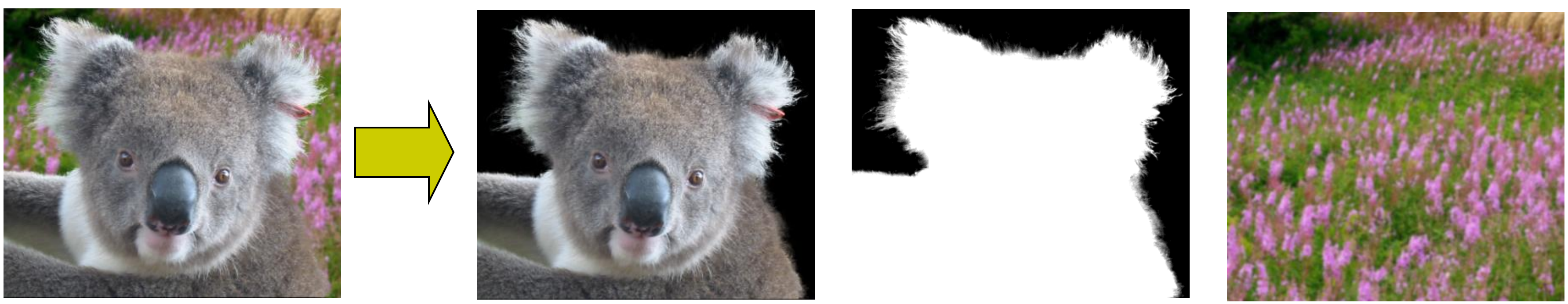
$$I = \alpha F + (1 - \alpha) B$$

I $\qquad$ αF $\qquad$ α $\qquad$ B

Matting – pulling of Matte

- Matting Equation

$$I = \alpha F + (1 - \alpha) B$$



I αF α B

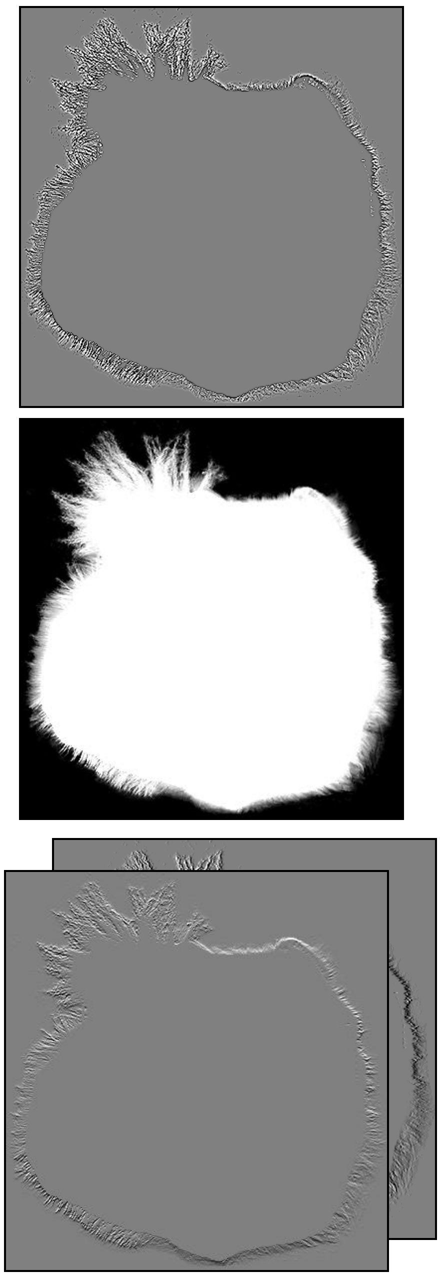
Analogy for Matte

$\rho(\mathbf{x})$ Matte Density

α Matte (Potential)

$\mathbf{g}$ Matte Gradient

$$\mathbf{g} = -\nabla\alpha$$



Relationships

$$\mathbf{g}$$

Field

$$\mathbf{g} = -\nabla\alpha$$

$$\rho(\mathbf{x}) = div(\mathbf{g}) \equiv \lim_{V\to 0} \frac{\oint_S \mathbf{g}\cdot d\mathbf{s}}{V}$$

$$\alpha$$

Potential

$$\rho$$

Density

$$\Delta\alpha = -\rho(\mathbf{x})$$

Poisson Matting (Global)

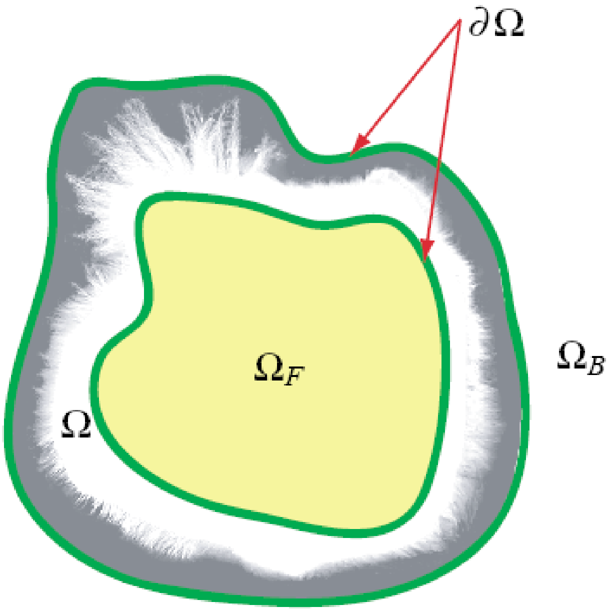
$$I = \alpha F + (1-\alpha)B$$

$$\nabla I = (F-B)\nabla\alpha + \alpha\nabla F + (1-\alpha)\nabla B$$

$$\nabla I \approx (F-B)\nabla\alpha$$

$$\nabla\alpha \approx \frac{\nabla I}{F-B} \qquad \rho(\mathbf{x}) \approx div(\frac{\nabla I}{F-B})$$

Matting by solving Poisson Equation



$$\Delta\alpha = div\left(\frac{\nabla I}{F-B}\right) \quad \text{s.t.}\ \alpha|_{\partial\Omega} = \begin{cases} 1 & \mathbf{x} \in \Omega_F \\ 0 & \mathbf{x} \in \Omega_B \end{cases}$$

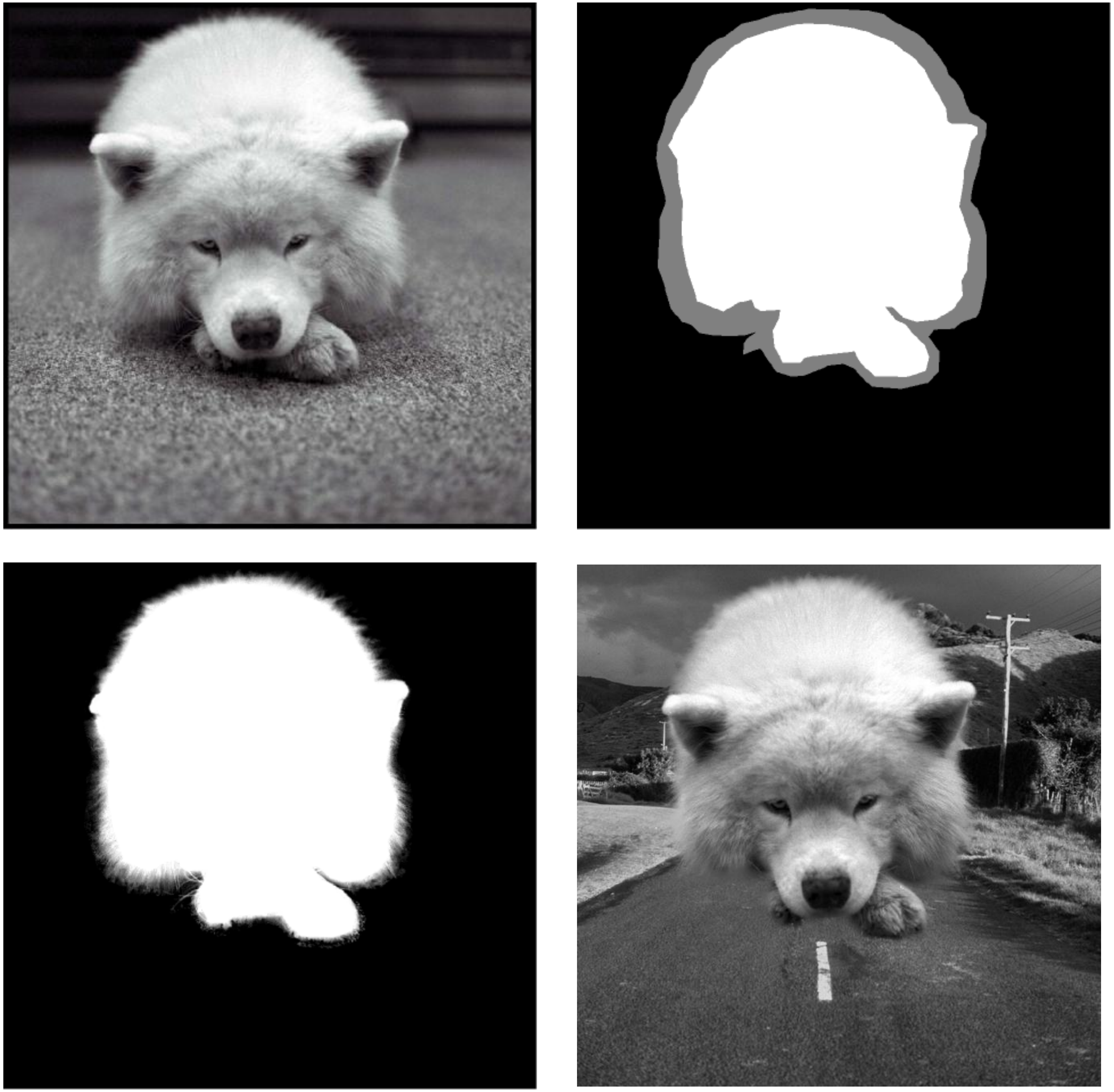
Variational interpretation

$$\alpha^* = \arg\min_f \iint_\Omega \underbrace{\left\| \nabla\alpha - \frac{\nabla I}{F-B} \right\|^2}_{J} \quad \text{s.t } \alpha^*|_{\partial\Omega} = \alpha|_{\partial\Omega}$$

Euler Equation: $J_f - \frac{\partial}{\partial x} J_{f_x} - \frac{\partial}{\partial y} J_{f_y} = 0$

$$\boxed{\Delta\alpha = div\left(\frac{\nabla I}{F-B}\right) \quad \text{s.t.} \quad \alpha|_{\partial\Omega} = \alpha|_{\partial\Omega}}$$

$\frac{\nabla I}{F-B}$ is a guidance field, an approximation of matte gradient.



Local Poisson Matting

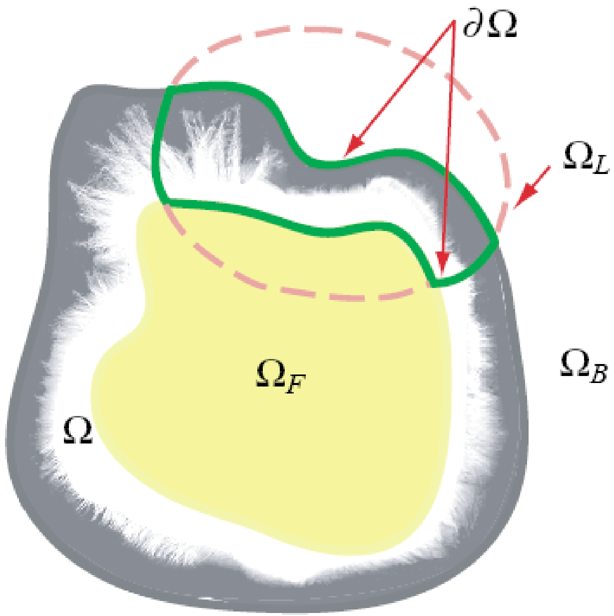
$$I = \alpha F + (1-\alpha)B$$

$$\nabla I = (F-B)\nabla\alpha + \alpha\nabla F + (1-\alpha)\nabla B$$

$$\nabla\alpha = \frac{1}{F-B}(\nabla I - \alpha\nabla F - (1-\alpha)\nabla B)$$

$$\nabla\alpha = A(\nabla I - D)$$

Matting by solving Poisson Equation



$$\Delta\alpha \approx div(A(\nabla I - D)) \quad \text{s.t.}\ \alpha|_{\partial\Omega} = \begin{cases} 1 & \mathbf{x} \in \Omega_F \\ 0 & \mathbf{x} \in \Omega_B \\ \alpha_g & \mathbf{x} \in \Omega \cap \Omega_L \end{cases}$$

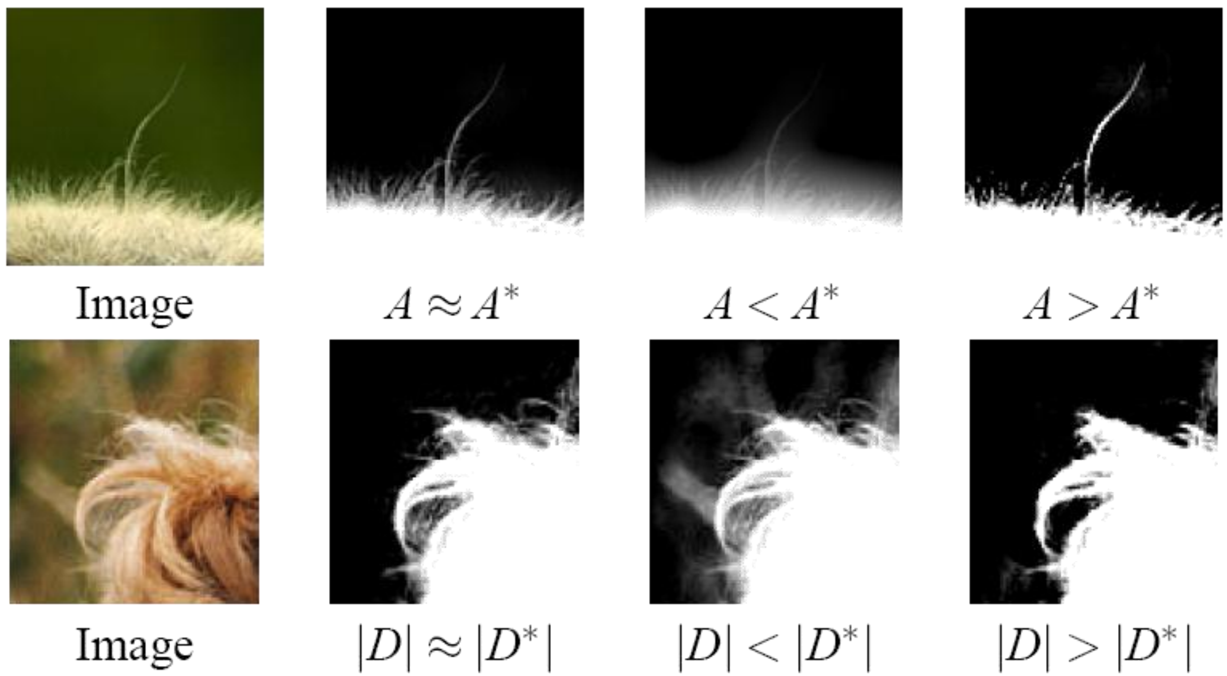
Why we do analogy for Matte?

- Editing Matte directly at pixel-level is very tedious and impractical!

  - *Local* editing → *global* effects

  - *Seamless* local refinement

Key Observation

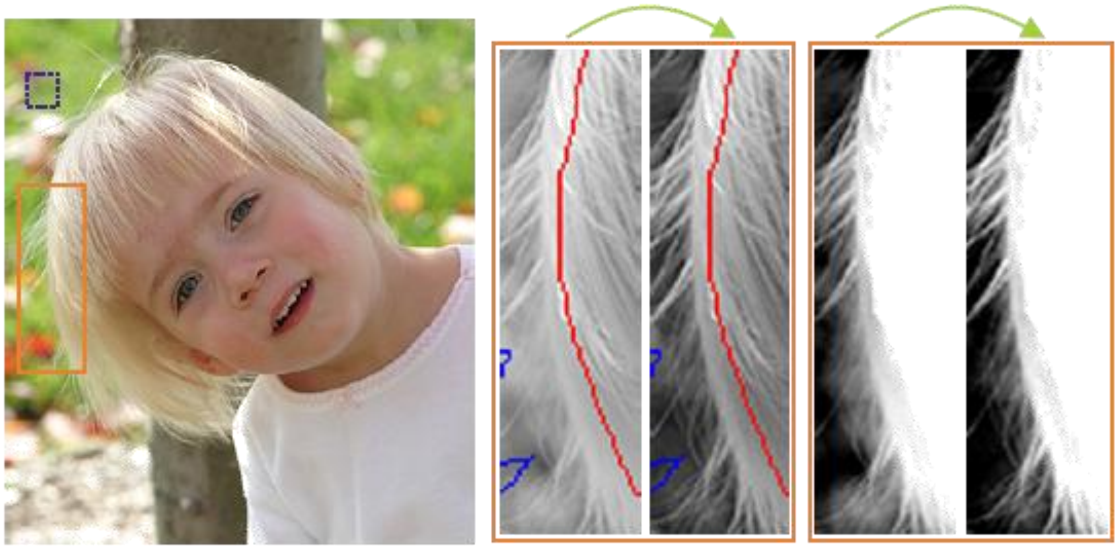


Human-in-the-loop

- In Global Poisson Matting:
  - *A* and *D* are estimated automatically.

- In Local Poisson Matting:
  - *A* and *D* are manipulated interactively.
  - *A* and *D* are manipulated more efficiently by a set of tools we designed.

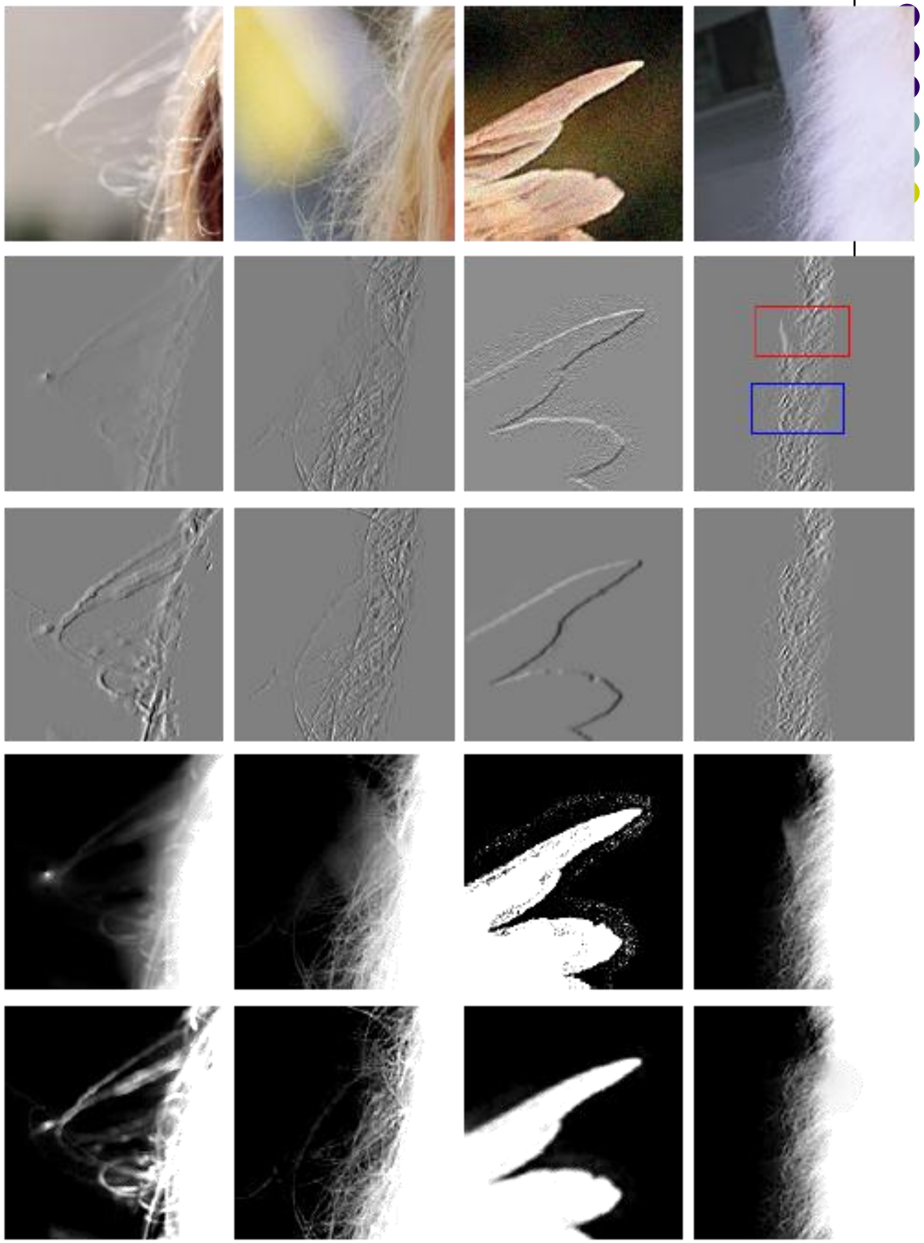
Tool I:
channel selection



Minimize the variance of the foreground or background colors.

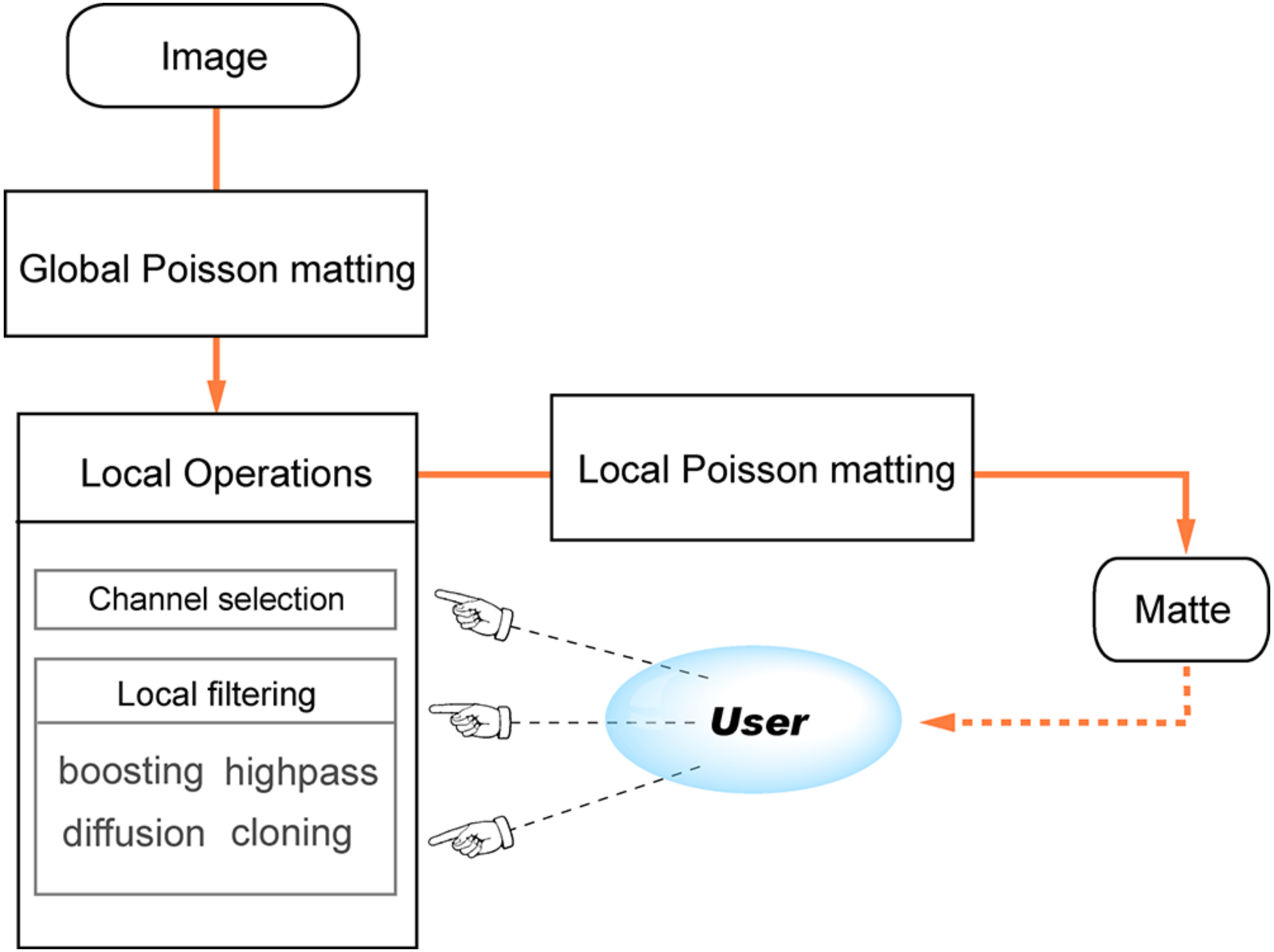
Tool 2: local filtering

- Boosting
- Highpass
- Diffusion
- Clone

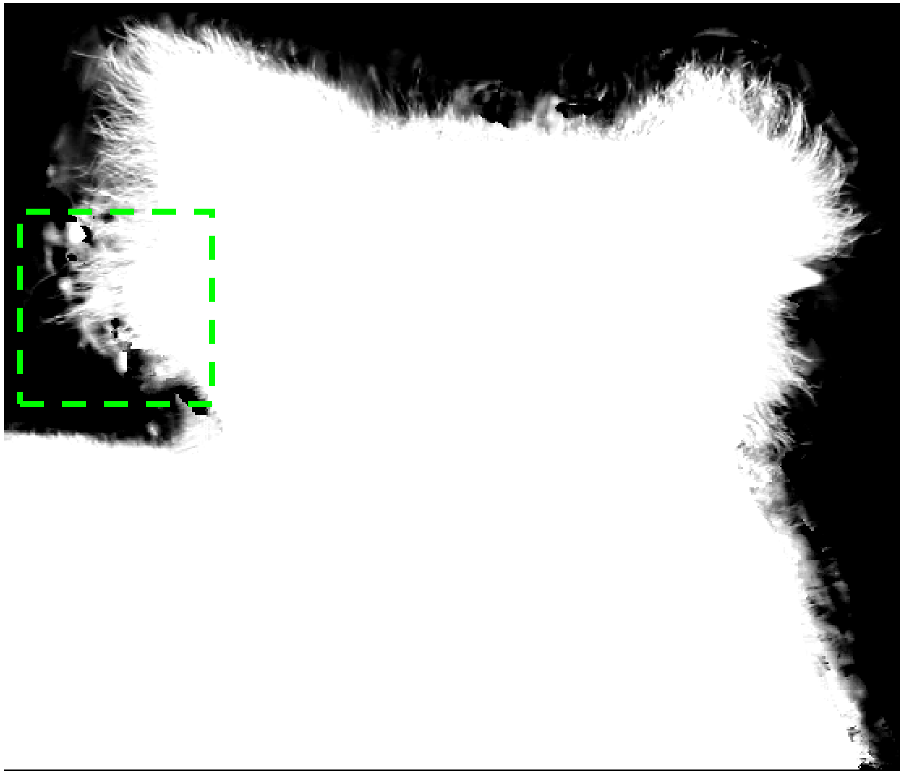


(a) Boosting (b) Highpass (c) Diffusion (d) Clone

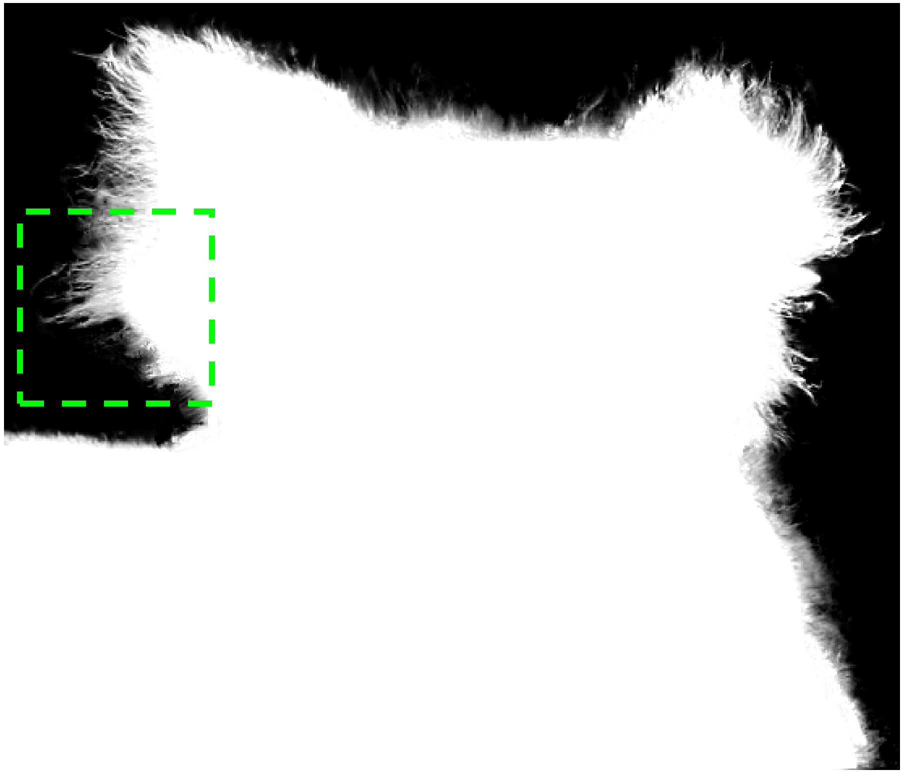
Poisson Matting

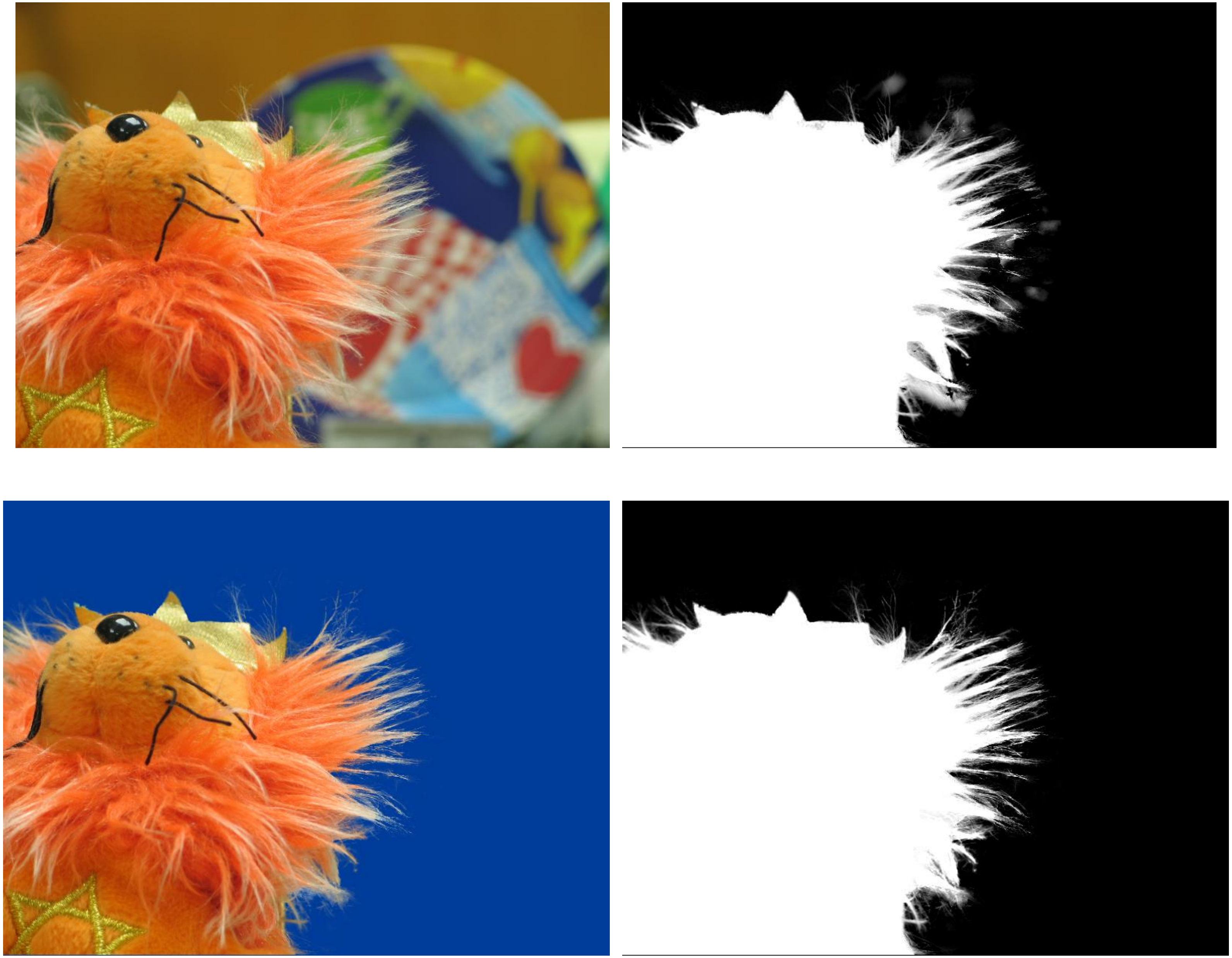


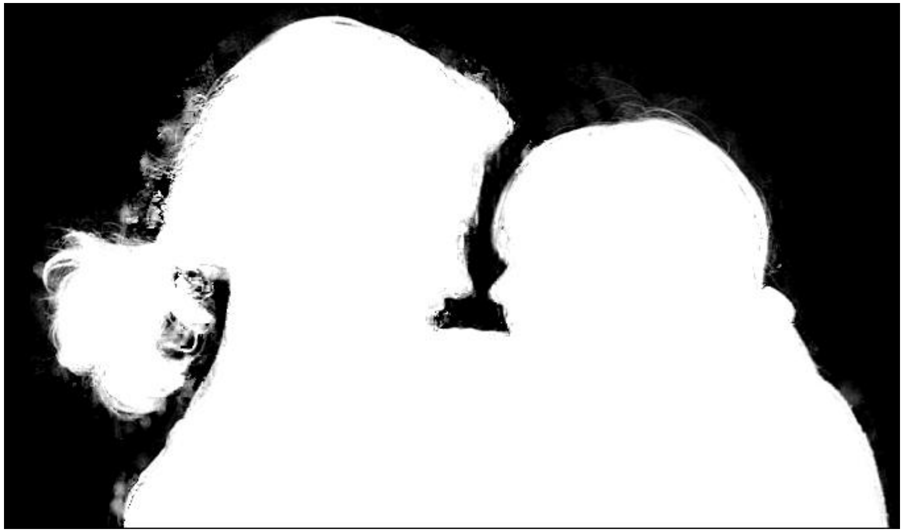
Bayesian Matting result

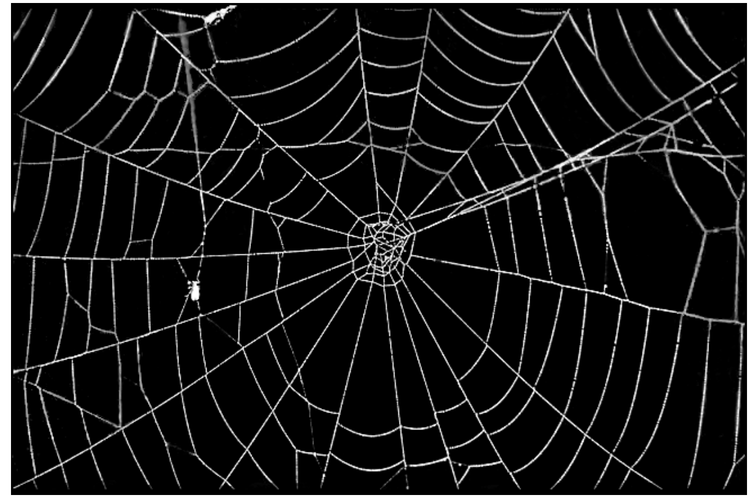


Poisson Matting result








Rambouillet

De-fogging

Video Demo

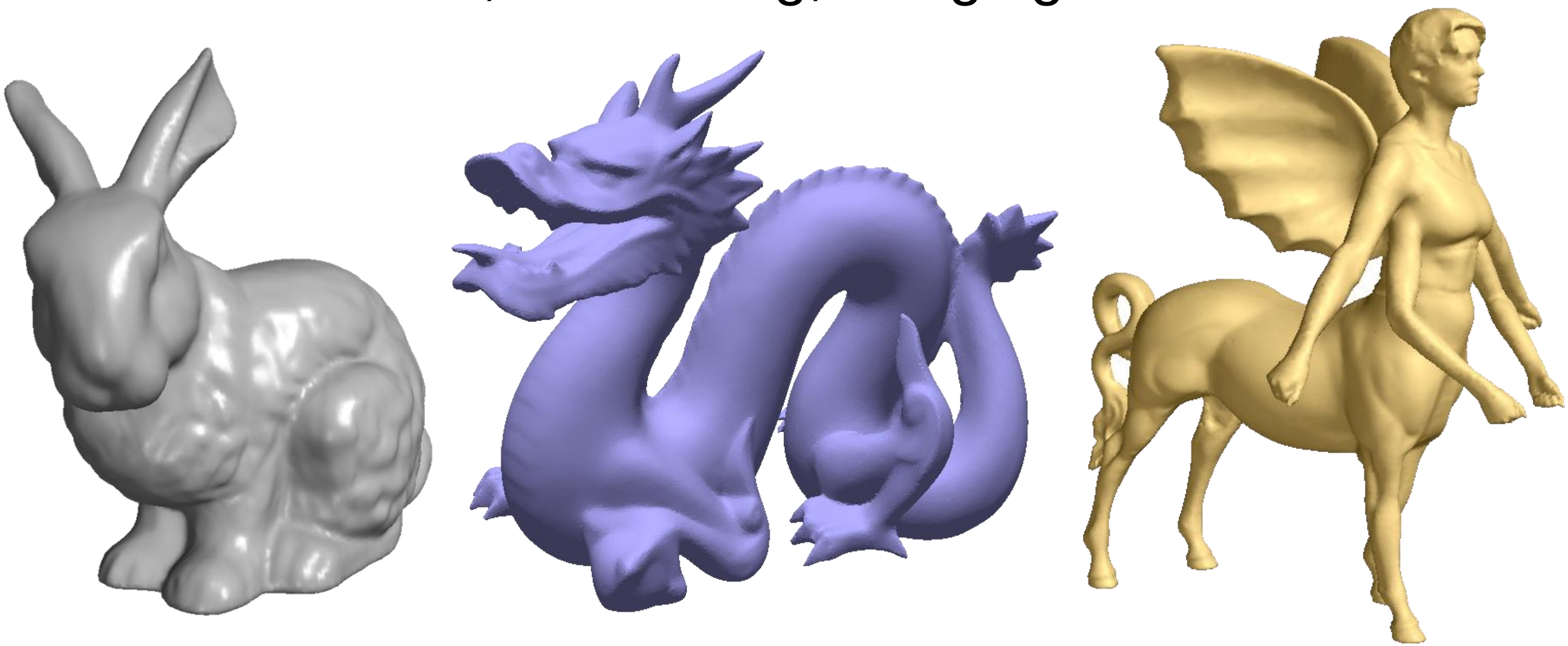
Poisson Mesh Editing

Yizhou Yu, et al.

Poisson Mesh Editing. *SIGGRAPH 2004.*

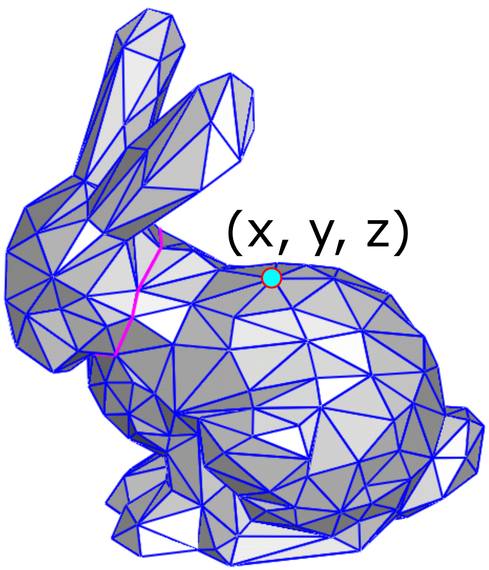
Mesh Editing

- Tools for 3D Content Creation
  - Deformation, Smoothing, Merging

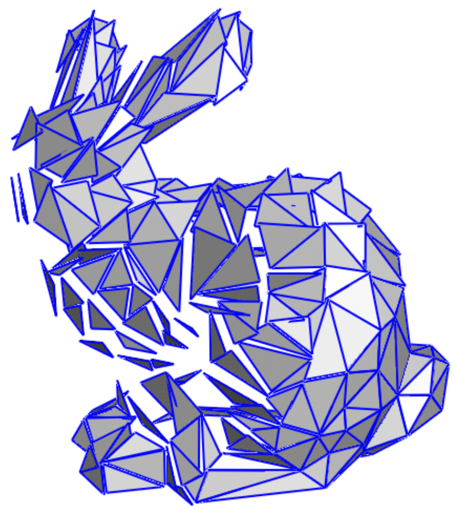


Poisson Mesh Editing

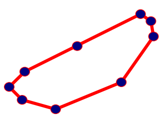
$$\Delta \boxed{f} = div\,\boxed{\mathbf{W}} \quad \text{s.t.} \quad f\,|_{\partial\Omega} = \boxed{f^{*}\,|_{\partial\Omega}}$$



Mesh Geometry



Guidance Field



Boundary Condition

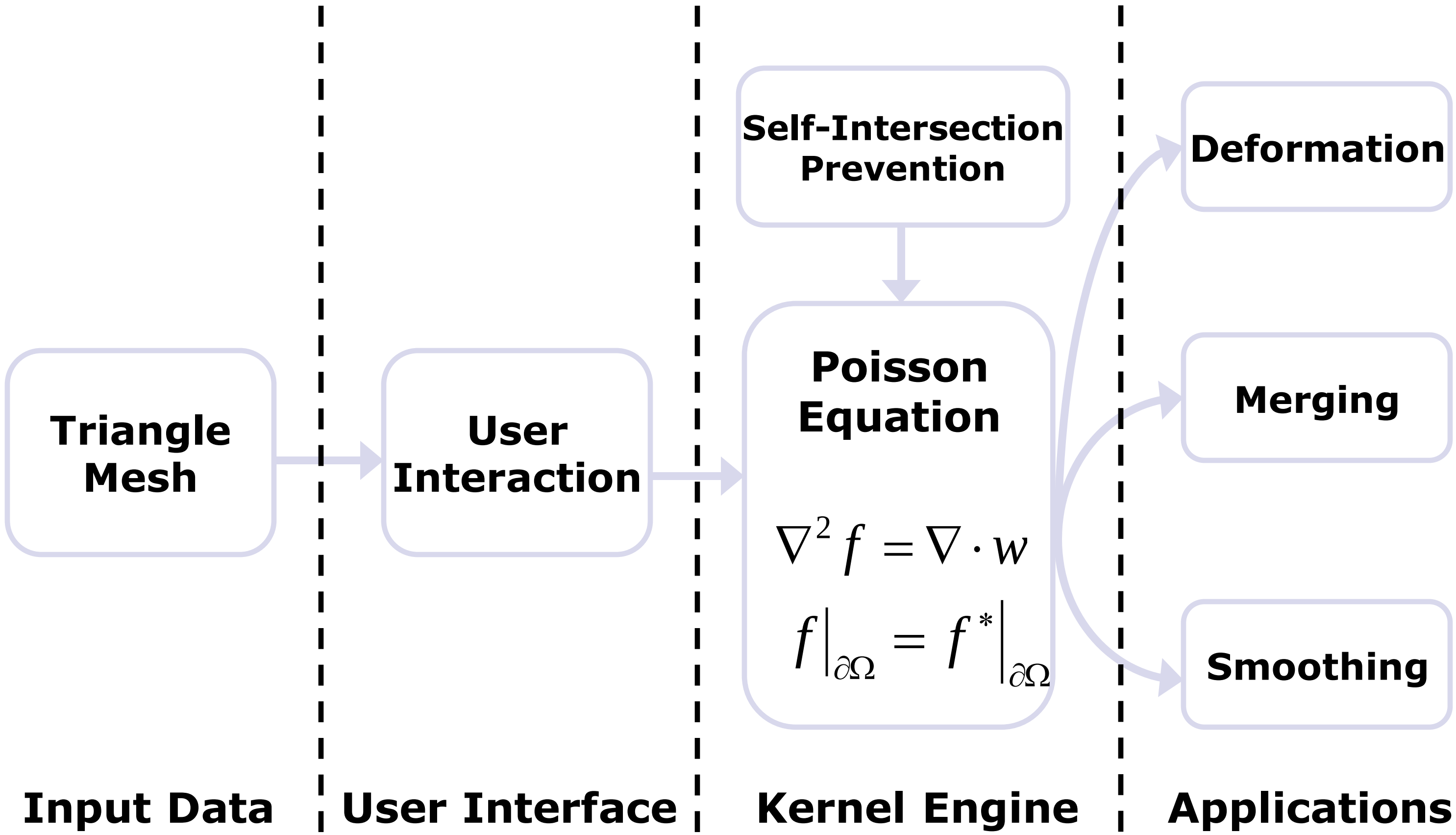
Why Poisson Mesh Editing?

- Editing mesh geometry explicitly is very tedious and impractical!
  - *Local* editing → *global* effects
  - *Avoid artifacts* by distributing errors using least-square minimization

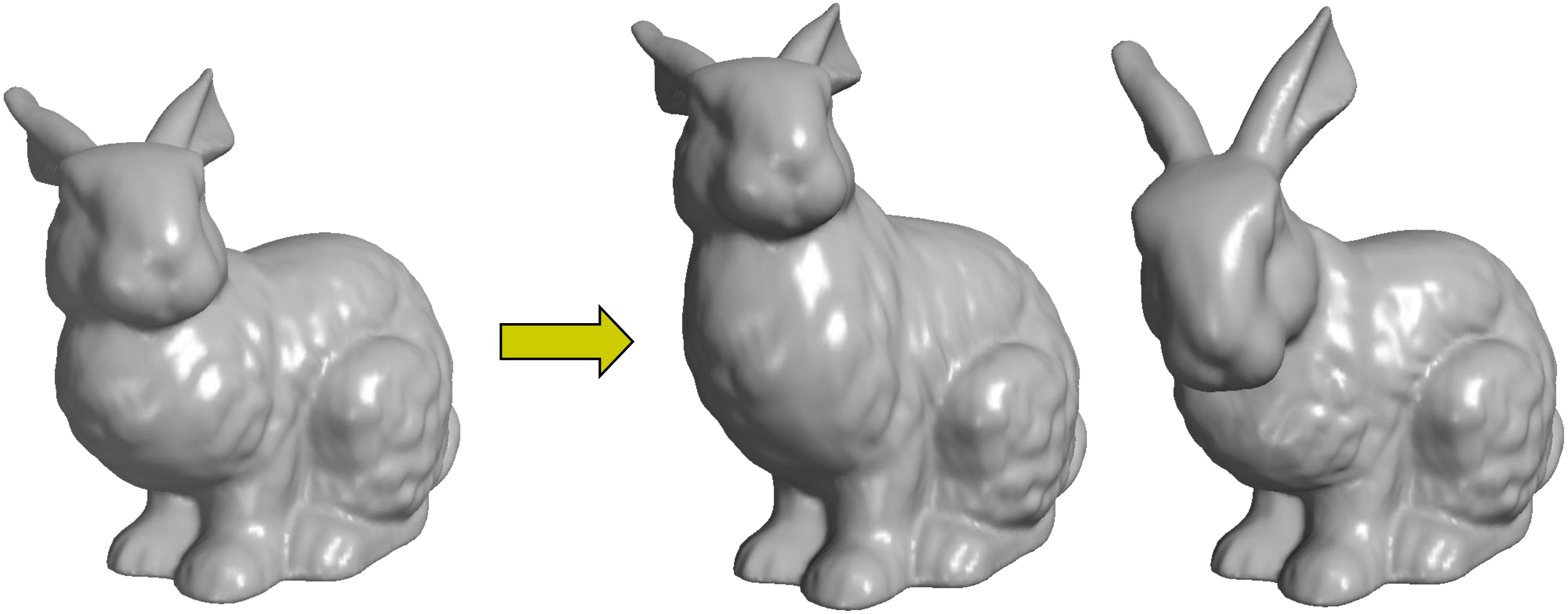
Challenges

- Apply Poisson equation to mesh
  - *Parameterization mesh*: define vertex positions as signal on 2d-manifold
  - Define discrete Laplace and divergence operators for mesh
- How to modify the guidance field and boundary conditions?
  - *Local frame propagation*

A Generic Editing Framework

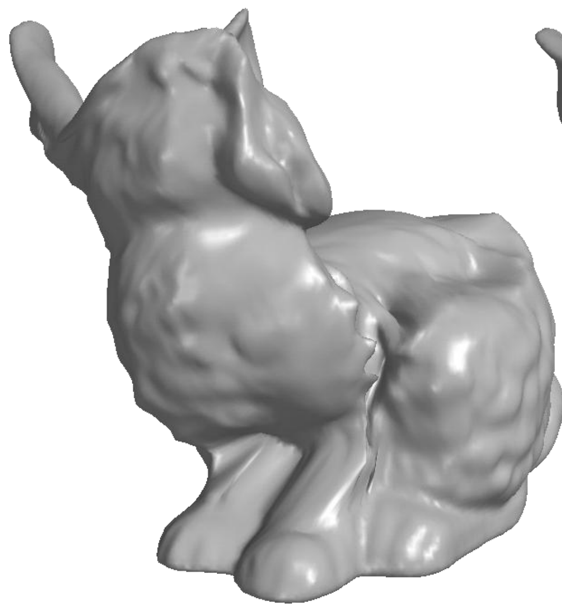


Deformation

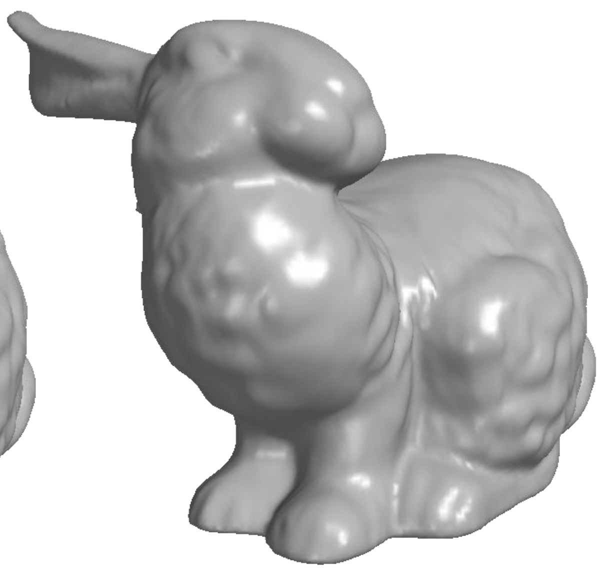


Deformation

Naïve Poisson



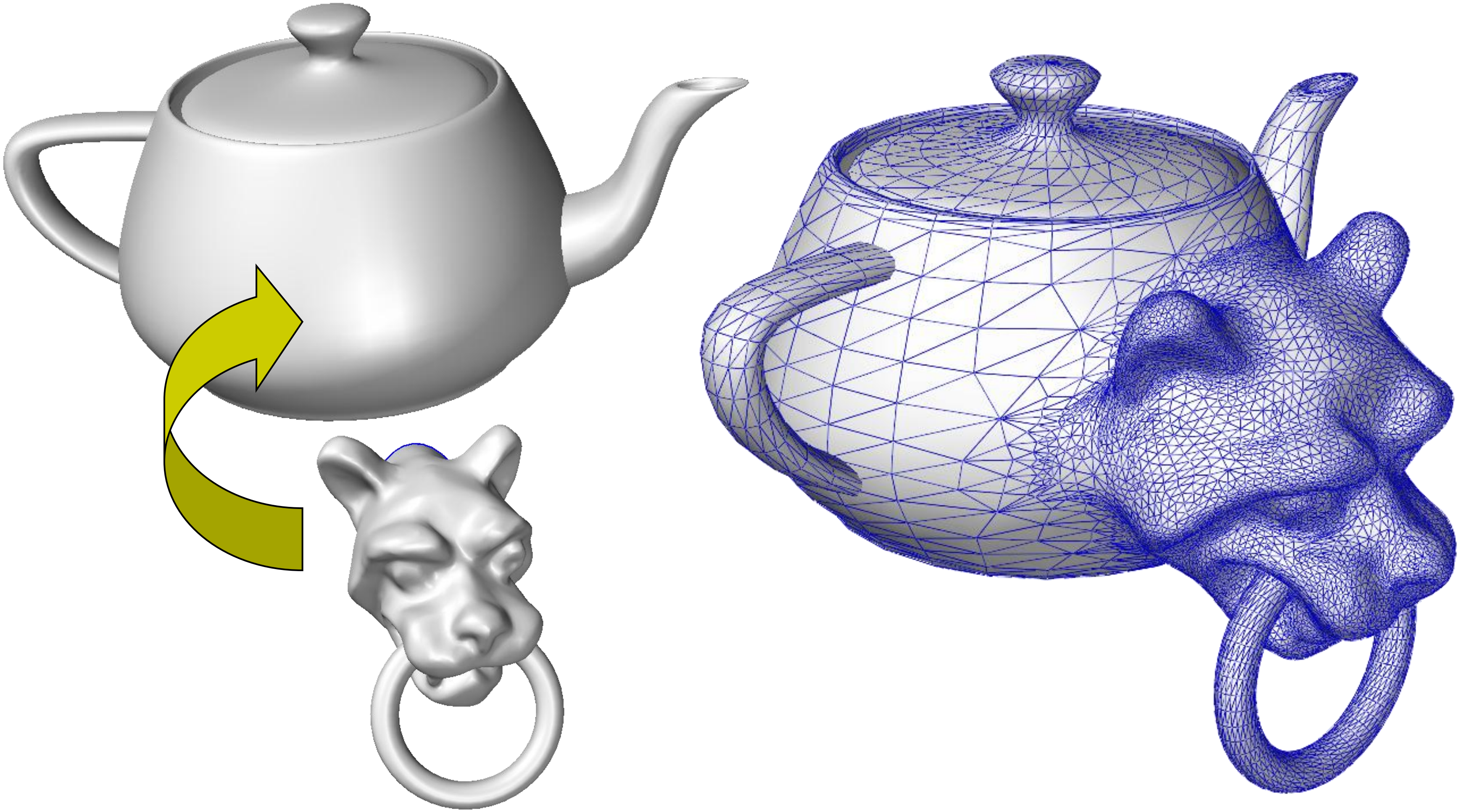
WIRE (Maya)



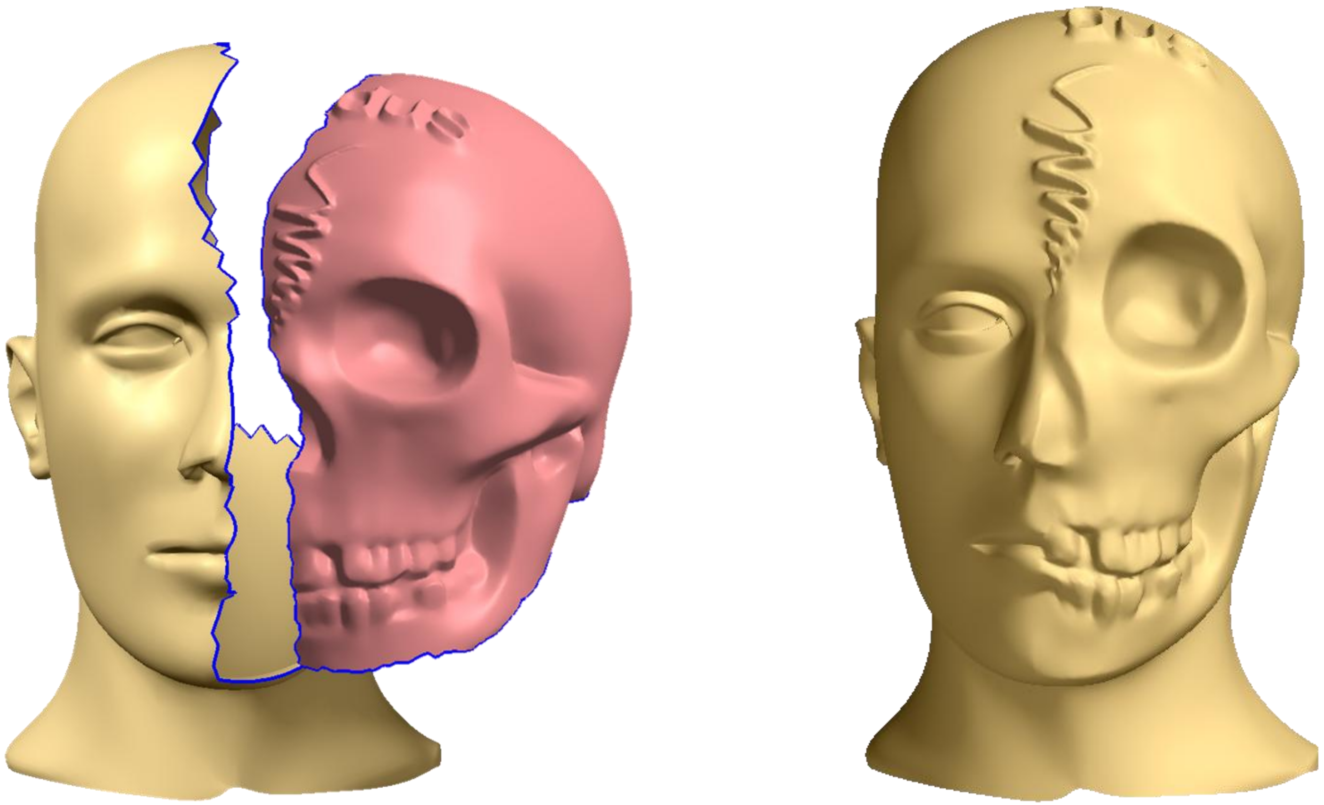
Poisson

Deformation

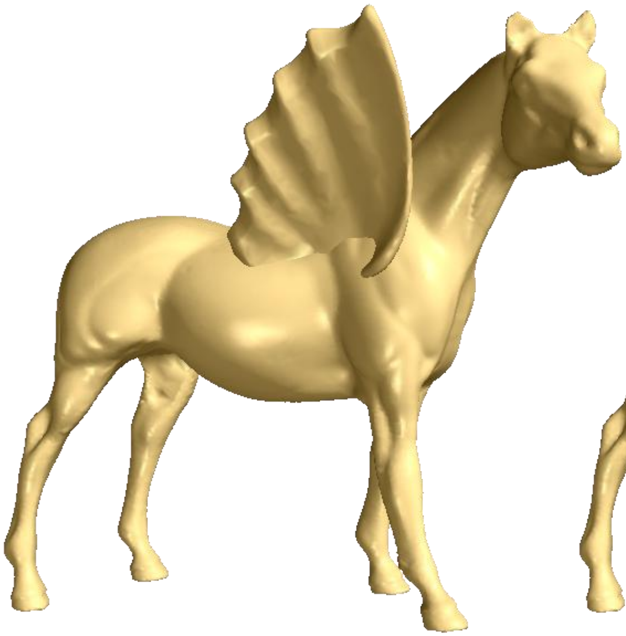
Object Merging



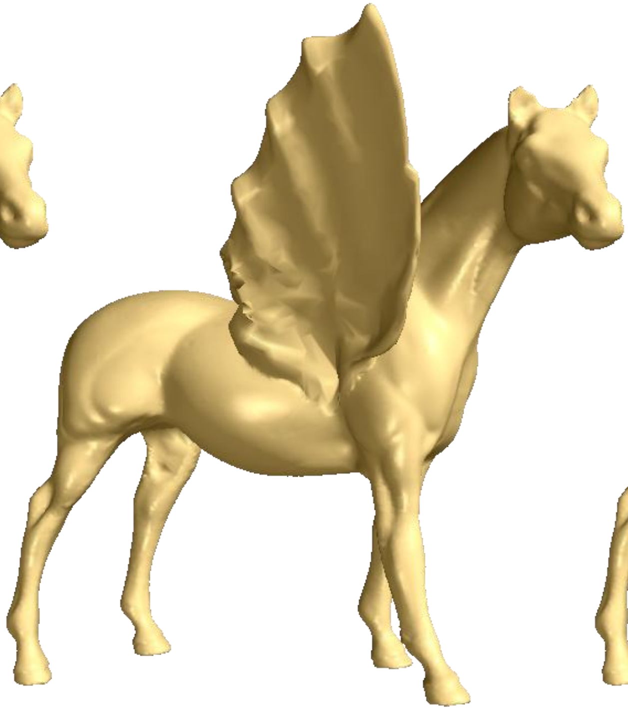
Object Merging



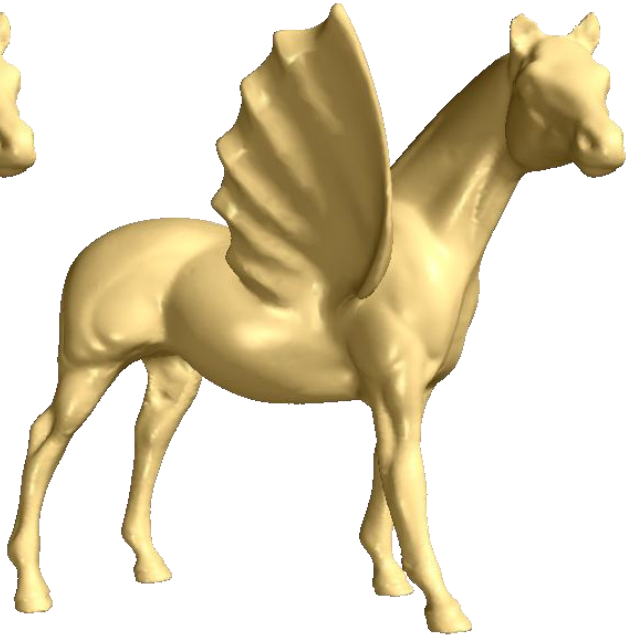
Object Merging



Boolean Operation

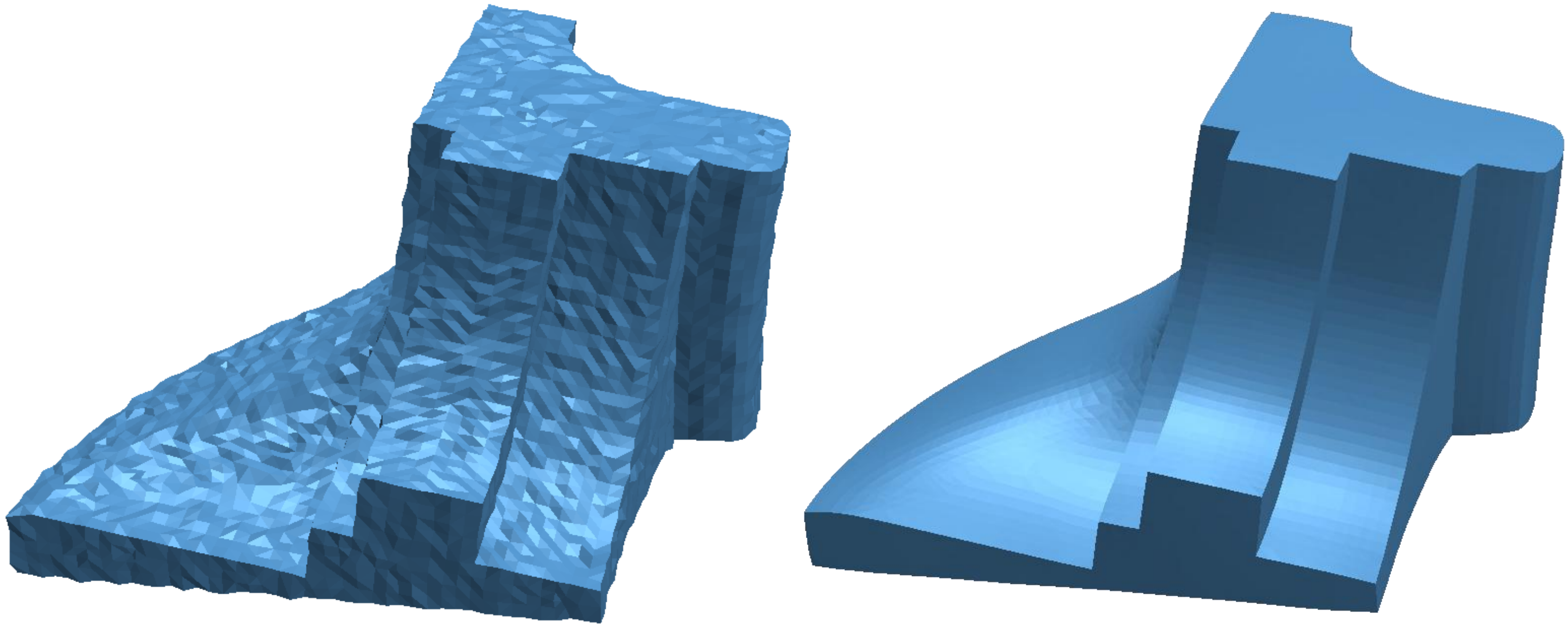


WIRE (Maya)



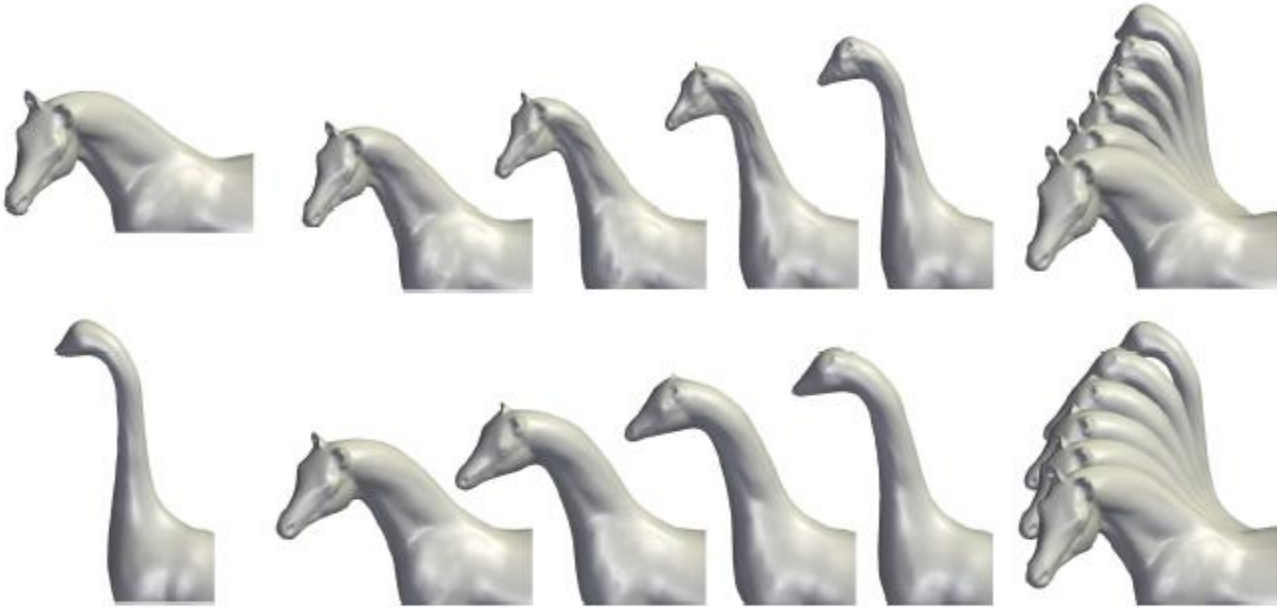
Poisson

Smoothing



Other Applications

- Mesh Morphing



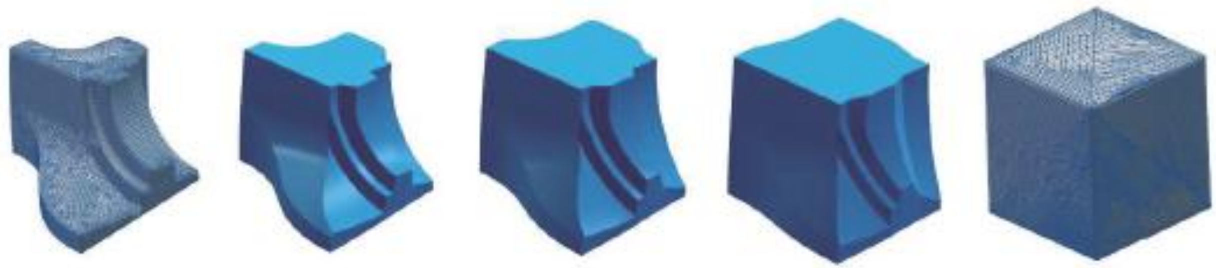


Fig. 5. Morphing between the fandisk and the cube model.

Fig. 6. Morphing between the bunny and the rabbit model.

Poisson

- Poisson equation is important
  - so is Laplacian …
- Understanding the graphics problem
  - Formulating Image editing using Poisson
- Extending to similar problems
  - Poisson mesh editing
  - From 2D image to mesh
- Creatively applying to different problems
  - Poisson matting