Point Estimation

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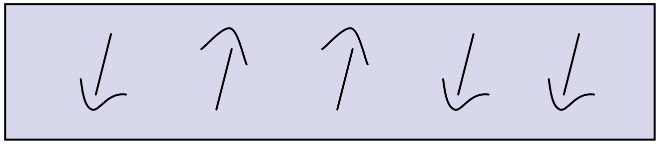
2007-03-01

What you need to know

- Point estimation:
  - **M**aximal **L**ikelihood **E**stimation (MLE)
  - Bayesian learning
  - **M**aximize **A** **P**osterior (MAP)
- Gaussian estimation
- Regression
  - Basis function = features
  - Optimizing sum squared error
  - Relationship between regression and Gaussians
- Bias-Variance trade-off

Your first consulting job

- A billionaire from Beijing asks you a question:
  - B: I have thumbtack, if I flip it, what's the probability it will fall with the nail up?
  - Y: Please flip it a few times …



  - Y: The probability is 3/5
  - B: Why???
  - Y: Because…

Binomial Distribution

- P(Heads) = θ , P(Tails) = 1- θ

$$P(D \mid \theta) = (1-\theta)\theta\theta(1-\theta)(1-\theta)$$

- Flips are i.i.d.
  - Independent events
  - Identically distributed according to Binomial distribution
- Sequence D of α_H Heads and α_T Tails

$$P(D \mid \theta) = \theta^{\alpha_H}(1-\theta)^{\alpha_T}$$

Maximum Likelihood Estimation

- **Data**: Observed set D of α_H Heads and α_T Tails
- **Hypothesis**: Binomial distribution
- Learning θ is an optimization problem
  - What's the objective function?

$$D = \{T, H, H, T, T\}$$

- MLE: Choose θ that maximizes the probability of observed data:

$$\begin{aligned} \hat{\theta} &= \arg\max_{\theta} P(D \mid \theta) \\ &= \arg\max_{\theta} \ln P(D \mid \theta) = \ldots \end{aligned}$$

Maximum Likelihood Estimation (cont.)

$$\begin{aligned}
\hat{\theta} &= \arg\max_{\theta} P(D \mid \theta) \\
&= \arg\max_{\theta} \ln(\theta^{\alpha_H} (1-\theta)^{\alpha_T}) \\
&= \arg\max_{\theta} (\alpha_H \ln\theta + \alpha_T \ln(1-\theta))
\end{aligned}$$

- Set derivative to zero:

$$\boxed{\frac{d}{d\theta} \ln P(D \mid \theta) = 0} \qquad \hat{\theta} = \frac{\alpha_T}{\alpha_H + \alpha_T} = \frac{3}{2+3}$$

How many flips do I need?

$$\hat{\theta} = \frac{\alpha_T}{\alpha_H + \alpha_T}$$

- B: I flipped 2 heads and 3 tails.
- Y: θ = 3/5, I can prove it!
- B: What if I flipped 20 heads and 30 tails?
- Y: Same answer, I can prove it!
- B: What's better?
- Y: Humm… The more the merrier???
- B: Is this why I am paying you the big bucks???

Simple bound (based on Höffding's inequality)

- For $N = \alpha_H + \alpha_T$ and $\hat{\theta} = \frac{\alpha_T}{\alpha_H + \alpha_T}$

http://omega.albany.edu:8008/machine-learning-dir/notes-dir/vc1/vc-l.html

- Let θ * be the true parameter, for any ε >0:

$$\boxed{P(\left|\hat{\theta} - \theta^*\right| \geq \varepsilon) \leq 2e^{-2N\varepsilon^2}} \leq \delta$$

$$N \geq \frac{1}{2\varepsilon^2}[\ln 2 - \ln \delta]$$

$$N \geq 270; (\varepsilon = 0.1, \delta = 0.01)$$

PAC Learning

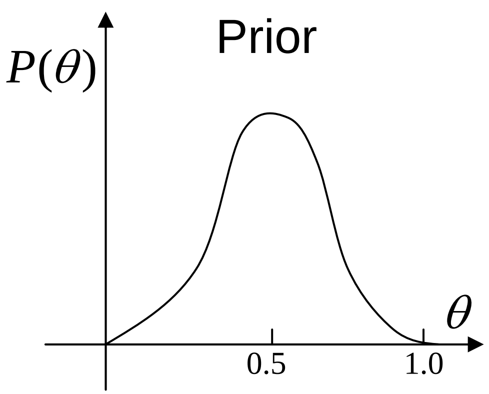
- PAC: Probably Approximate Correct
- B: I want to know the thumbtack parameter θ, within ε = 0.1, with probability at least 1- δ = 0.99. How many flips?
- Y: 270, ☺

Prior: knowledge before experiments

- B: Wait, I know that the thumbtack is “close” to 50-50. What can you …?
- Y: I can learn it the Bayesian way…

- Rather than estimating a single θ , we obtain a distribution over possible values of θ

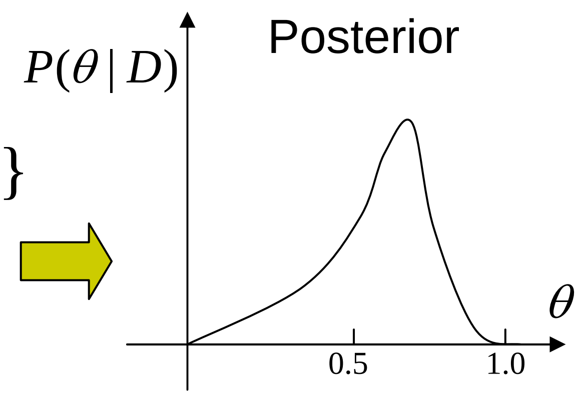
Prior



Data

$$D = \{T, H, H, T, T\}$$

Posterior



Bayesian Learning

- Bayes rule:

$$\text{Posterior} \longrightarrow \quad P(\theta \mid D) = \frac{\overset{\substack{\text{Prior} \\ \downarrow}}{P(\theta)}\,\overset{\substack{\text{Likelihood} \\ \downarrow}}{P(D \mid \theta)}}{P(D)} \quad \longleftarrow \text{Data distribution}$$

(Normalization constant)

- Or equivalently:

$$P(\theta \mid D) \propto P(\theta)P(D \mid \theta)$$

Bayesian Learning in our case

- Likelihood function is simply Binomial:

$$P(D \mid \theta) = \theta^{\alpha_H}(1-\theta)^{\alpha_T}$$

- What about prior?
  - Represent expert knowledge
  - Simple posterior form
- Conjugate priors:
  - Closed-form representation of posterior
  - For Binomial, conjugate prior is Beta distribution

Beta prior distribution – P(θ)

- Prior: Beta distribution

$$P(\theta) = \frac{\theta^{\beta_H - 1}(1-\theta)^{\beta_T - 1}}{B(\beta_H, \beta_T)} \sim Beta(\beta_H, \beta_T)$$

- Likelihood: Binomial distribution

$$P(D \mid \theta) = \theta^{\alpha_H}(1-\theta)^{\alpha_T}$$

- Posterior:

$$\begin{aligned} P(\theta \mid D) &\propto P(\theta)P(D \mid \theta) \\ &\propto \theta^{\alpha_H}(1-\theta)^{\alpha_T}\theta^{\beta_H - 1}(1-\theta)^{\beta_T - 1} \\ &\sim Beta(\alpha_H + \beta_H, \alpha_T + \beta_T) \end{aligned}$$

Using Bayesian posterior

- Posterior distribution:

$$P(\theta \mid D) \sim Beta(\alpha_H + \beta_H, \alpha_T + \beta_T)$$

- Bayesian inference:
  - No longer single parameter:

$$E[f(\theta)] \sim \int_0^1 f(\theta) P(\theta \mid D) d\theta$$

  - Integral, ☹

MAP: Maximum a posteriori approximation

$$P(\theta \mid D) \sim Beta(\alpha_H + \beta_H, \alpha_T + \beta_T)$$

$$E[f(\theta)] = \int_0^1 f(\theta) P(\theta \mid D) d\theta$$

approximation

- MAP: use most likely parameter

$$\hat{\theta} = \arg\max_{\theta} P(\theta \mid D) \qquad E[f(\theta)] \approx f(\hat{\theta})$$

MAP for Beta distribution

$$P(\theta \mid D) \sim Beta(\alpha_H + \beta_H, \alpha_T + \beta_T)$$

- MAP: use most likely parameter

$$\widehat{\theta} = \arg\max_{\theta} P(\theta \mid D) = \frac{\alpha_T + \beta_T - 1}{\alpha_H + \beta_H + \alpha_T + \beta_T - 2}$$

- Beta prior equivalent to extra thumbtack flips
- As $N = \alpha_T + \alpha_H \to \infty$, prior is “forgotten”
- But, for small sample size, prior is important!

Gaussian distribution

Continuous variable:

mean

$$P(x \mid \mu, \delta) \sim \frac{1}{\boxed{\delta}\sqrt{2\pi}} e^{-\frac{(x-\boxed{\mu})^2}{2\delta^2}}$$

variance

Normalize item

Consider the difference between continuous and discrete variables?

MLE for Gaussian

- Prob. of i.i.d. samples $D = \{x_1, x_2, \ldots, x_N\}$

likelihood $$P(D \mid \mu, \sigma) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^N \prod_{i=1}^{N} e^{\frac{-(x_i-\mu)^2}{2\sigma^2}}$$

- The magic of log (to likelihood)

$$\begin{aligned} \ln P(D \mid \mu, \sigma) &= \ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^N \prod_{i=1}^{N} e^{\frac{-(x_i-\mu)^2}{2\sigma^2}} \\ &= -N \ln(\sigma\sqrt{2\pi}) - \sum_{i=1}^{N} \frac{(x_i-\mu)^2}{2\sigma^2} \end{aligned}$$

MLE for mean of a Gaussian

$$
\begin{aligned}
\frac{\partial}{\partial\mu}\ln P(D\,|\,\mu,\sigma) &= \frac{\partial}{\partial\mu}\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{N}\prod_{i=1}^{N}e^{\frac{-(x_i-\mu)^2}{2\sigma^2}} \\
&= \frac{\partial}{\partial\mu}-\sum_{i=1}^{N}\frac{(x_i-\mu)^2}{2\sigma^2} \\
&= \sum_{i=1}^{N}\frac{(x_i-\mu)}{\sigma^2} \quad = 0
\end{aligned}
$$

$$
\mu = \frac{1}{N}\sum_{i} x_i
$$

MLE for variance of a Gaussian

$$\frac{\partial}{\partial\sigma}\ln P(D\mid\mu,\sigma) = \frac{\partial}{\partial\sigma}\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^N \prod_{i=1}^{N} e^{\frac{-(x_i-\mu)^2}{2\sigma^2}}$$

$$= \frac{\partial}{\partial\sigma}[-N\ln\sigma\sqrt{2\pi}] - \sum_{i=1}^{N}\frac{\partial}{\partial\sigma}[\frac{(x_i-\mu)^2}{2\sigma^2}]$$

$$= -\frac{N}{\sigma} + \sum_{i=1}^{N}\frac{(x-\mu)^2}{\sigma^3} = 0$$

$$\sigma^2 = \frac{1}{N}\sum_i (x_i-\mu)^2$$

Gaussian parameters learning

- MLE

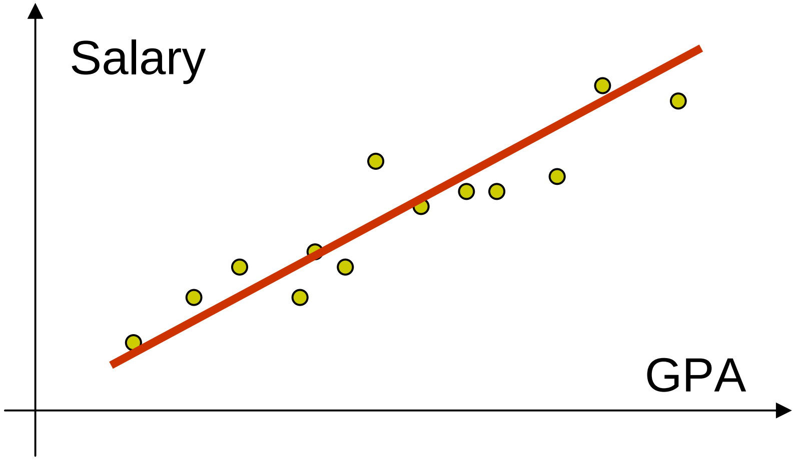
$$\hat{\mu} = \frac{1}{N}\sum_i x_i$$

$$\hat{\sigma}^2 = \frac{1}{N}\sum_i (x_i - \mu)^2$$

- Bayesian learning: prior?
- Conjugate priors:
  - Mean: Gaussian priors
  - Variance: Wishart Distribution

Prediction of continuous variable

- B: Wait, that's not what I meant!
- Y: Chill out, dude.
- B: I want to predict a continuous variable for continuous inputs: I want to predict salaries from GPA.
- Y: I can regress that…



The regression problem

- **Instances:** $<\mathbf{x}_i, t_i>$
- **Learn:** mapping from $\mathbf{x}$ to $t(\mathbf{x})$.
- **Hypothesis space:** $t(\mathbf{x}) \approx \hat{f}(x) = \sum_{i=1}^{k} w_i h_i$
  - Given, basis functions $H = \{h_1, \ldots, h_k\}$
  - Find coefficients $\mathbf{w} = \{w_1, \ldots, w_k\}$
- **Problem formulation:**

$$\mathbf{w}^* = \arg\min_{\mathbf{w}} \sum_j [t(\mathbf{x}_j) - \sum_{i=1}^{k} w_i h_i(x)]^2$$

But, why sum squared error?

- Model:

$$P(t \mid \mathbf{x}, \mathbf{w}, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-[t - \sum_i w_i h_i(x)]^2}{2\sigma^2}}$$

- Learn **w** using MLE

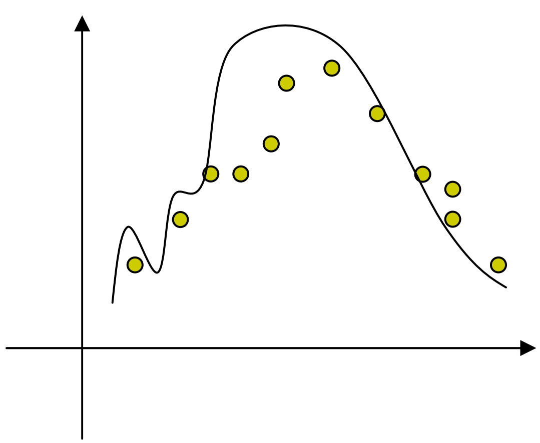
Maximizing log-likelihood

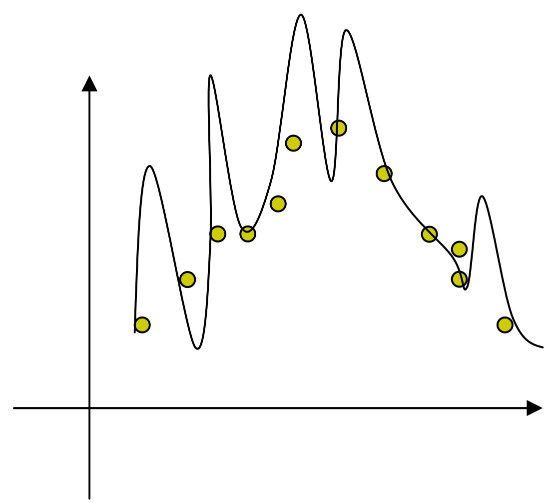
$$\ln P(D \mid \mathbf{w}, \sigma) = \ln \prod_j \left(\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-[t_j - \sum_i w_i h_i(x_j)]^2}{2\sigma^2}} \right)$$

$$\Rightarrow \min \sum_j \frac{-[t_j - \sum_i w_i h_i(x_j)]^2}{2\sigma^2}$$

Bias-Variance Tradeoff

- Choice of hypothesis basis introduce learning bias:
  - More complex basis:
    - Less bias
    - More variance (over-fitting)





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Homework

- Finish the “Gaussian parameters learning”
  - Please use google, ^_*