Implicit Surface Modeling

Hongxin Zhang and Jieqing Feng

2007-01-04

State Key Lab of CAD&CG
Zhejiang University

Contents

- Introduction of Implicit Surface Modeling
- Implicit Surface Modeling for Computer Graphics and Animation: Field Function Defined Implicit Surface
- Quadric Surfaces
- Download Course

Introduction of Implicit Surface Modeling

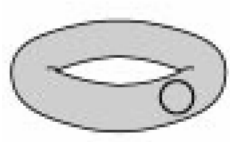
- What is implicit surface?
- Comparison: implicit v.s. parametric surfaces
- Implicit methods for graphics and animation
- Implicit methods for CAD/CAM
- Other topics in implicit modeling

What is implicit surface?

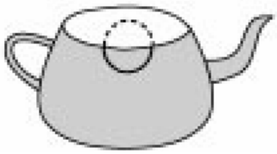
- Implicit surfaces are two-dimensional, geometric shapes that exist in three dimensional space.
  - Defined in $\mathbf{R}^3$:
  - 2-D manifold:
    - A surface embedded in $\mathbf{R}^3$
    - Infinitesimal neighborhood around any point on the surface is topologically equivalent (‘ locally diffeomorphic’) to a disk.

What is implicit surface?

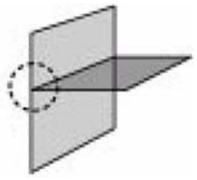
- 2D manifold can be
  - bounded (or closed): sphere
  - Unbounded: plane
- A manifold with boundary: topologically equivalent to either a disk or a half-disk locally.
- Nonmanifold



Manifold



Manifold with boundary



Nonmanifold

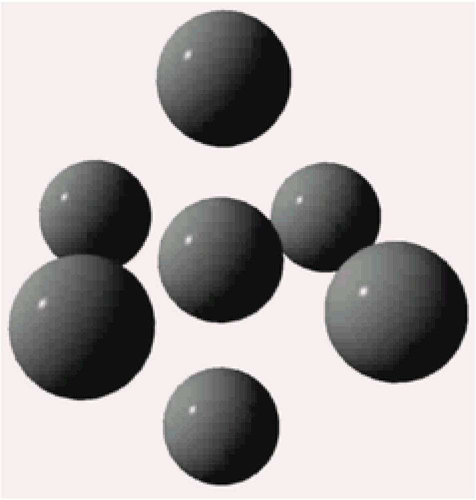
What is implicit surface?

- Manifold or not? Eule-Poincaré formula

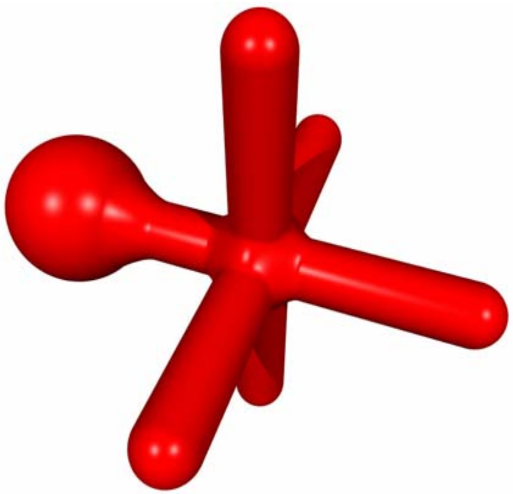
$$v-e+f=2-2h$$

  - v: number of vertices
  - e: number of edges
  - f: number of faces
  - h: number of holes

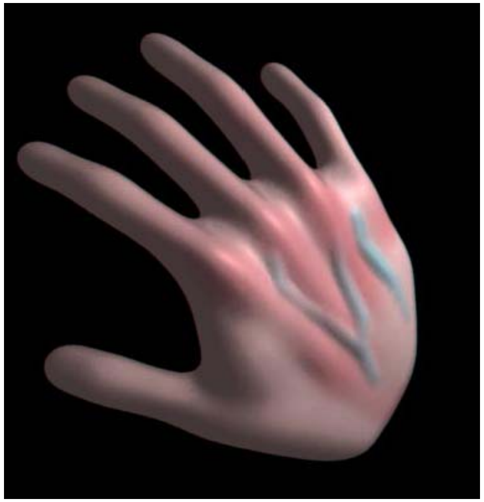
Examples of implicit surfaces



Metaball

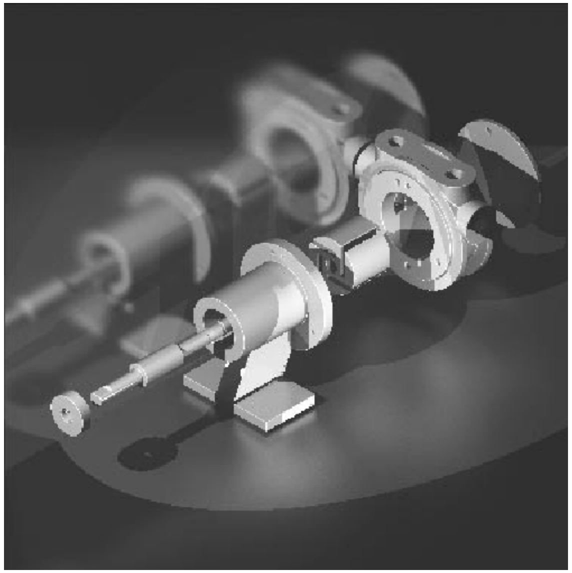


Skeleton Surface and Metaball

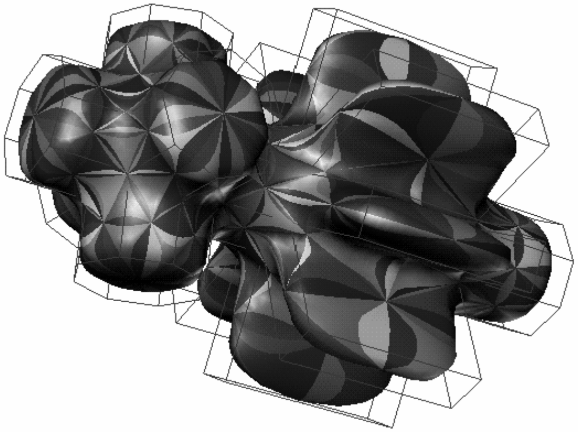


Convolution Surface

Examples of implicit surfaces



Quadric Surface



A-Patch

Compactly Supported Radial Basis Functions

Definition of implicit surface

- Definition

$$\{p=(x,y,z): f(p)=0,\ p\in\mathbf{R}^3\}$$

  - Implicit function f : three classes of specification (definition)
    - Discrete sampling: a set of points on or within an object
    - Mathematical function: one or more equations are used to compute the coordinates of points on or within an object
    - Procedural methods: an algorithmic process computes points on or within an object

Algebraic Function

- Algebraic surface: when f is algebraic function, i.e., polynomial function
  - The coefficients are not unique

$$f=ax+by+cz=0$$

    - $a=b=c=1$
    - $a=b=c=1/\sqrt{3}$
  - Algebraic distance: The value of $f(p)$ is the approximation of distance from p to the algebraic surface $f=0$

Transcendental Function

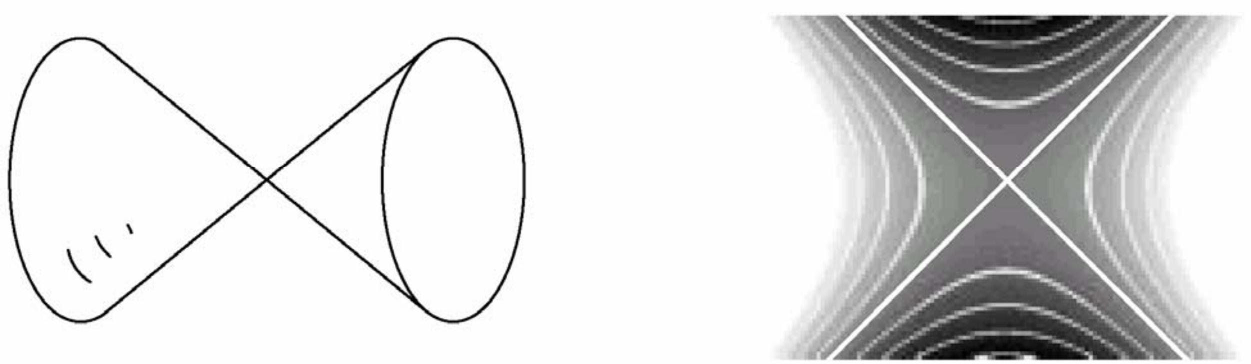
- Transcendental Function
  - Trigonometric, exponential, logarithmic, hyperbolic functions, etc.
  - Approximated by convergent power series: Taylor series

Definition of implicit surface

- Regular point p on the surface

$$\nabla f(p) = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z) \neq \mathbf{0}$$

  - For cone, 0 is not the regular value for f



The cone $f=-x^2+y^2+z^2$ is regular with the exception of a singularity at the origin $(0,0,0)$.

Mathematical Foundation

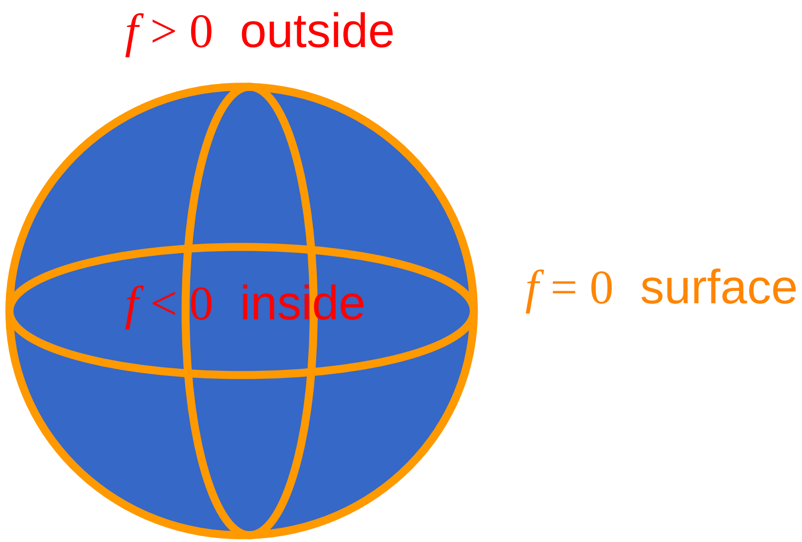
- Implicit Function Theorem:

  if 0 a regular value of continuous function $f(p)$, the implicit surface $f^{-1}(0)$ is a two-dimensional manifold

- Jordan-Brouwer Separation Theorem

  A $2D$ manifold separates $\mathbf{R}^3$ into surface itself and two connected open sets: an infinite(finite) `outside' and a finite(infinite) ‘inside'.

Mathematical Foundation

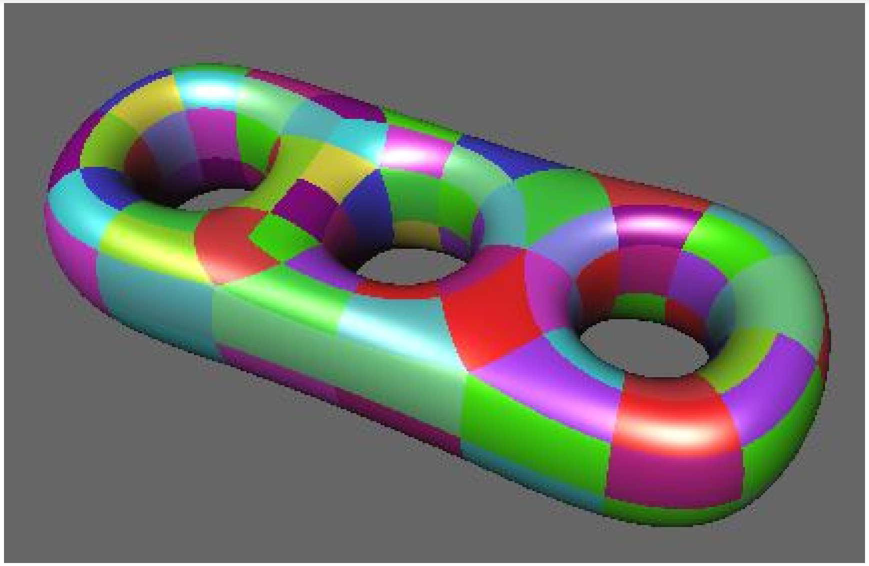


$$f(x,y,z) - 1 = x^2+y^2+z^2-1 \qquad f(x,y,z) = x^2+y^2+z^2$$

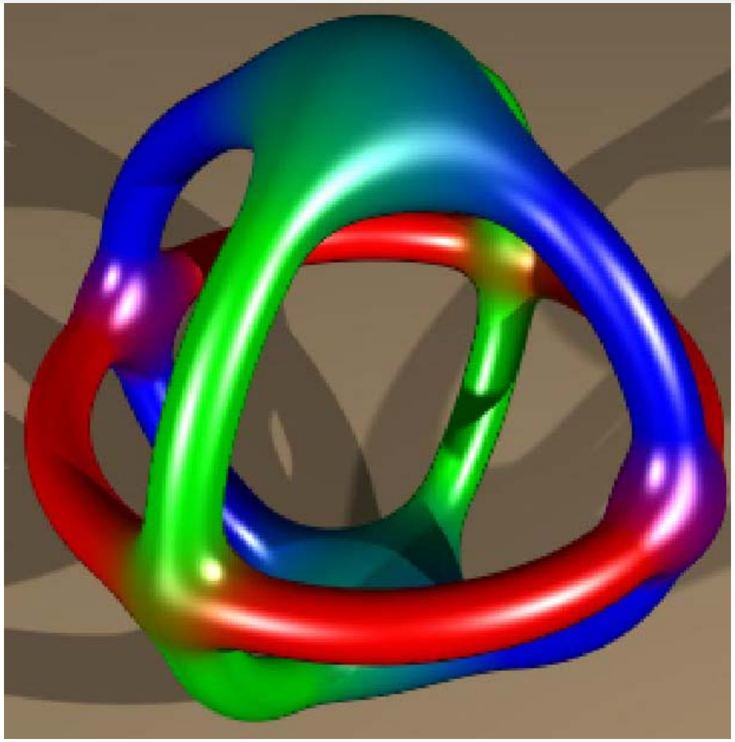
Implicit v.s. Parametric Surfaces

Disadvantages		
Explicit	Implicit	Parametric
• Infinite slopes are impossible if $f(x)$ is a polynomial. • Axis dependent (difficult to transform). • Closed and multivalued curves are difficult to represent.	• Difficult to fit and manipulate free form shapes. • Axis dependent. • Complex to trace.	• High flexibility complicates intersections and point classification.
Advantages		
Explicit	Implicit	Parametric
• Easy to trace.	• Closed and multivalued curves and infinite slopes can be represented. • Point classification (solid modeling, interference check) is easy. • Intersections/offsets can be represented.	• Closed and multivalued curves and infinite slopes can be represented. • Axis independent (easy to transform). • Easy to generate composite curves. • Easy to trace. • Easy in fitting and manipulating free-form shapes.

Complex Surface Modeling

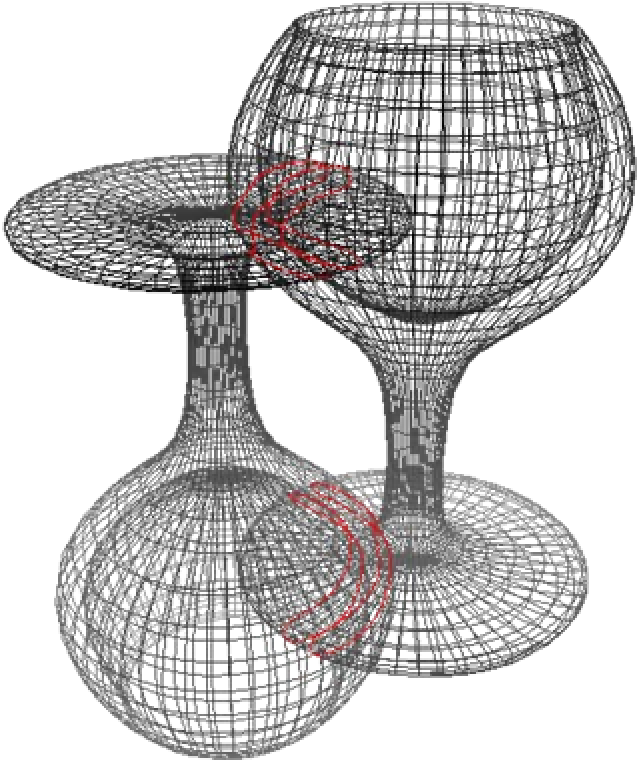


Complex surface constructed by smoothly joined B-spline surface patches

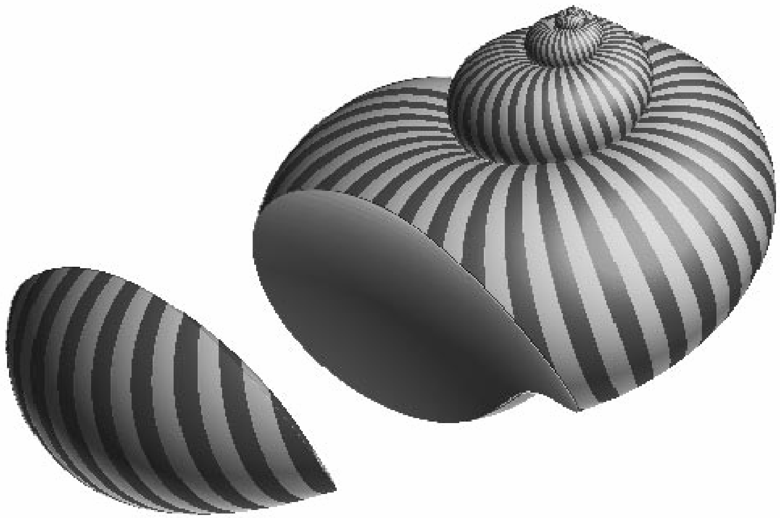


Complex surface constructed by convolution surface method

Surfaces Intersection

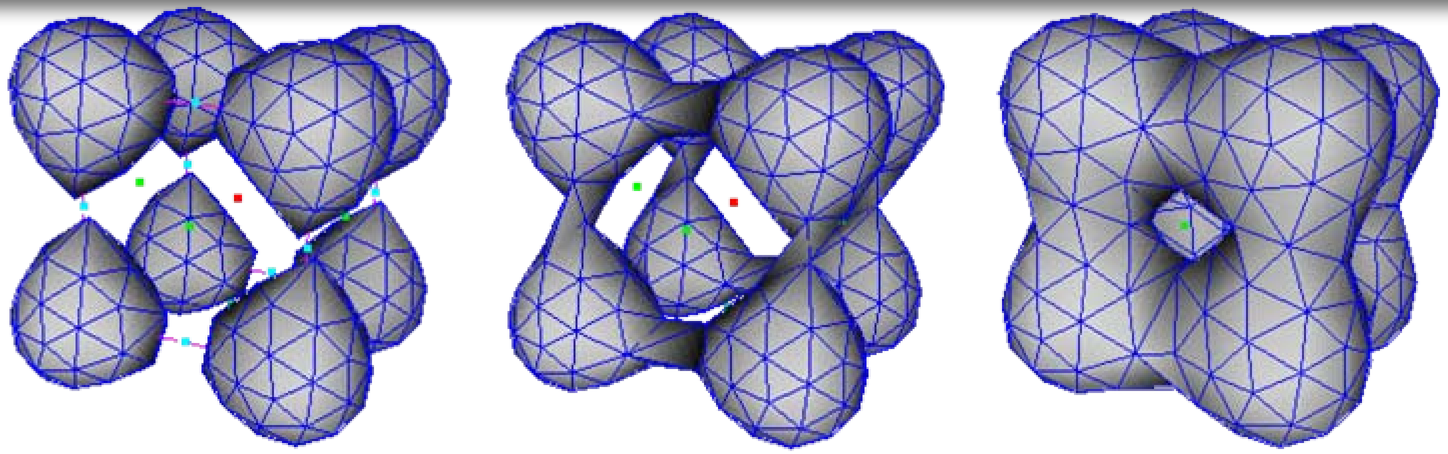


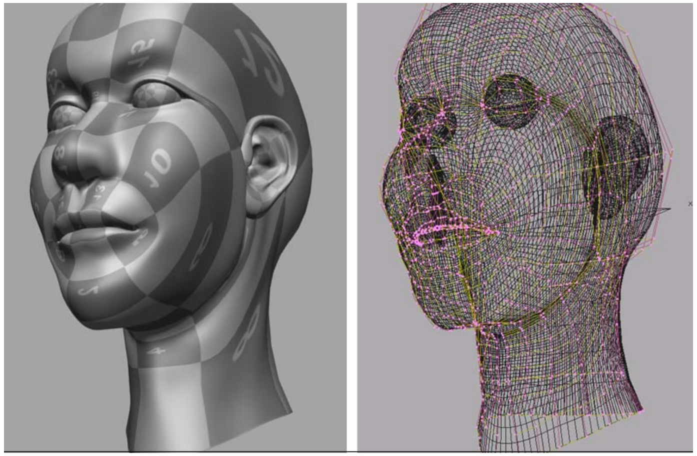
Parametric Surface Intersection



Implicit Surface Intersection

Tessellation

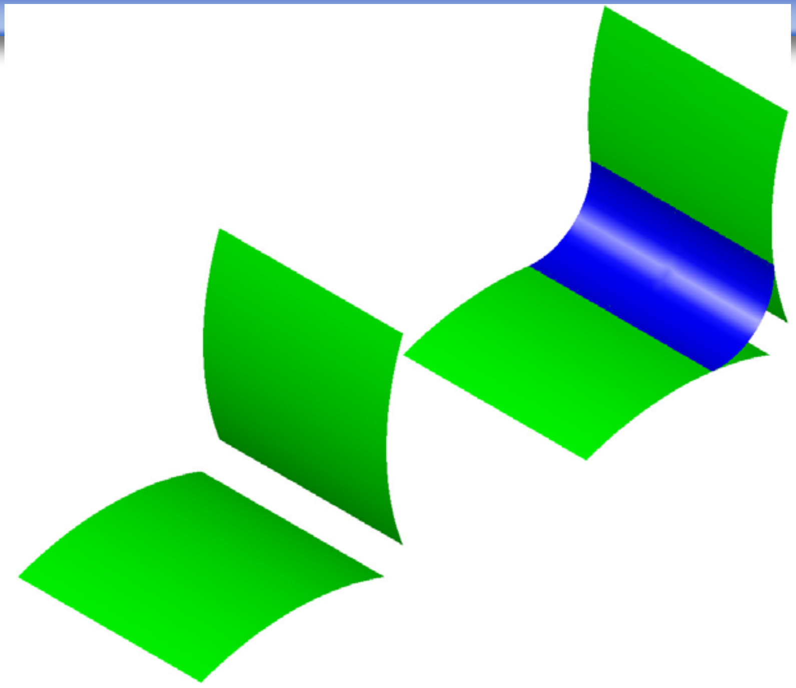


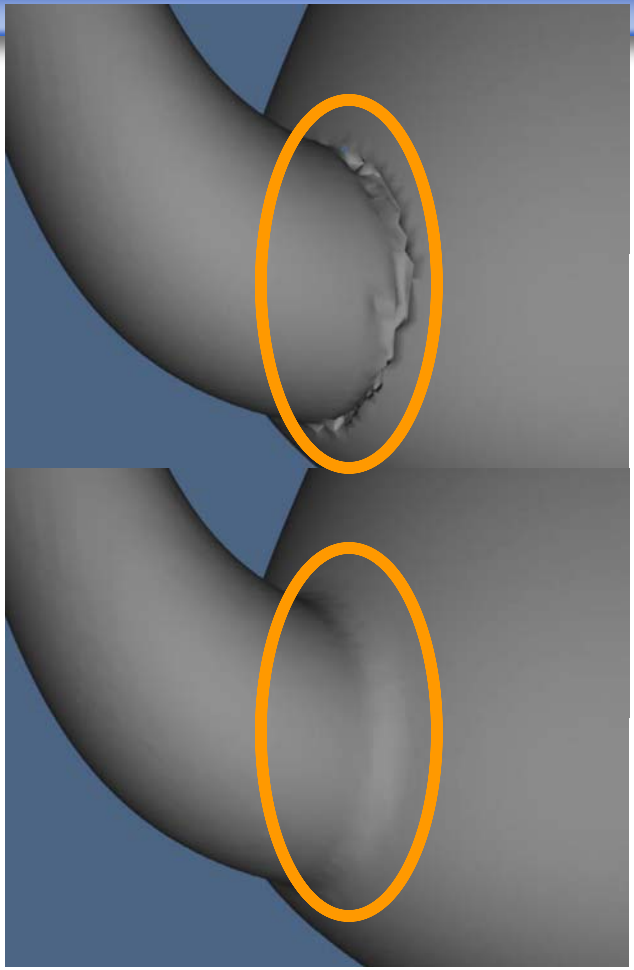


Up: tessellation of implicit surface

Bottom: tessellation of NURBS surface

Blending or rounding

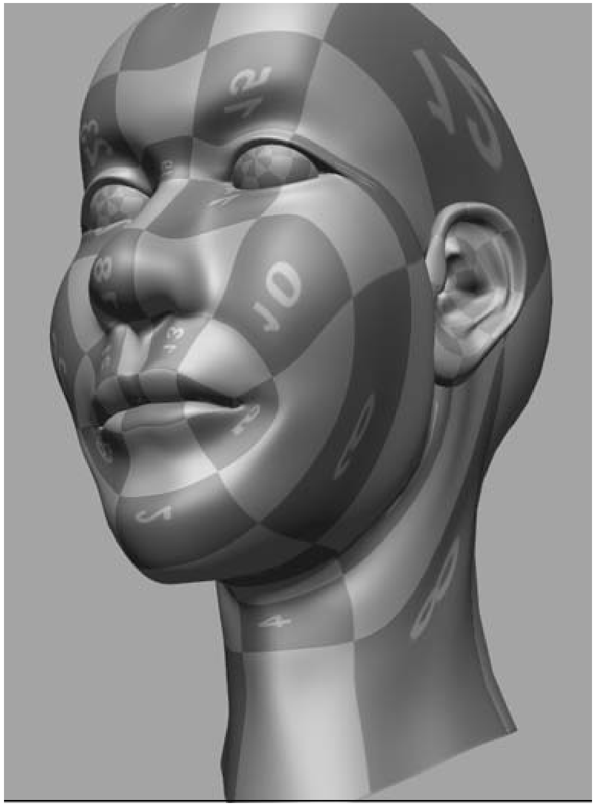




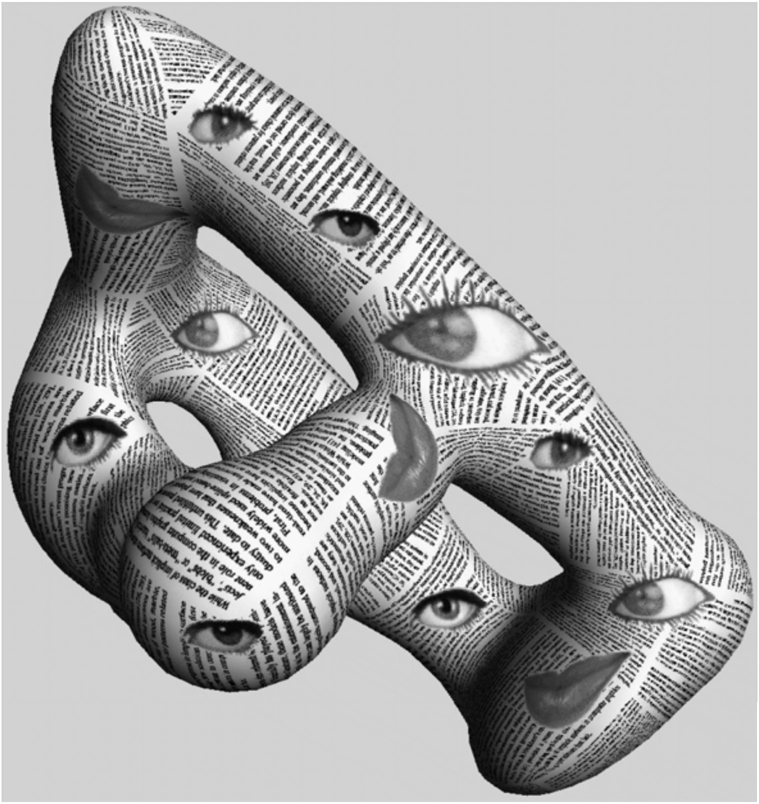
Left: Blending by using parametric surface

Right: Blending by using implicit surface

Texture mapping



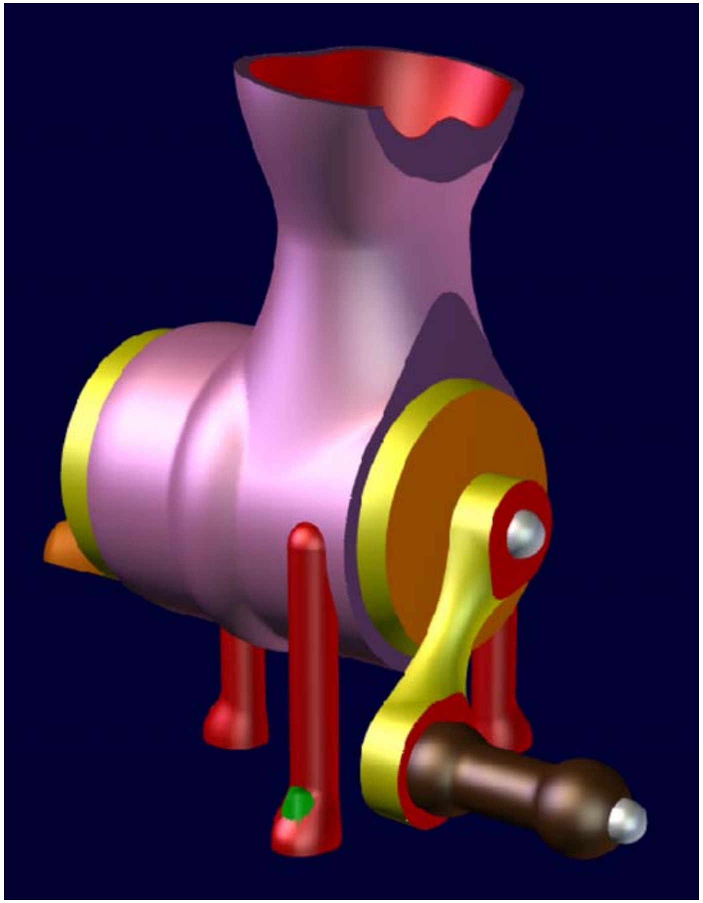
Texture mapping for NURBS surface



Texture mapping for implicit surface

Solid Modeling

Solid Modeling via NURBS Surface



Solid Modeling via Implicit Surface

Implicit v.s. Parametric Surfaces

- Implicitization:

Parametric representation	→	Implicit representation

- Parameterization

Implicit representation	→	Parametric representation

About conversions

- Implicitization
  - Any rational parametric surface can be implictized by elimination of parameters in the parametric form
  - Implicitization is often computational demanding
  - The degree of implicit form is higher than its parametric form, the implicit representation of
    - A parametric triangular patch of degree n is degree n^2
    - A tensor product surface of degree m by n is degree $2mn$
    - The number of terms in algebraic surface of degree n is $3C_2^n$ (bicubic patch is degree 18, with 1330 terms!)
    - Approximated implicitization

About conversions

- Parameterization: not always possible
  - Algebraic surfaces of fourth and higher degree cannot be parameterized by rational functions
  - Parameterization is always possible for non-degenerate quadrics and for cubics that have a singular point.

Example:sphere

- Trigonometric:

$$f(\alpha,\beta)=(\cos\alpha\cos\beta,\ \cos\alpha\sin\beta,\ \sin\alpha)$$

$$\alpha\in[0,\pi],\ \beta\in[0,2\pi]$$

- Rational:

$$x=4st,\ y=2t(1-s^2),\ z=(1-t^2)(1+s^2),\ w=(1+t^2)(1+s^2)$$

- Implicit:

$$f(x,y,z)=x^2+y^2+z^2-1$$

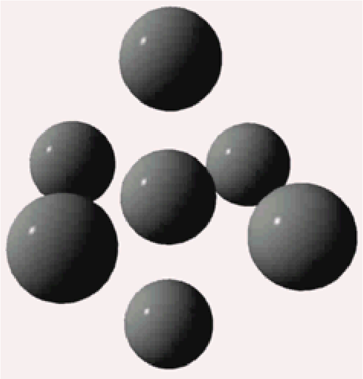
Example: sphere

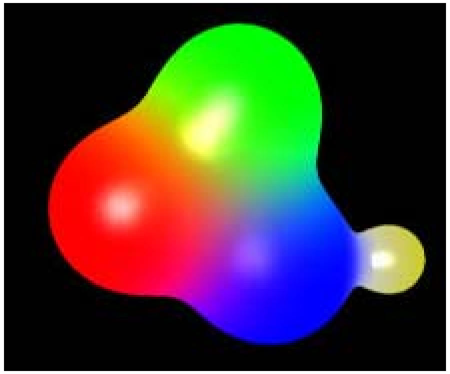
- Points on the parametrically defined sphere are readily found by substitution of α and β into the equations for x, y, and z (similarly for s and t).
- The mapping from parametric space to geometric space is convenient.
- There is no obvious mapping for implicit form!

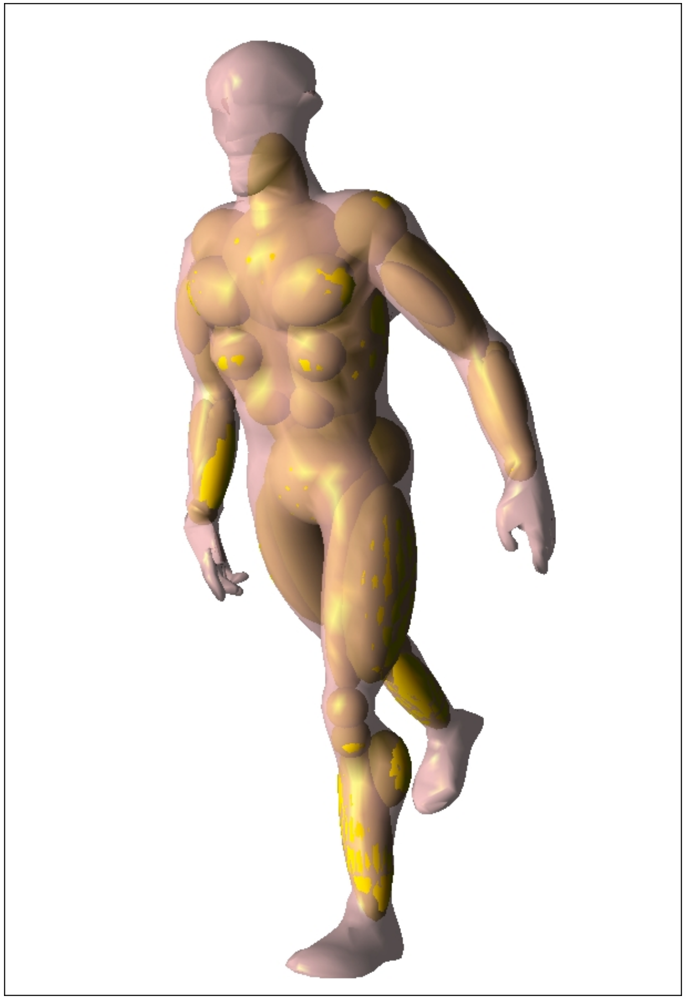
Implicit methods for graphics and animation

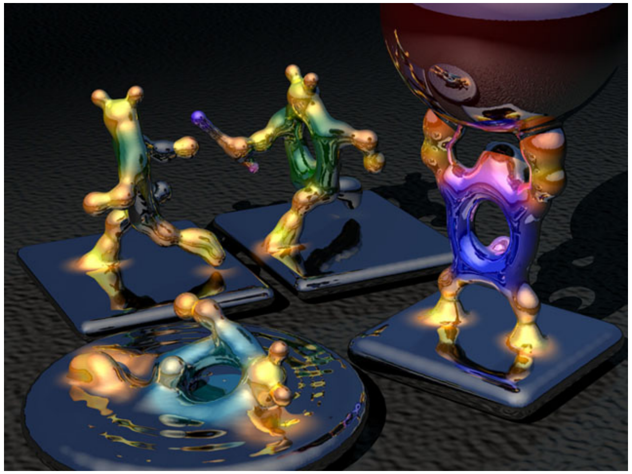
- Blobby (metaball, soft objects)
- Implicit surface defined by skeletons
  - Distance surface
  - Convolution surface
- Variational Implicit Surfaces
- Level-Set Methods
- Procedural Models
- Animation applications

Blobby (metaball, soft objects)

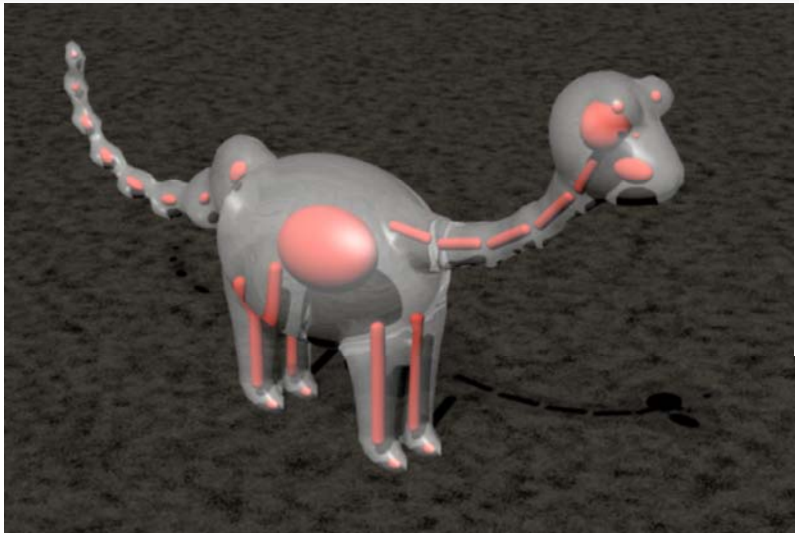


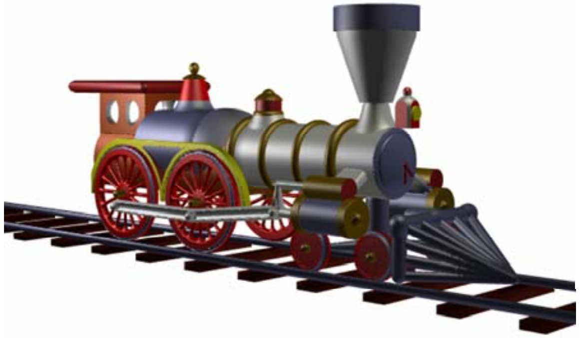




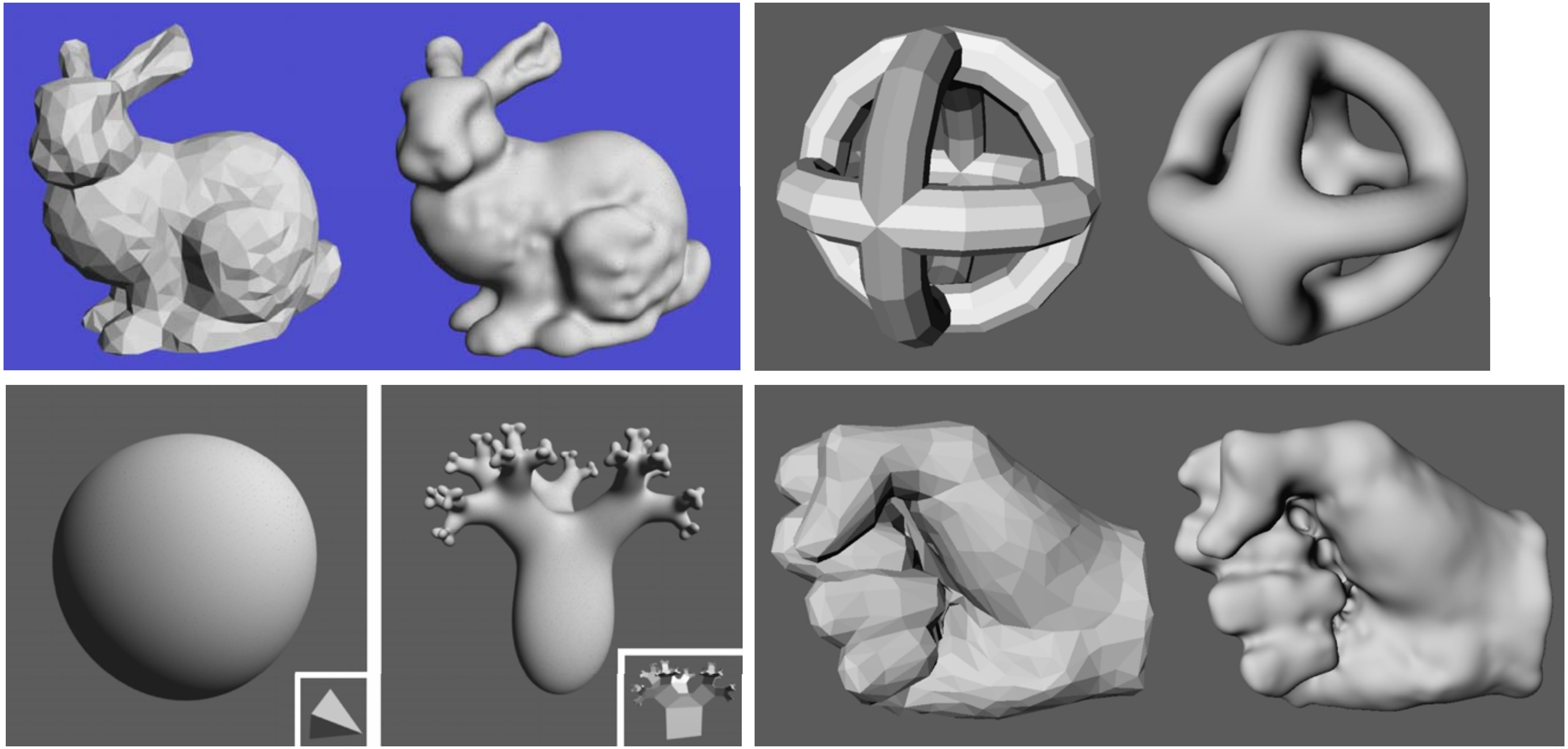


Implicit surface defined by skeletons

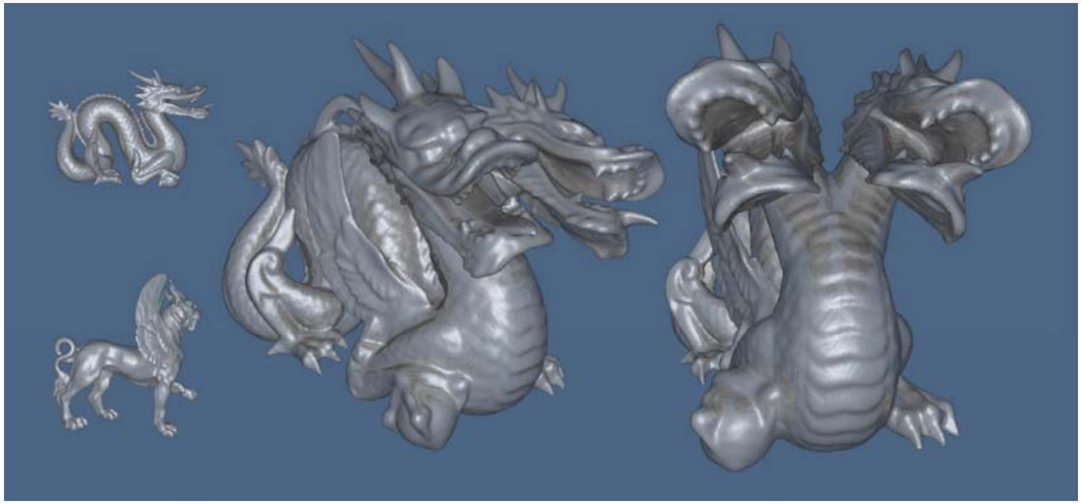


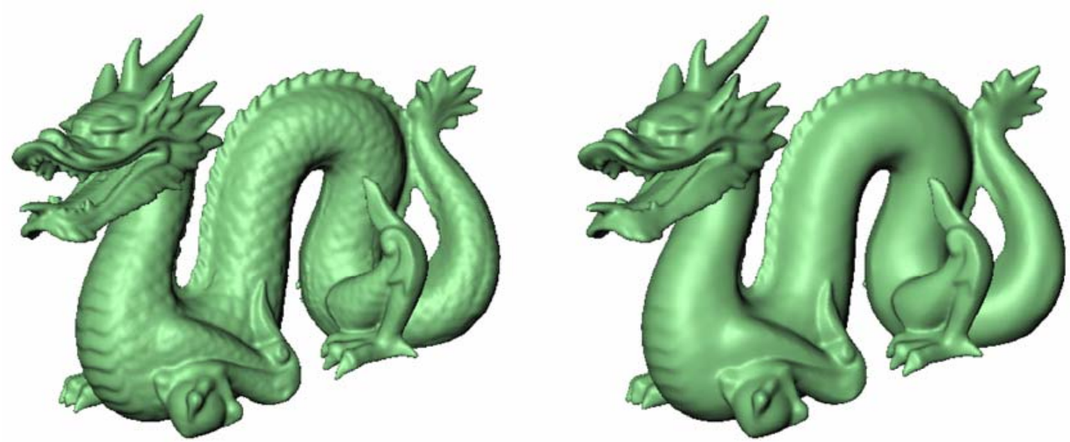


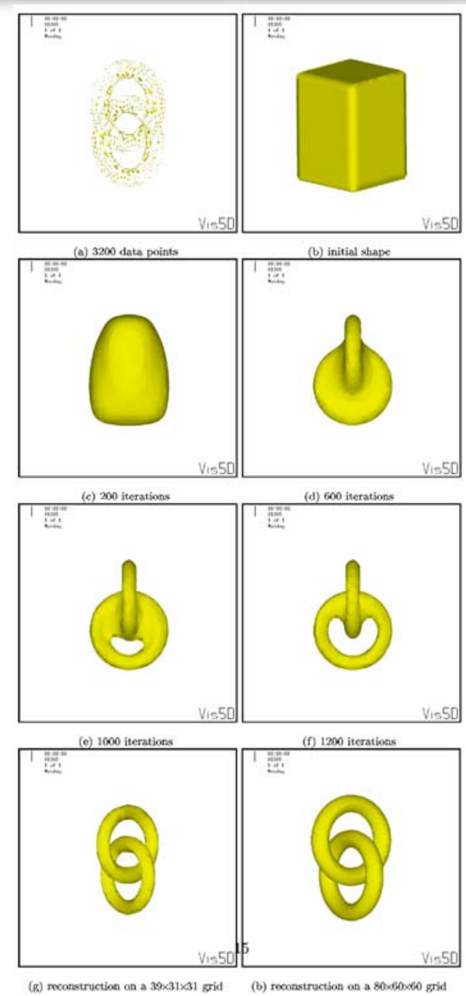
Variational Implicit Surfaces



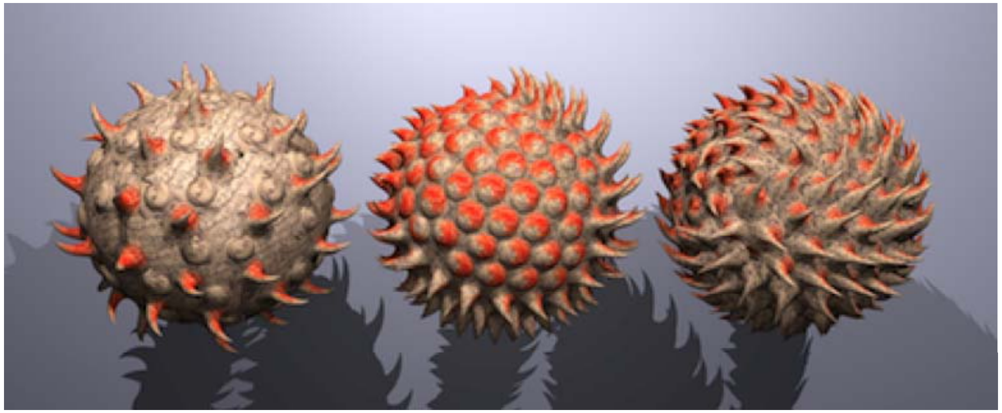
Level-Set Surface Models



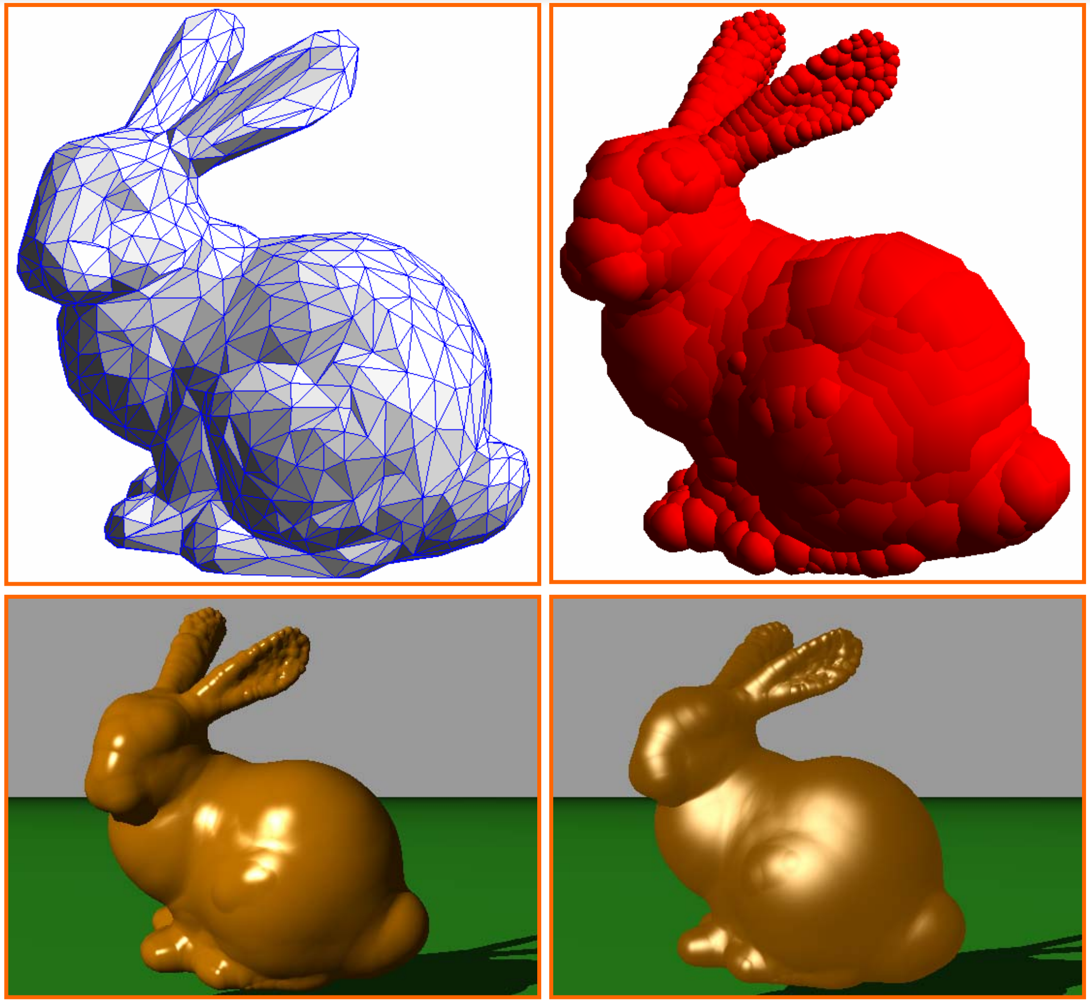




Procedural Models

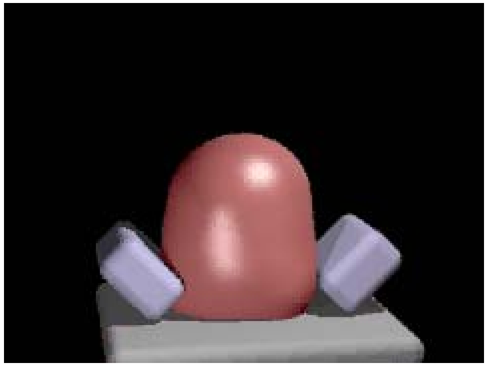


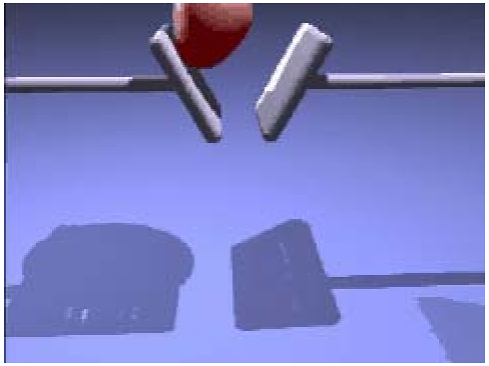
Reconstruction

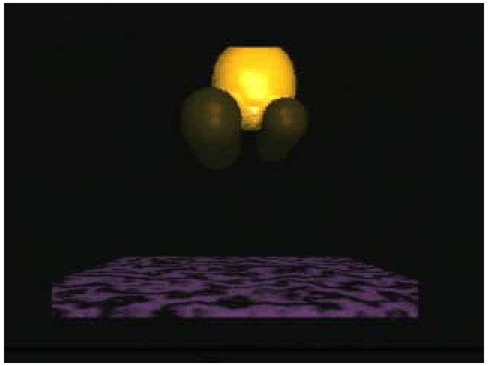


The polygonal bunny model was approximated by metaballs.

Animation applications



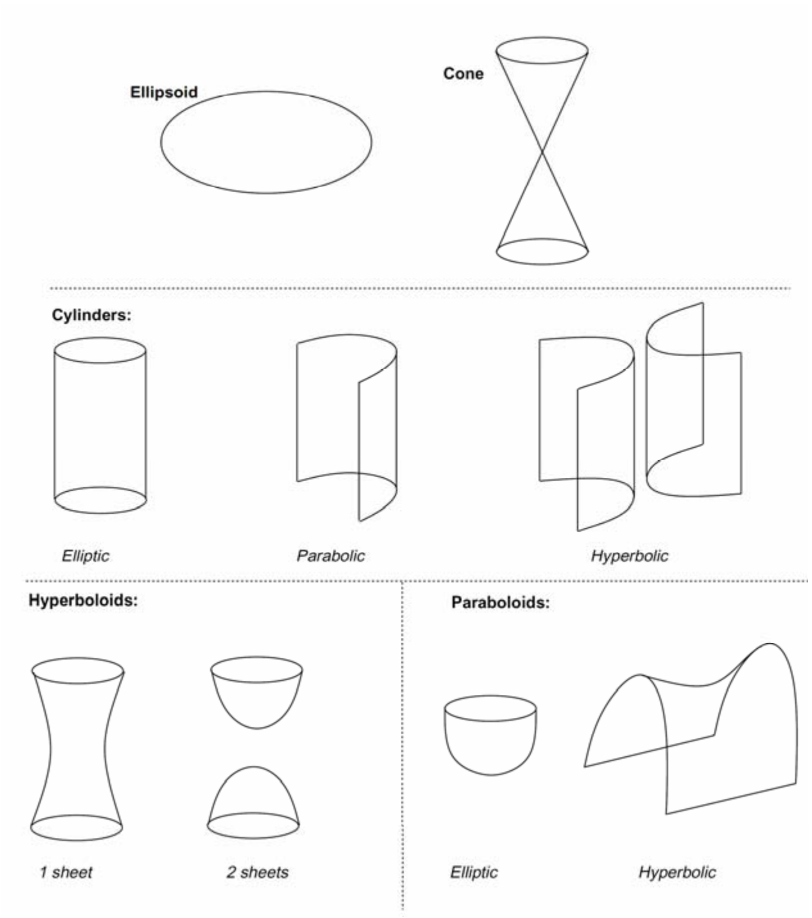


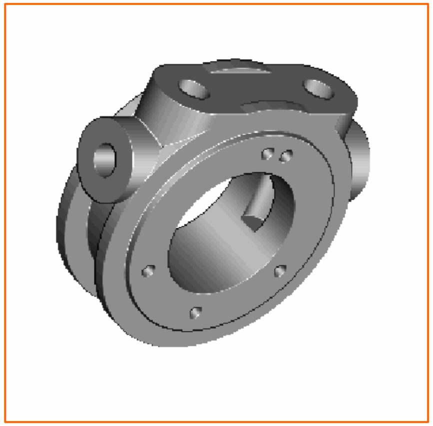


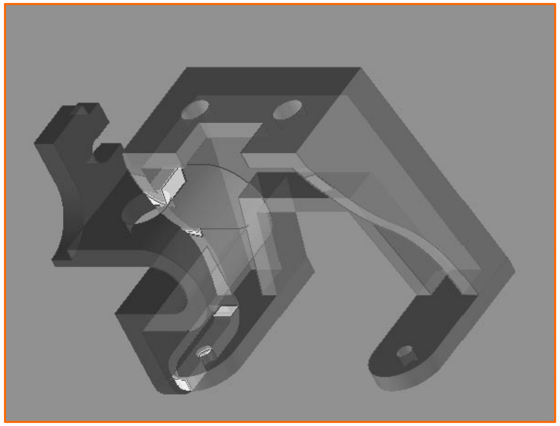
Implicit methods for CAD/CAM

- Quadric surface
- Algebraic surface patches
  - A-patch
  - Tensor-product algebraic surface patches

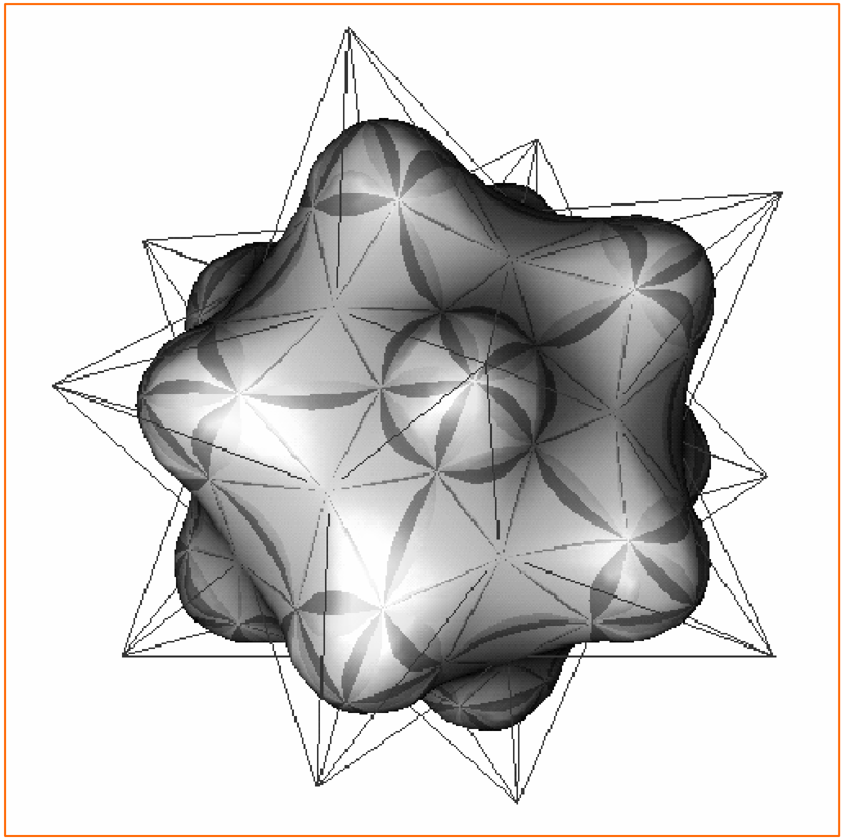
Quadric Surface



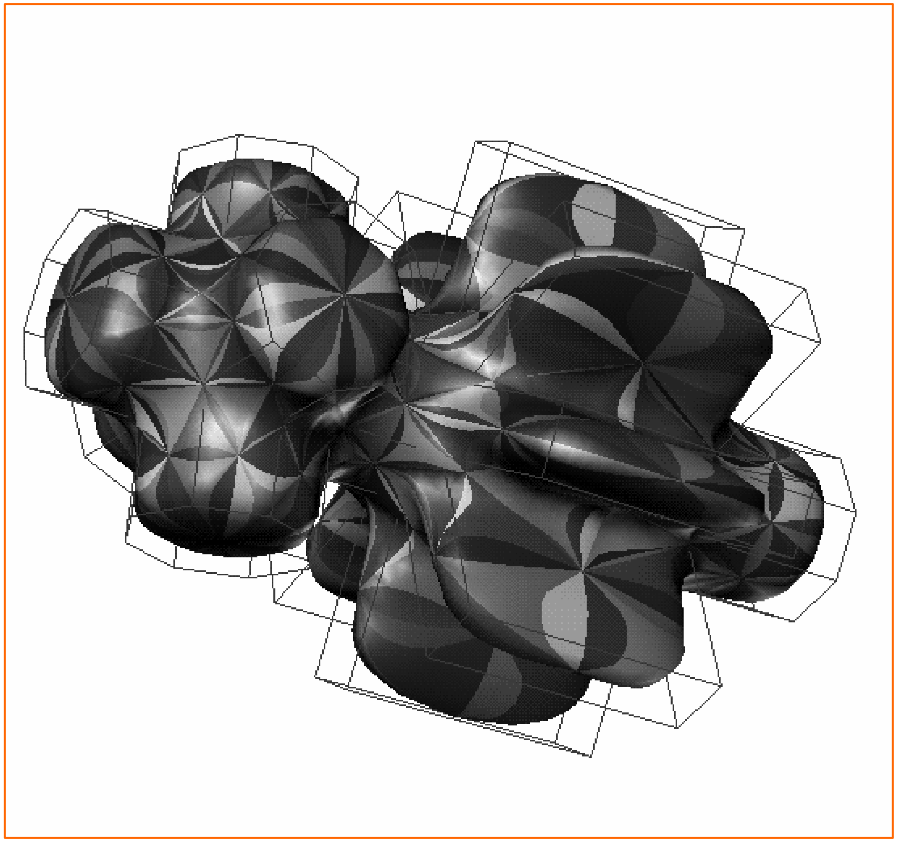


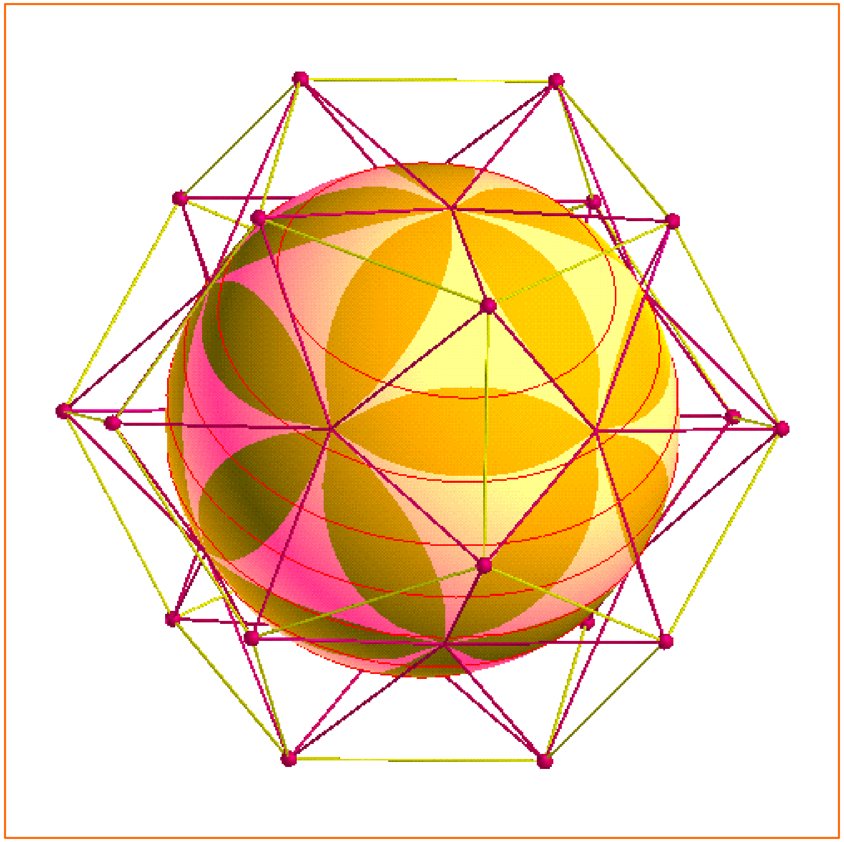


A-patch: Algebraic Surface Patch

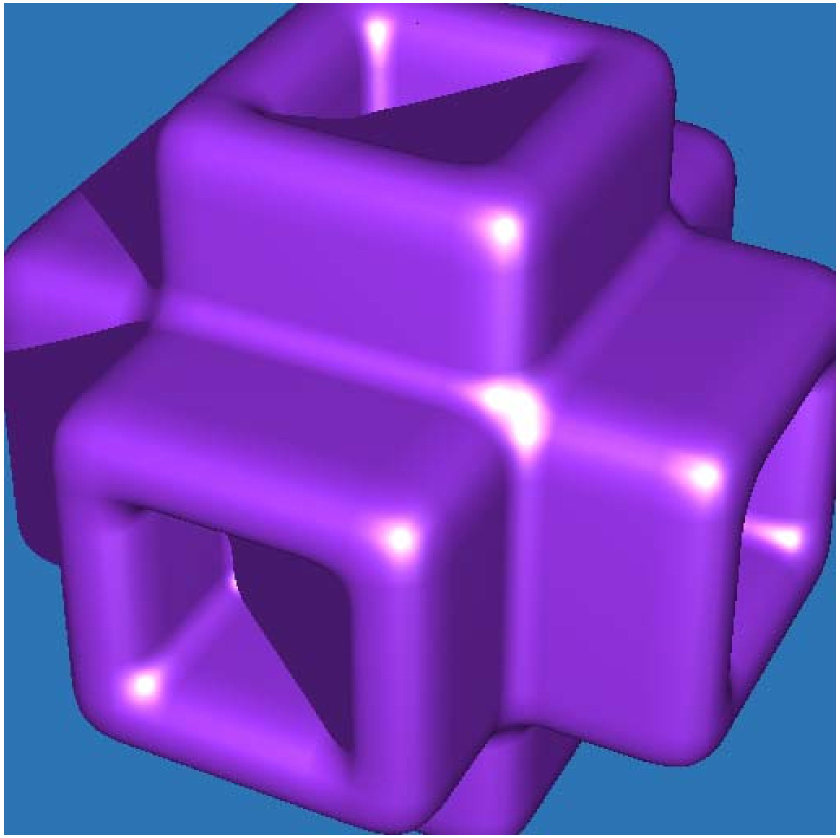


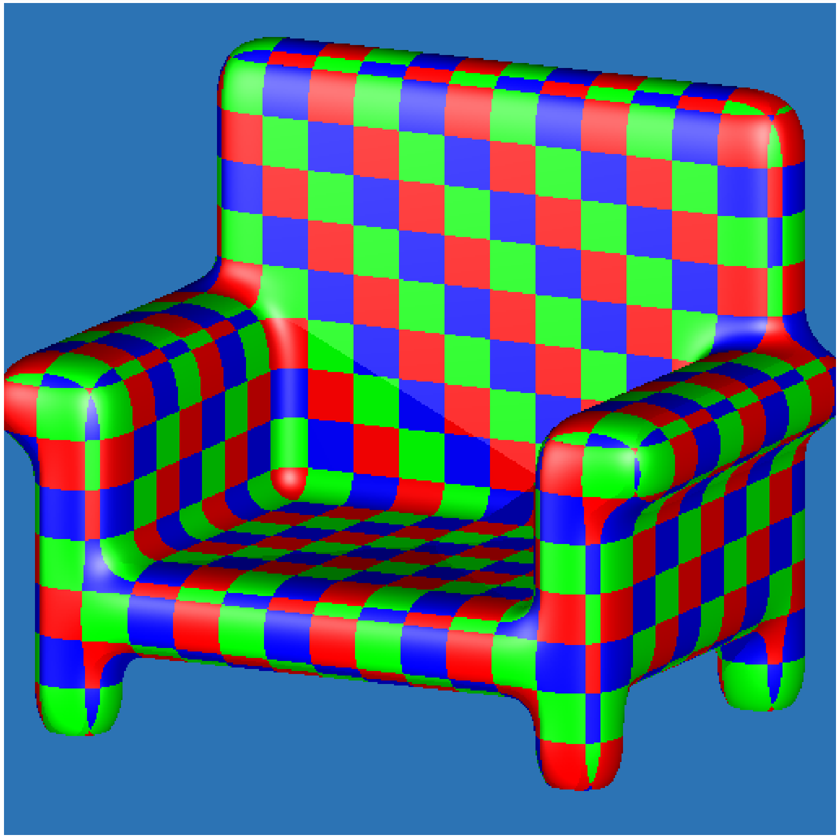
A-patch: Algebraic Surface Patch



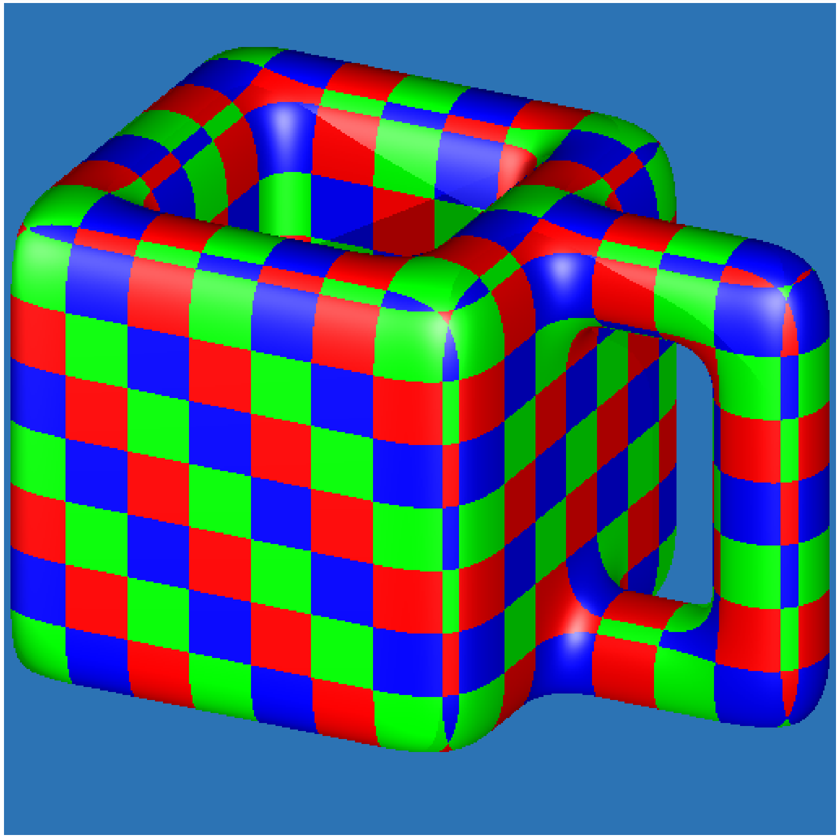


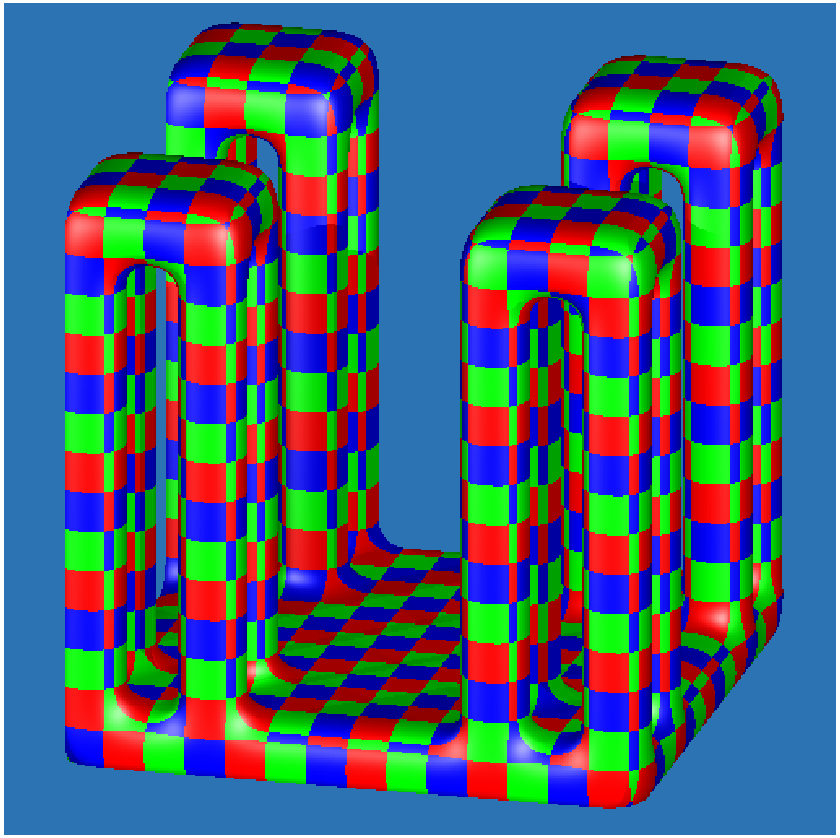
Tensor-product algebraic surface patches





Tensor-product algebraic surface patches





Other Topics in Implicit Modeling

- Functional representation
- Visualizing implicit surfaces
- Animation by using implicit models
- Texture mapping

Functional Representation (F-rep)

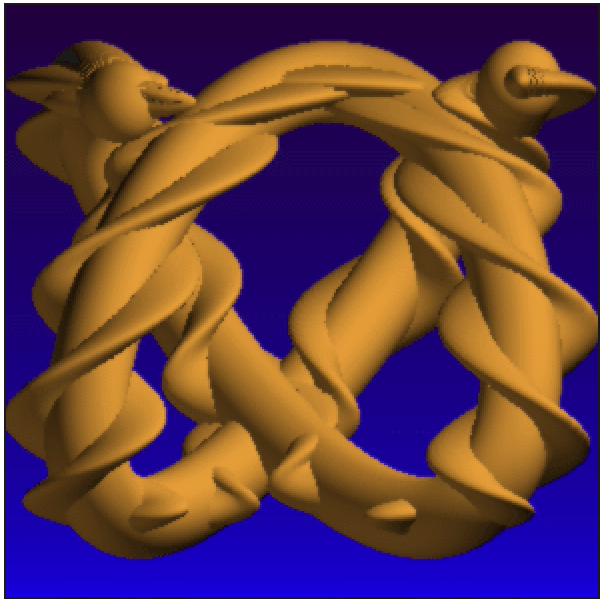
- F-rep: denfines a whole geometric object by a single real continuous function of several variables as $F(\mathbf{X}) \geq 0$
  - At least C^0 continuous
  - F: a formula or an evaluation procedure
  - F-rep: combines classic implicits, skeleton based implicits, set-theoretic solids, sweeps, volumetric objects, parametric and procedural models

Functional Representation (F-rep)

- Set-theoretic operations
- Blending set-theoretic operations
- Offsetting
- Cartesian product
- Bijective mapping
- Metamorphosis
- Relations
- Sweeping by a moving solid
- Deformation with algebraic sums
- Three-dimensional texture modeling
- Interaction

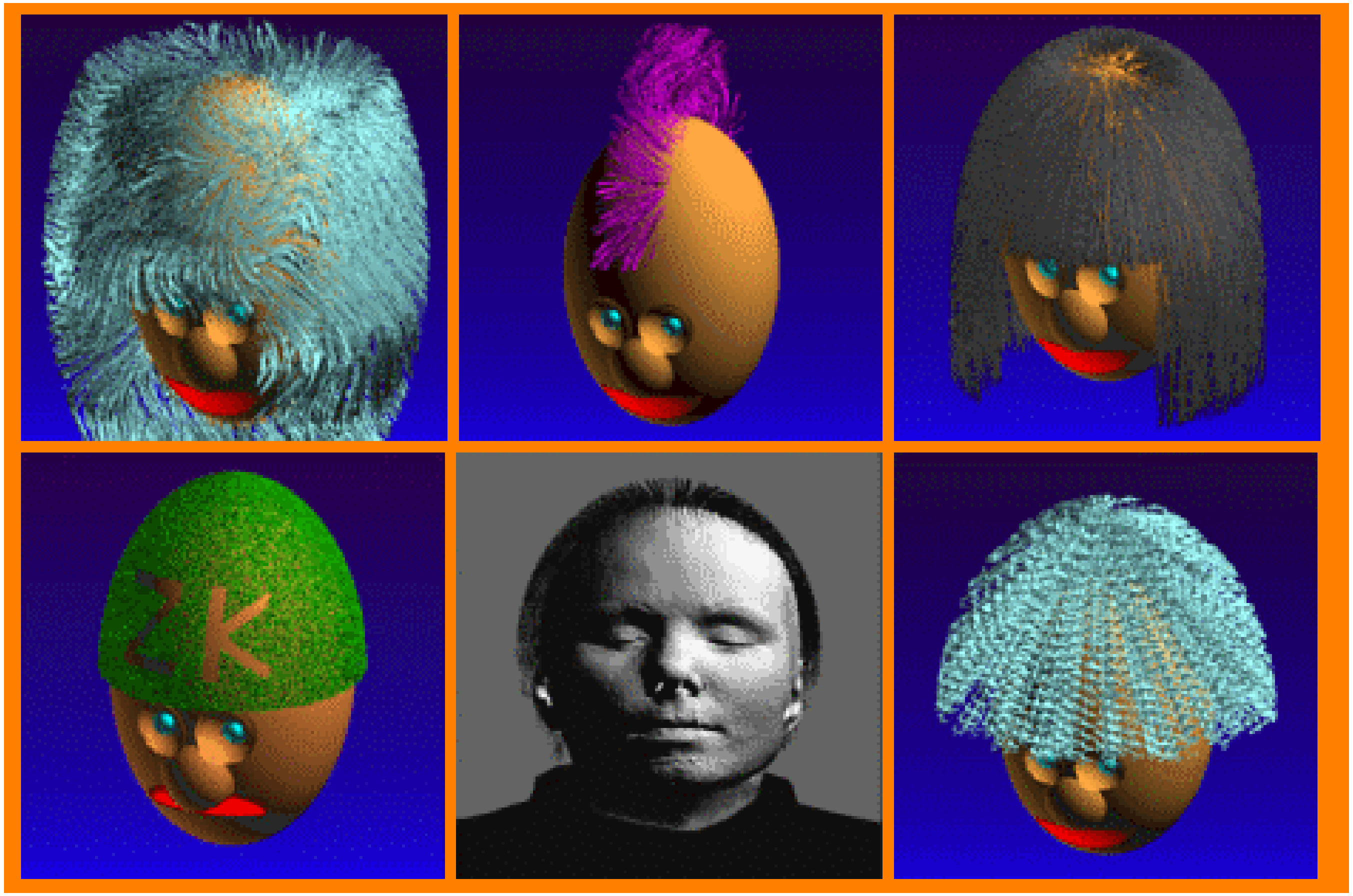
F-rep: examples

Artistic shape modeling using real functions



Moving solid

F-rep: hairs

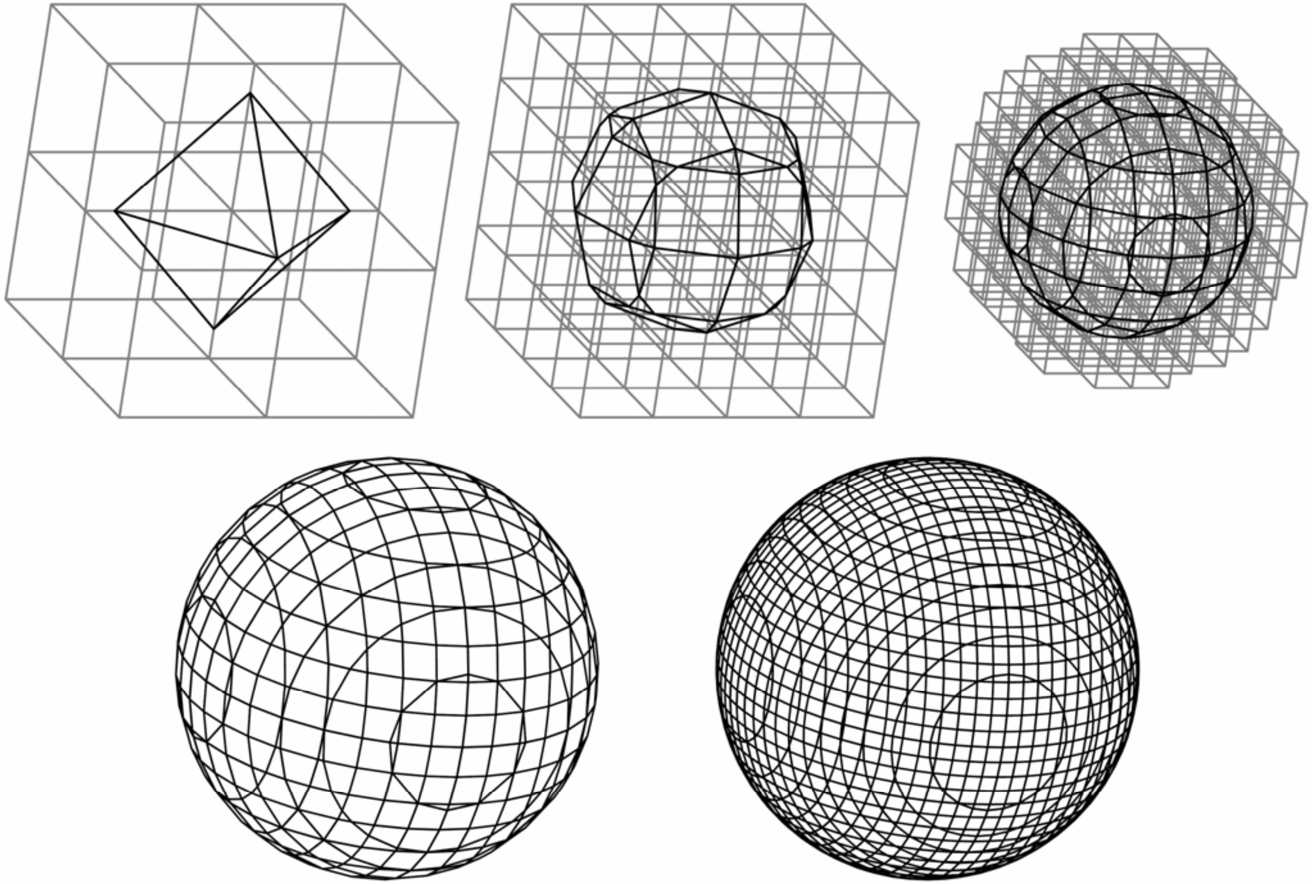


Using real functions with application to hair modeling

Visualizing implicit surfaces

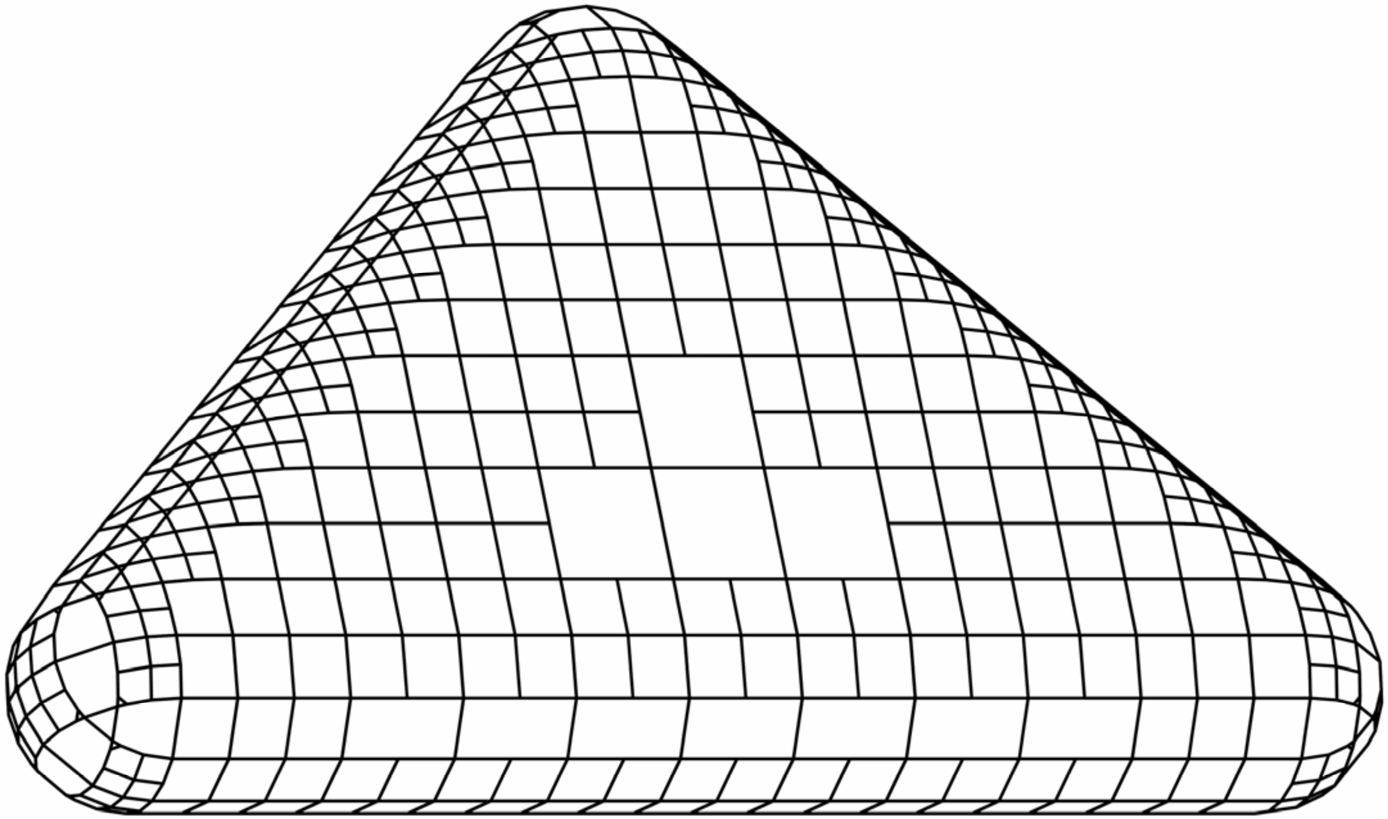
- Polygonization: Generation of polygons from implicit surface or volume data
  - Uniform
  - Adaptive
- Ray tracing
  - Classical
  - Sphere tracing
- Particle system

Uniform Polygonization

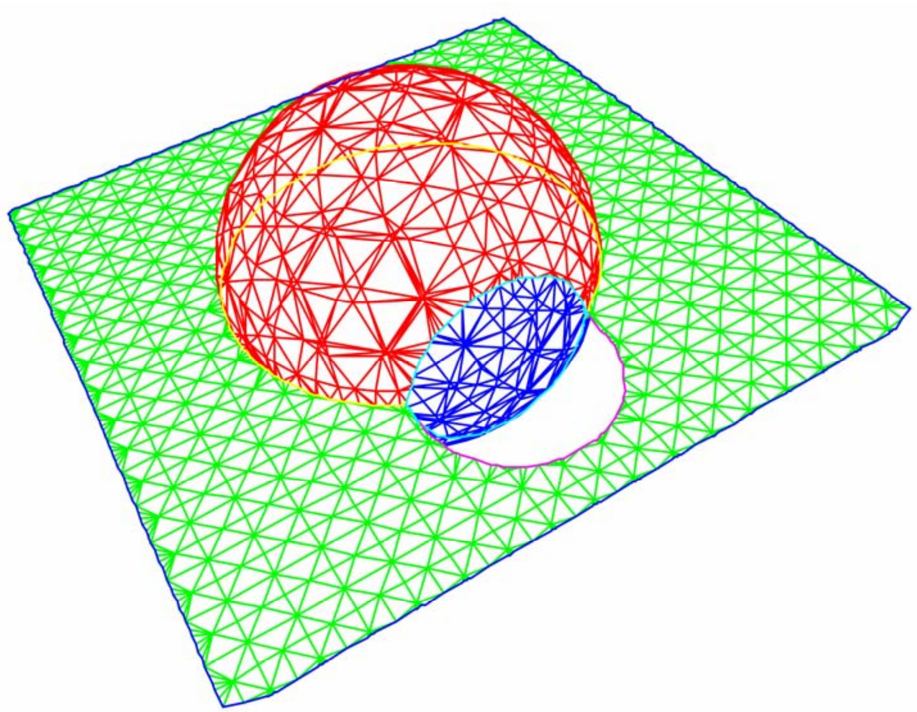


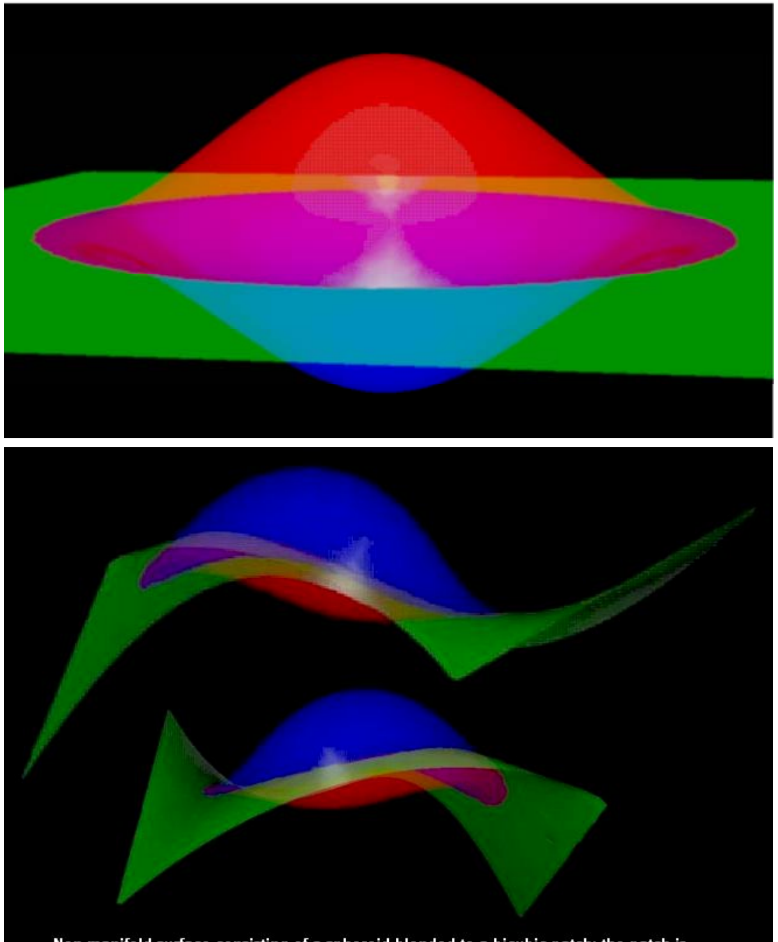
A sphere approximated from depths 1 through 5.

Adaptive Polygonization



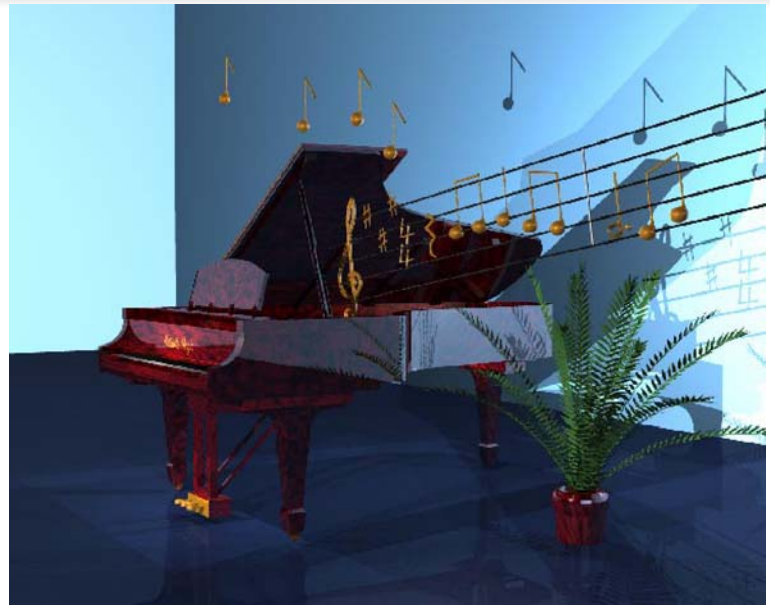
Polygonization of non-manifold

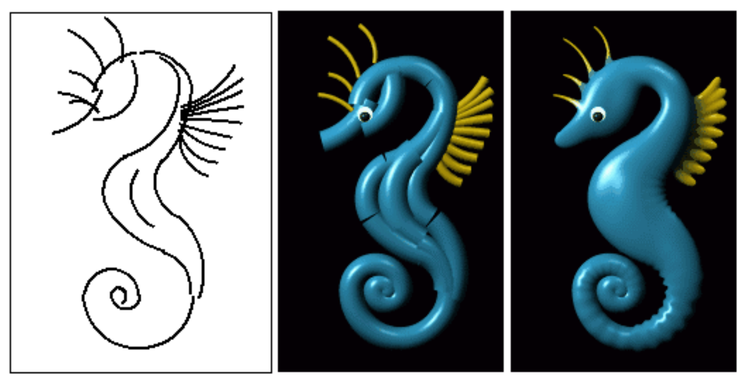




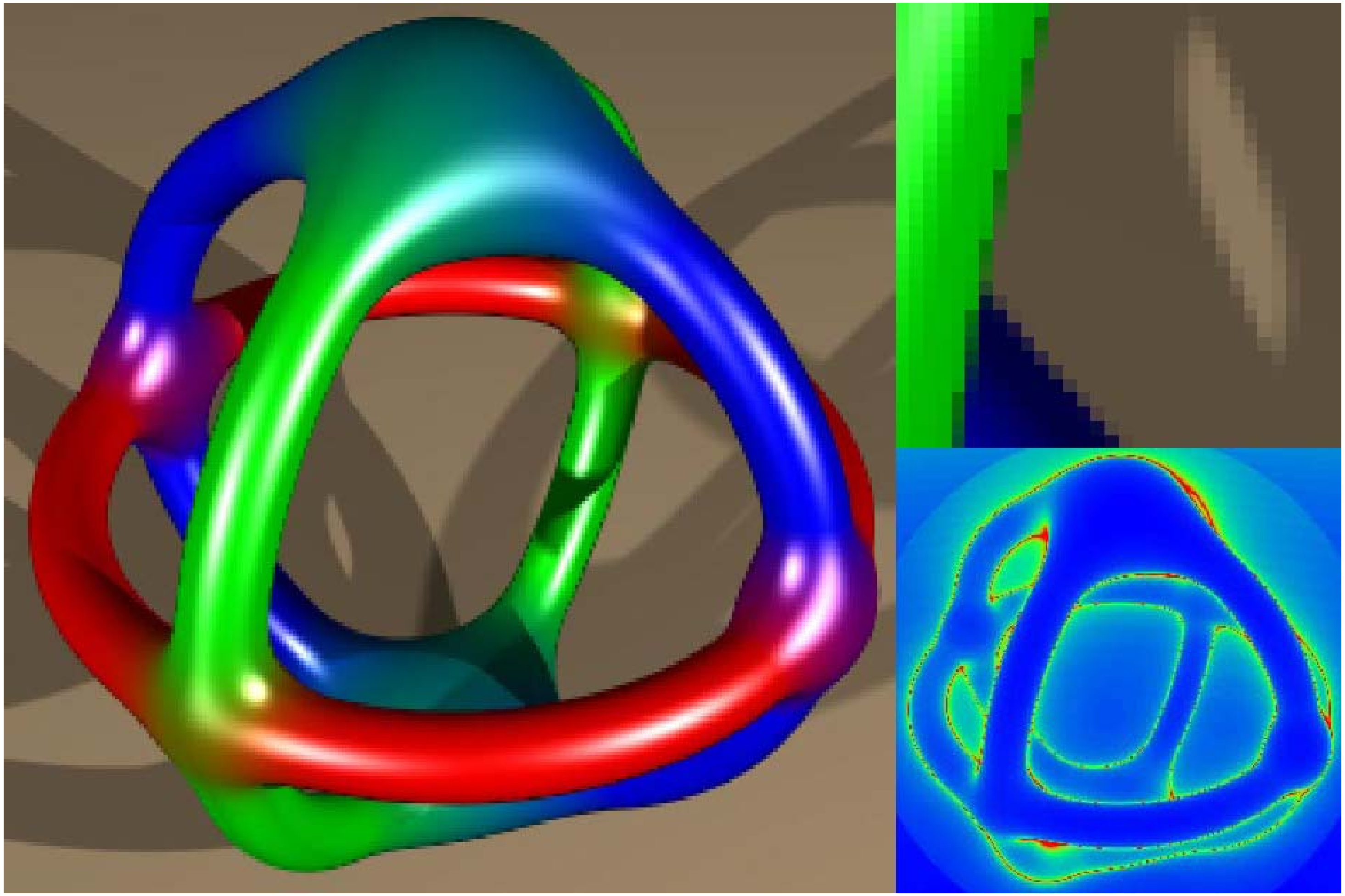
Non-manifold surface consisting of a spheroid blended to a bicubic patch; the patch is automatically trimmed to the seam between spheroid and patch (see SIGGRAPH '95)

Classic Ray Tracing

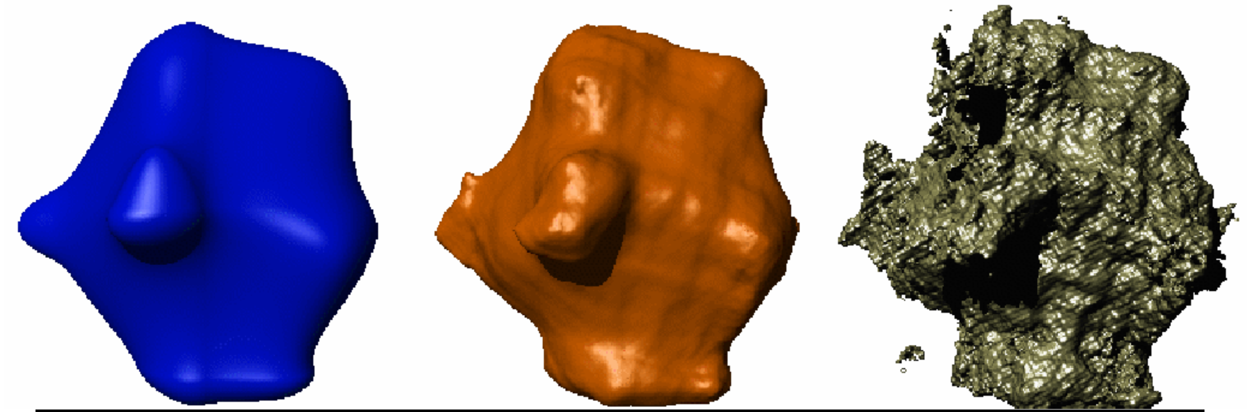


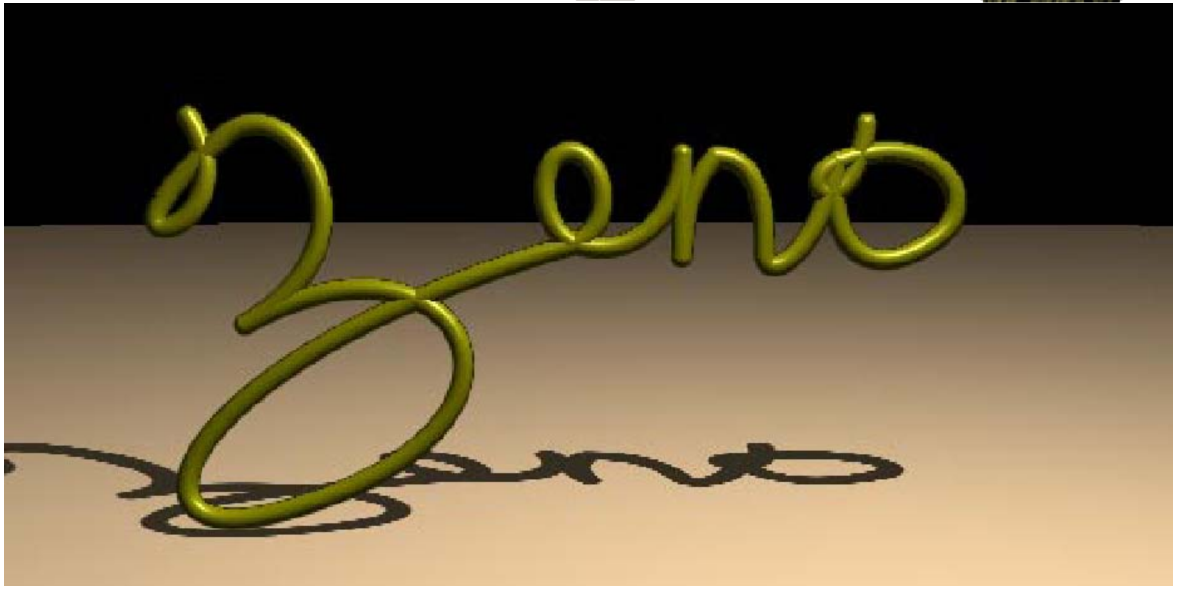


Sphere tracing

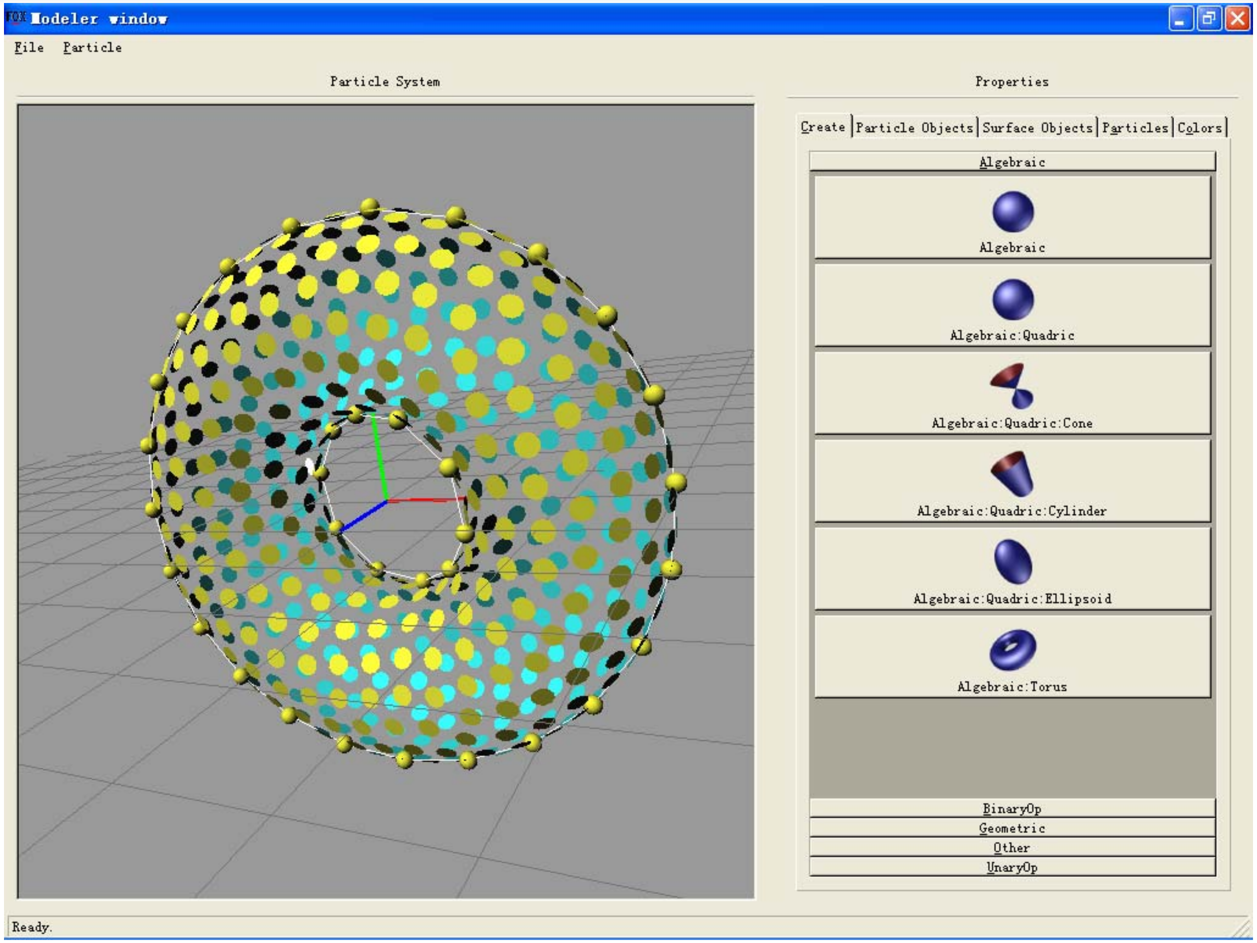


Sphere tracing





Particle system: display and control



http://graphics.cs.uiuc.edu/projects/surface/

Field Function Defined Implicit Surface

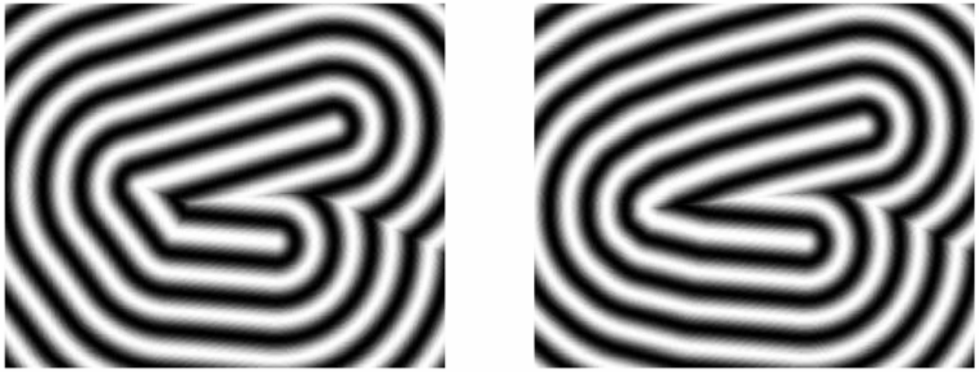
- Field Function in 3D
- Simple Control Primitives: point *etc.*
  - Blobby Molecules
  - Metaball
  - Soft Objects
  - Discussion and Extensions
- Skeletal Primitives
  - Distance surface
  - Convolution surface

Field Function in 3D

- Consider implicit surface

$$D(x,y,z)-Iso=0$$

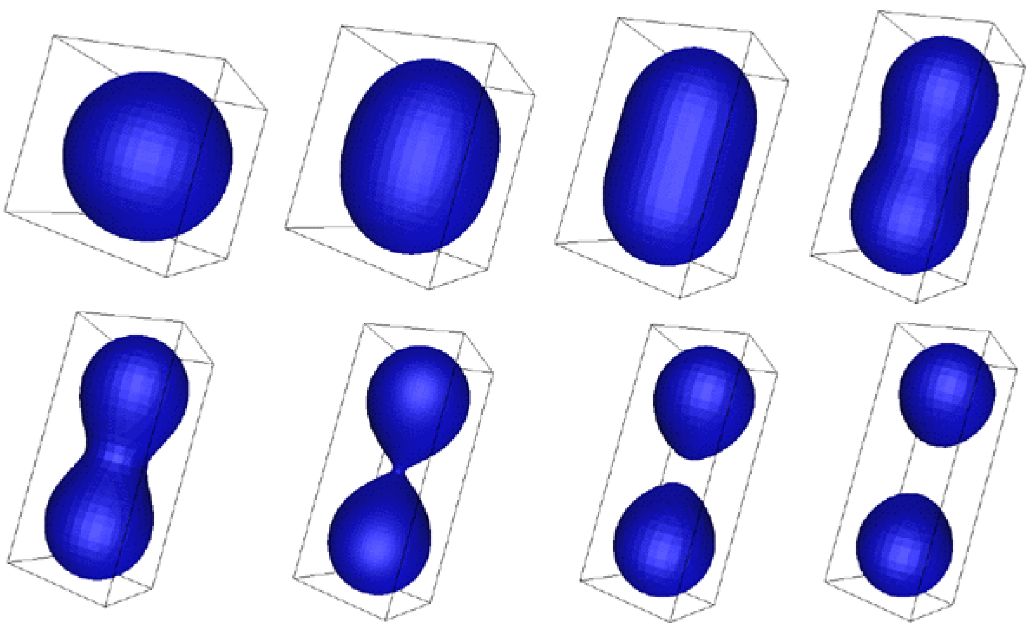
  - D is called scalar field function
  - Distance field is one of good choice



Distance to a Curve: 3 and 9 linear segments

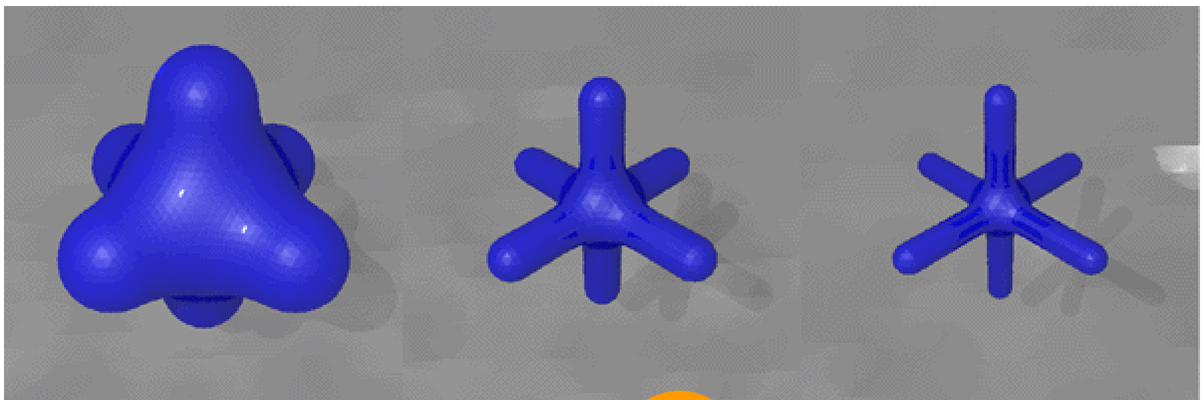
Example of Distance Field

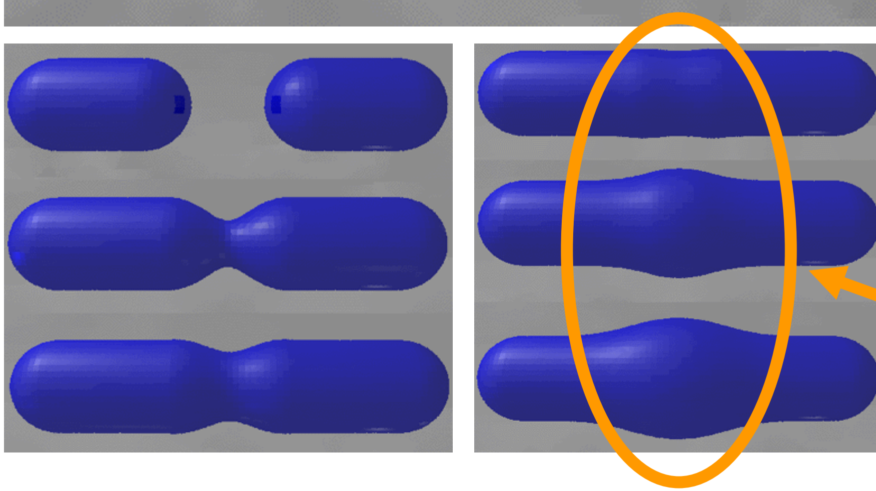
- Consider function $D(r)=1/r^2$, control points in 3D space, where r is the distance to one control points
- The total field strength at any point in space is the sum of the field strengths due to each control point



Example of Distance Field

- Changing control points as line segments





Simple summation of distance fields will result in bulge!

Blobby Molecules by Blinn

Jim Blinn

- Second Graphics Fellow at Microsoft
- 1983 -- The first Siggraph Computer Graphics Achievement Award for work in lighting and surface modeling techniques.
- 1989 -- IEEE Outstanding Contribution Award for Jim Blinn's corner

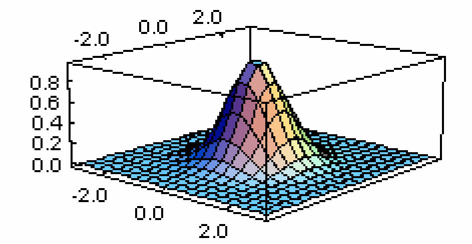
http://research.microsoft.com/users/blinn/

Blobby Molecules

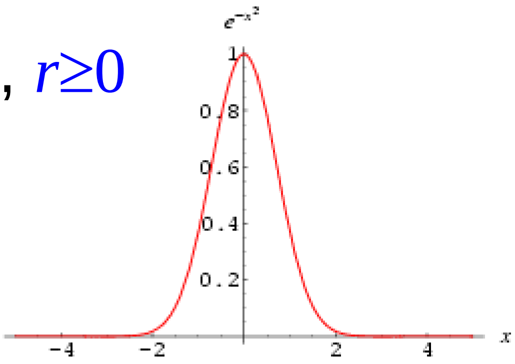
- Hydrogen atoms electron density fields (Gaussian Distribution)

$$D(r) = ae^{-br^2}$$

  - "b" : standard deviation of Gaussian curve
  - "a" : height of Gaussian Curve
  - "r" : the distance to atom center, $r \geq 0$



2D Gaussian Function



1D Gaussian Function

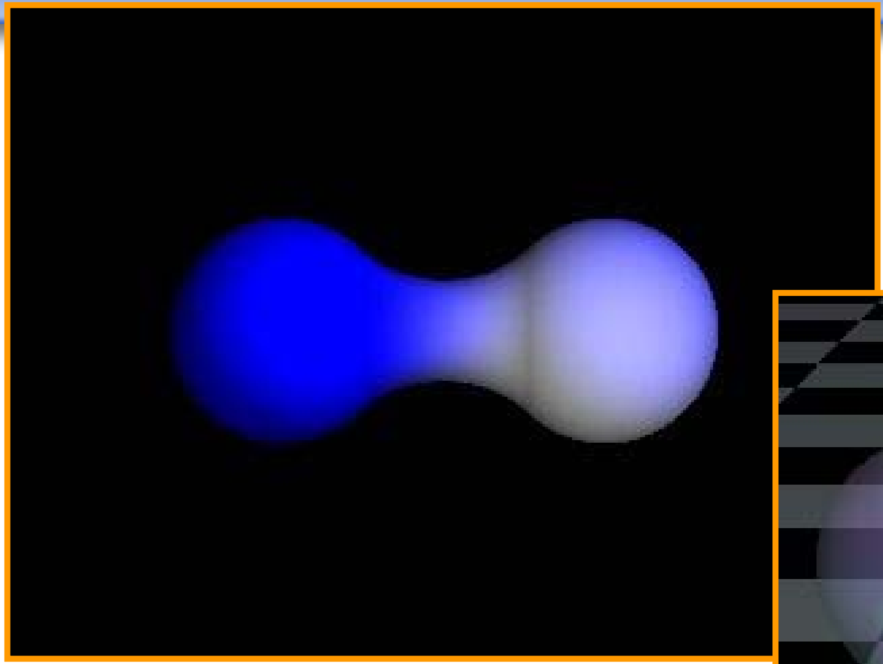
Multiple Blobby Molecules

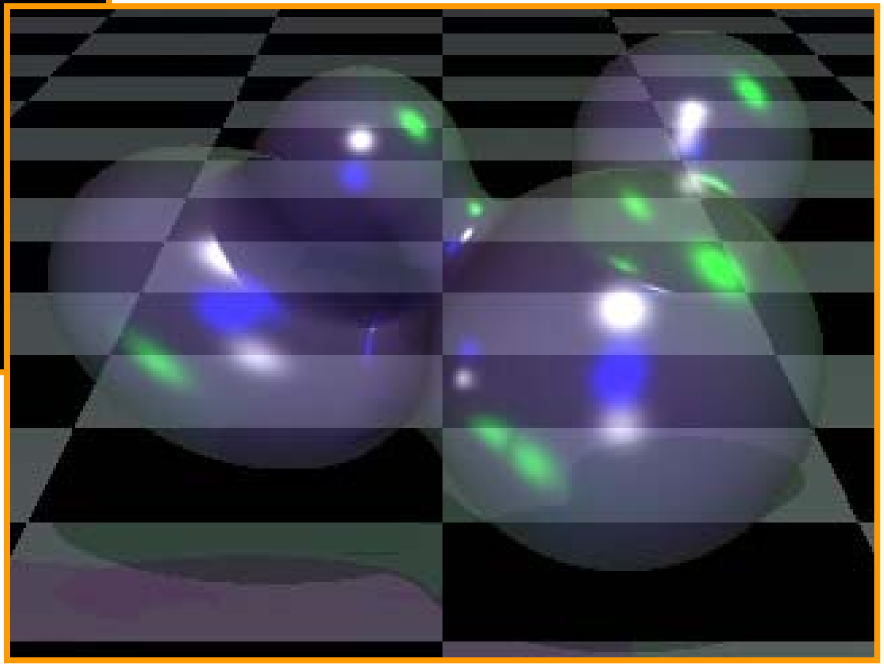
- The overall density at a given point for multiple atoms is summation of individual ones:

$$D(p) = \sum_i a_i e^{-b_i r^2} = Iso$$

  - The “blobbiness” of a model can be controlled by adjusting the parameters a_i and b_i.
    - a_i is weight/strength of ith blobby molecule
    - b_i is influence radius of ith blobby molecule
  - The implicit surface is defined as all points where the density is equal to some threshold value Iso.

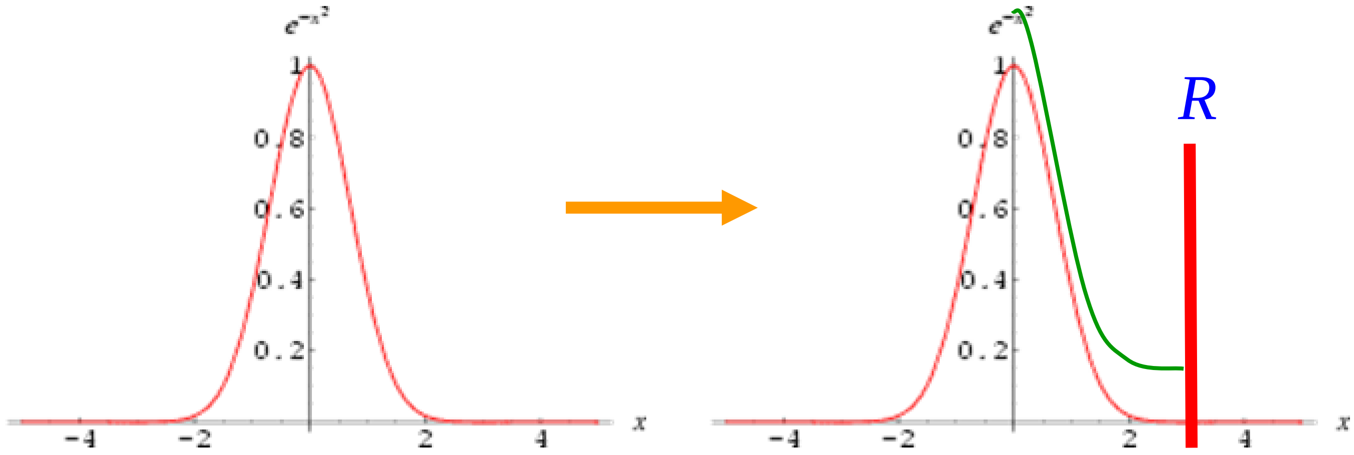
Examples of Blobby Models





Blobby Molecules

- Disadvantages of Blobby Molecules
  - Exponential distribution of density is Global
    - Computationally expensive!
  - Blinn: Cut off after $r > R$



Metaball by Nishimura

- Nishimura at Osaka University in Japan proposed Metaball in 1983 (Almost same with Blinn)

Metaball by Nishimura

- Piecewise quadrics to approximate the Gaussian function

$$D(r) = \begin{cases} a\left(1 - \dfrac{3r^2}{b^2}\right) & 0 \le r \le b/3 \\ \dfrac{3a}{2}\left(1 - \dfrac{r}{b}\right)^2 & b/3 \le r \le b \\ 0 & r \ge b \end{cases}$$

  - "b" : Influence radius of metaball
  - "a" : weight of metaball
  - "r" : the distance to metaball center, $r \geq 0$

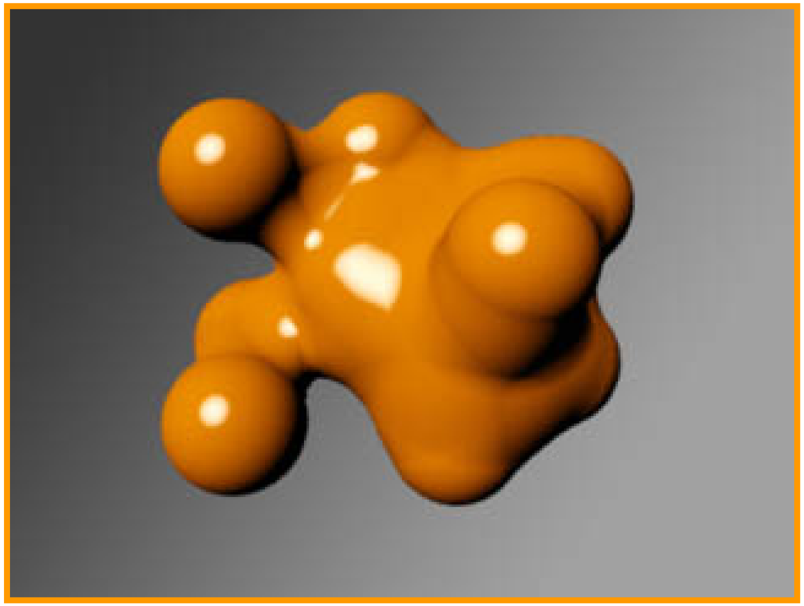
Metaball Systems

- The system is composed of several metaball

$$D(p) = \sum_i D_i(p) = Iso$$

  - Each metaball has its own parameters (a_i, b_i)
  - Fusion effects

Examples of Metaballs



Metaball

- Advantages
  - Field function is locally defined in $[0,b]$
  - Fast ray and metaball intersection computation!
- Disadvantage
  - Distance r: square root , expensive

Soft Object by Wyvill Brothers

- Soft objects is proposed by Wyvills in Canada and New Zealand: truncated Taylor expansion series of exponential function

Brian Wyvill

Geoff Wyvill

Field Function for Soft Object

- Definition

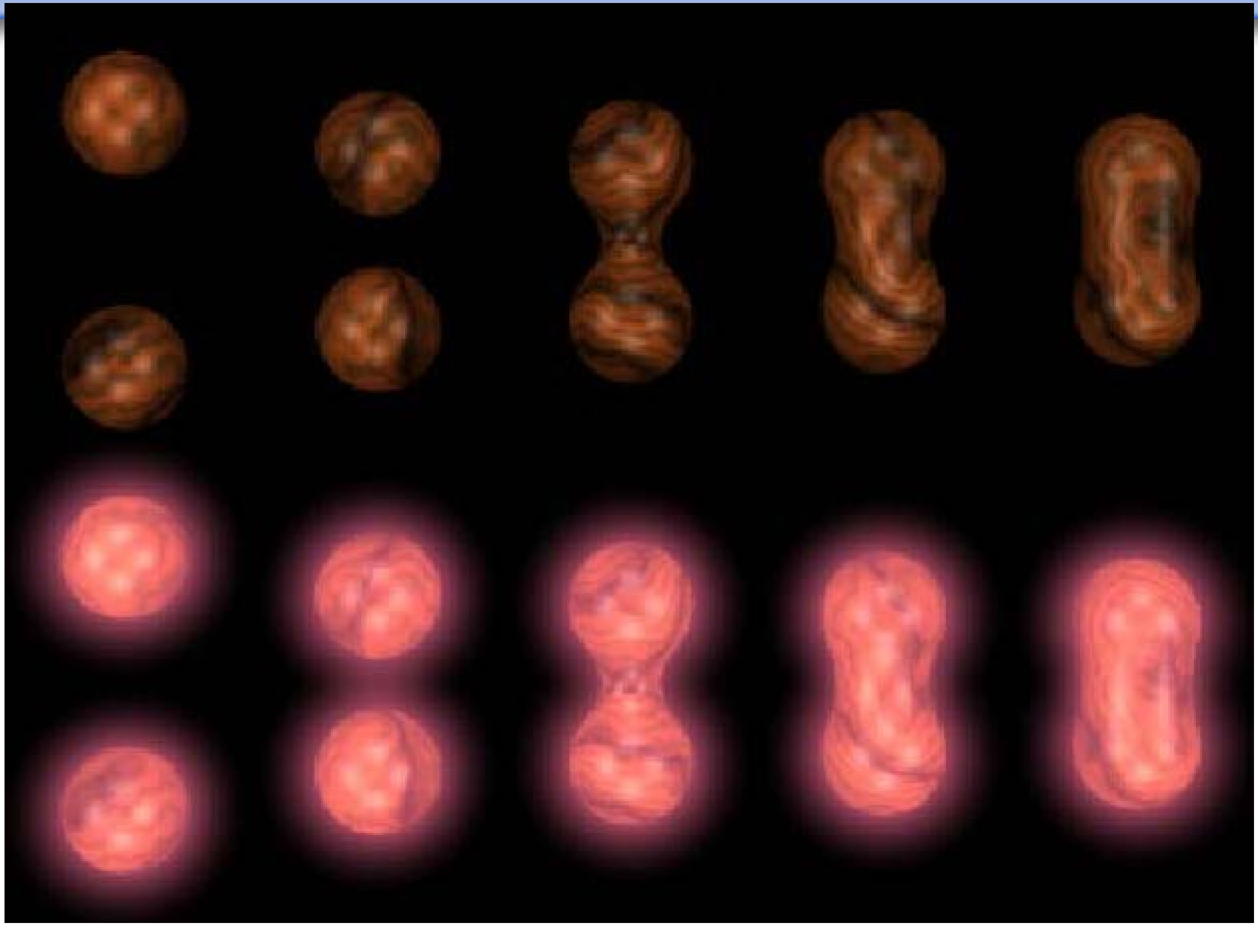
$$D(r)=\begin{cases} a\left(1-\dfrac{4r^6}{9b^6}+\dfrac{17r^4}{9b^4}-\dfrac{22r^2}{9b^2}\right) & r\le b \\ 0 & r>b \end{cases}$$

  - The function has following properties

    $$D(0)=1,\ D(b)=0,\ D(b/2)=0.5,\ C'(0)=C'(b)=0$$

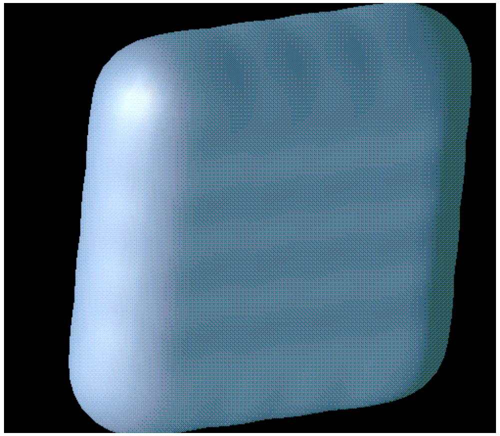
  - The volume bounded by $D(r)<m$ is one half of that bounded by $D(r)<m/2$

Examples of Soft Objects

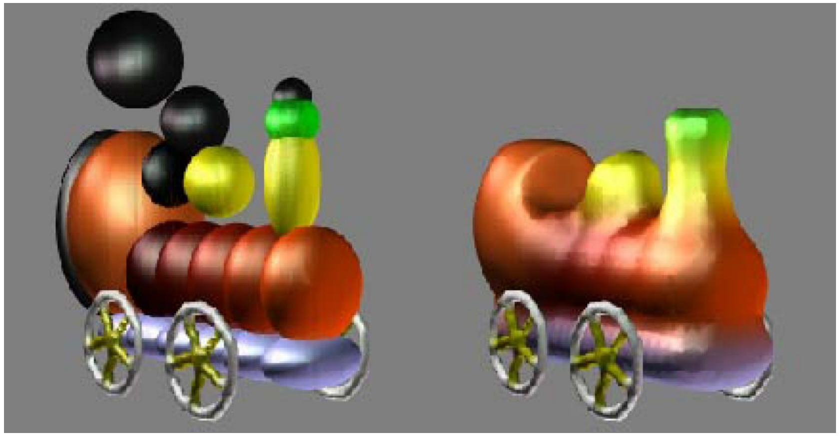


Examples of Soft Objects

Two blend soft objects with different threshold: 84, 44, 24, 11



25 balls, threshold is 0.5



Trains modeled by soft objects

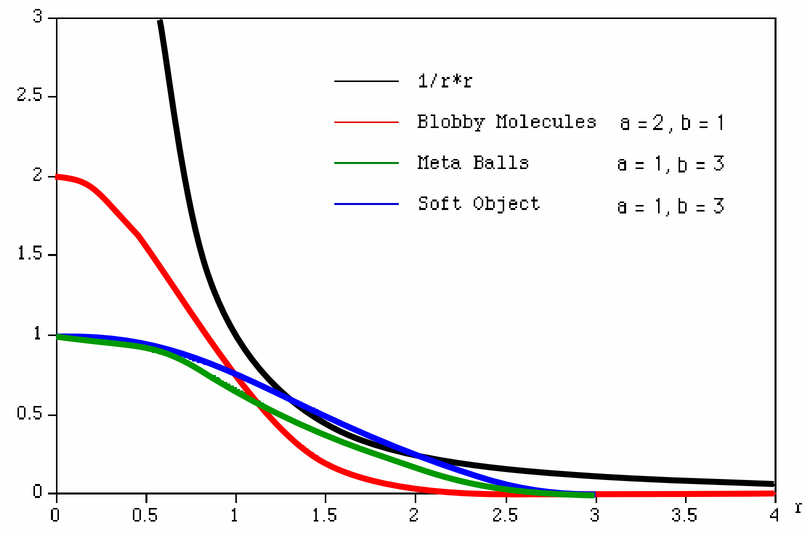
Soft Object

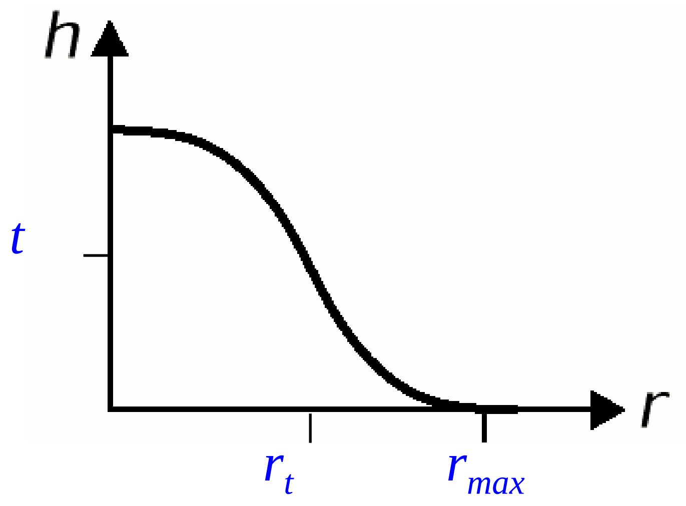
- Advantages
  - Only square term of the distance r, square root computation is removed!
  - Finite extent of each ball
  - Computational costs:

    Blobby >> Metaball > Soft object

Discussions

- The functions described here guarantee a continuously and smoothly changing surface
- Comparison of four field functions

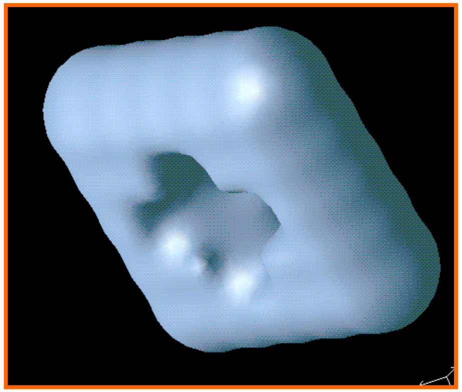
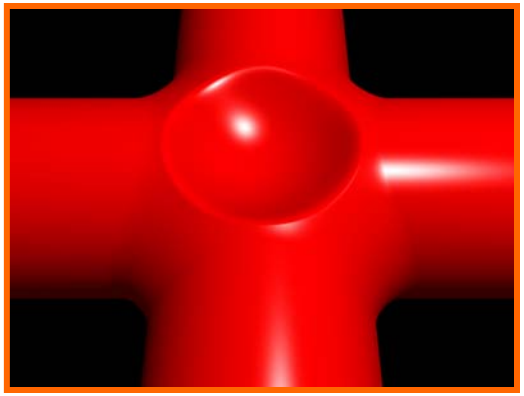
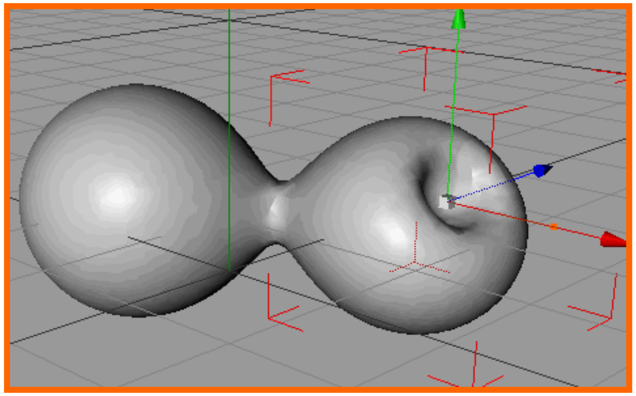


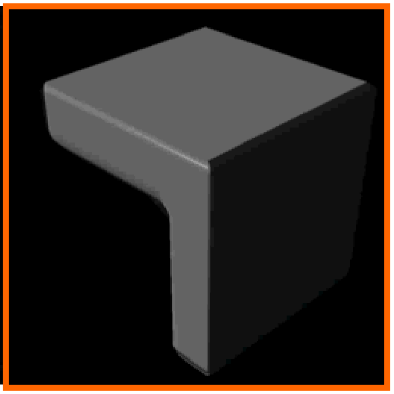


- The blending function can be designed freely!

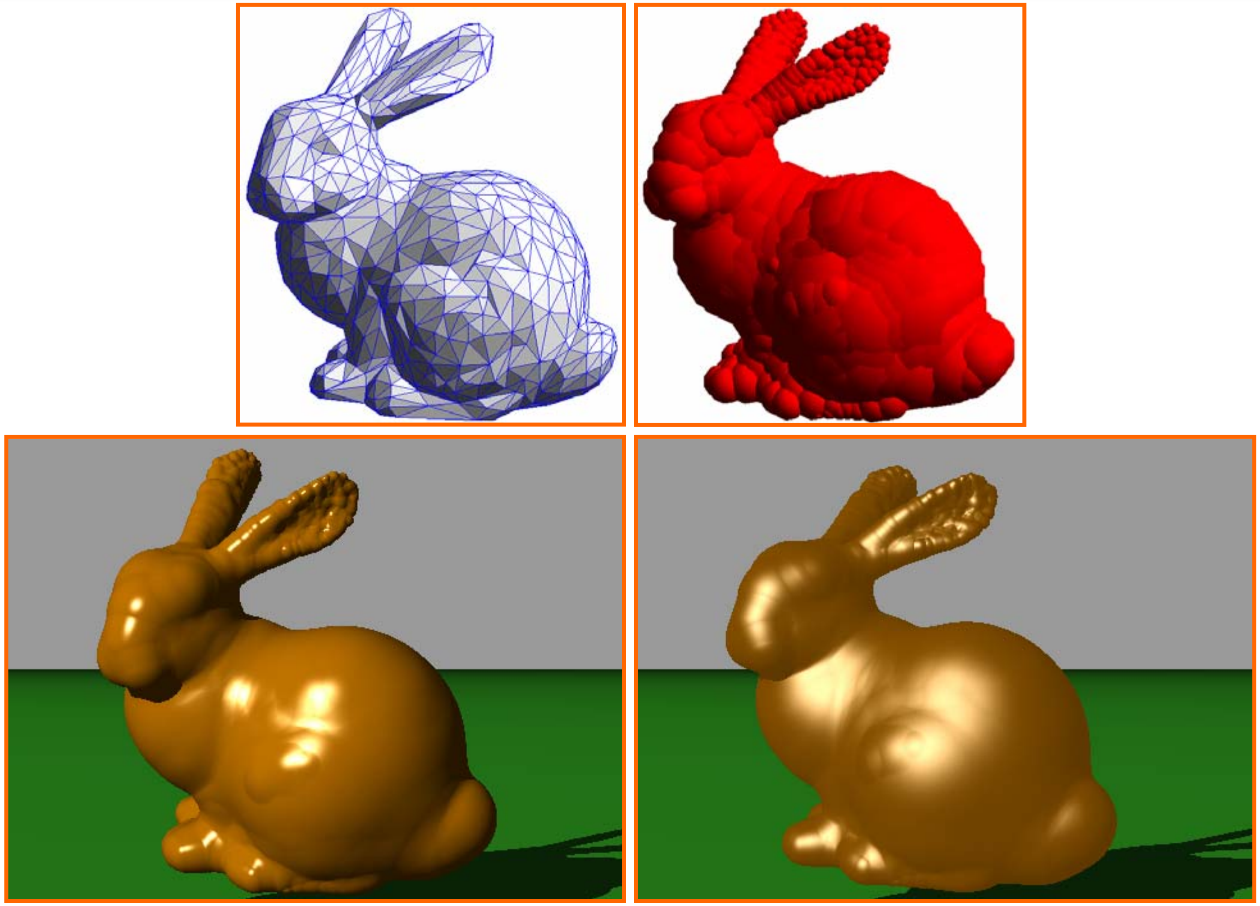
Extensions

- Minus primitives: set weight negative



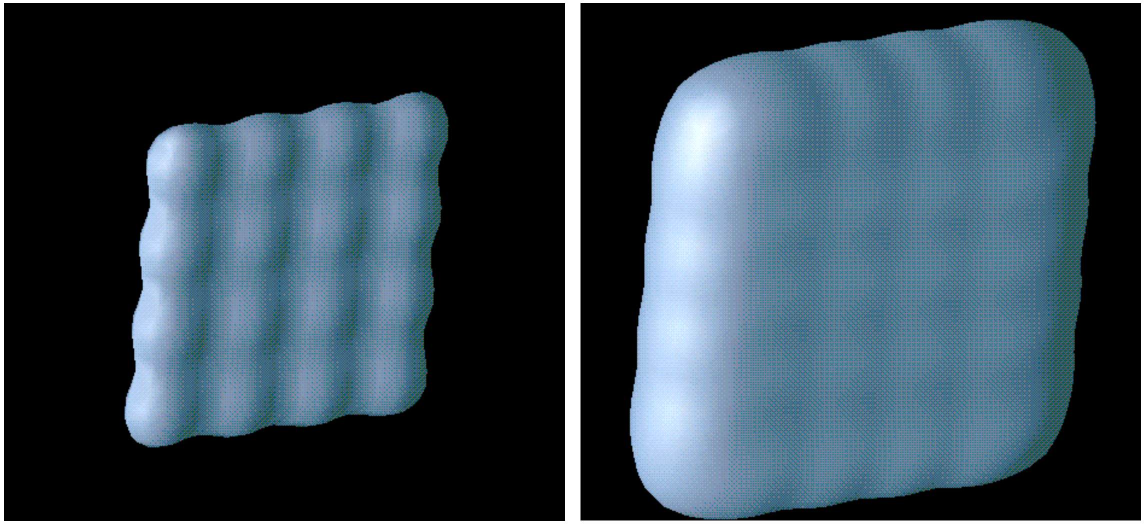
Applications of Metaball



Shape approximation and representation by metaballs

Skeletal Primitives

- Disadvantage of point primitives: flat surfaces can only be approximated.



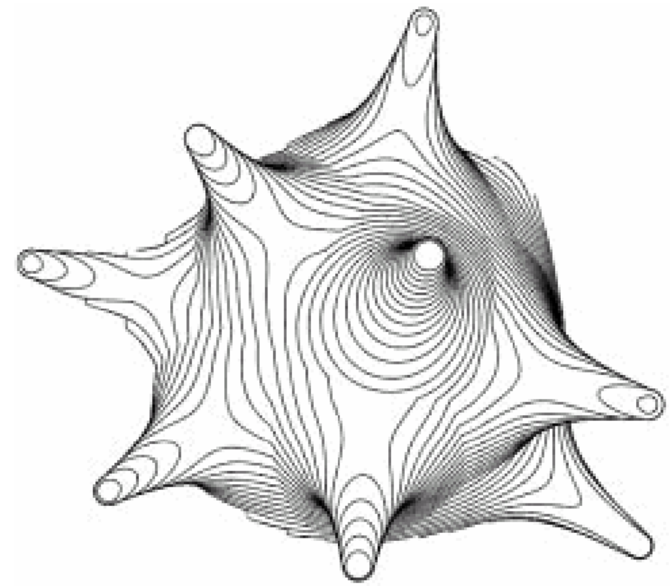
Evenly spaced grid of metaballs(16): threshold is 1.66 or 0.5

Skeletal Primitives

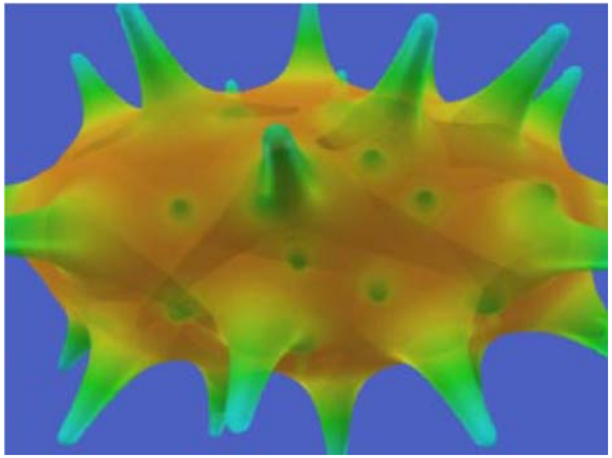
- The use of complex skeletons is one possible solution to this problem.
  - Point can be replace as lines, polygons, curves, surfaces , volumes and any other complex geometric skeletons
  - The shape of a primitive follows the shape of its skeletons
- Two Approaches
  - Distance surfaces
  - Convolution Surfaces

Examples of Skeletal Primitives

skeletons

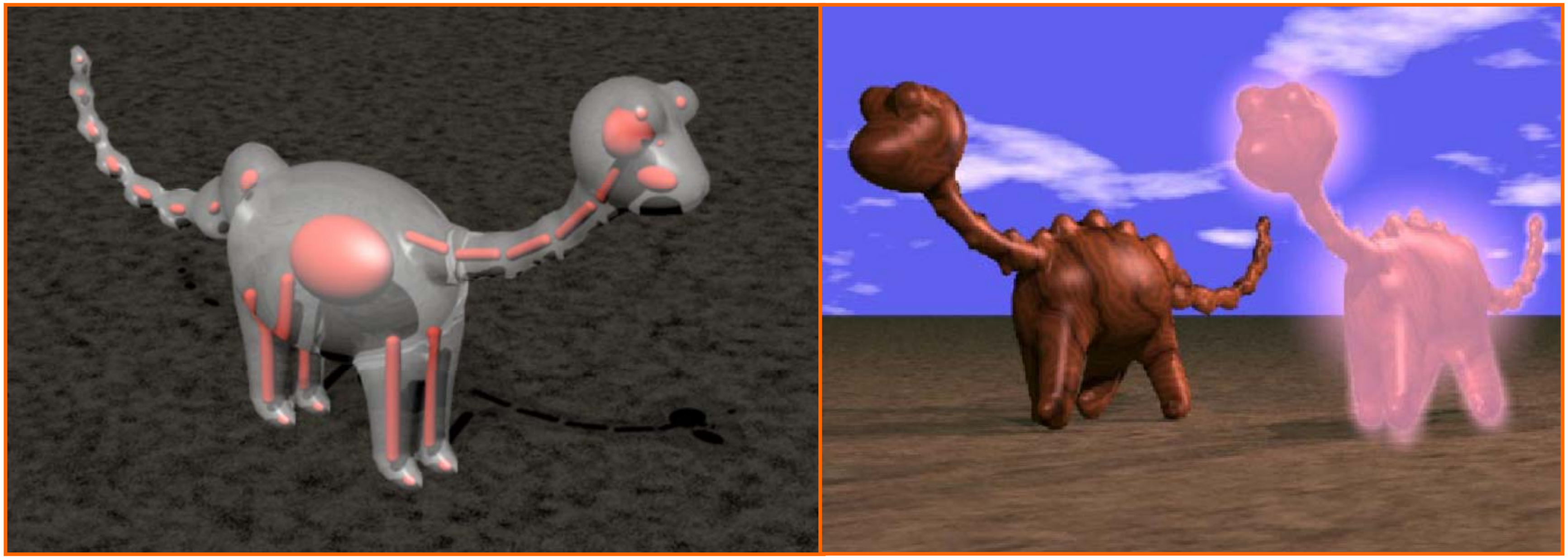


contour line drawing



ray traced image

Examples of Skeletal Primitives



Complex skeletons: ellipsoid, sphere, line segments, points

Distance surface

- A distance surface is a surface that is defined by distance to some set of base skeleton elements
  - Union of skeletons: Distance surface to the union of skeletons
  - Union of primitive surfaces: Boolean operation
  - Blending of primitive surfaces: Algebraic blending

Union of Skeletons

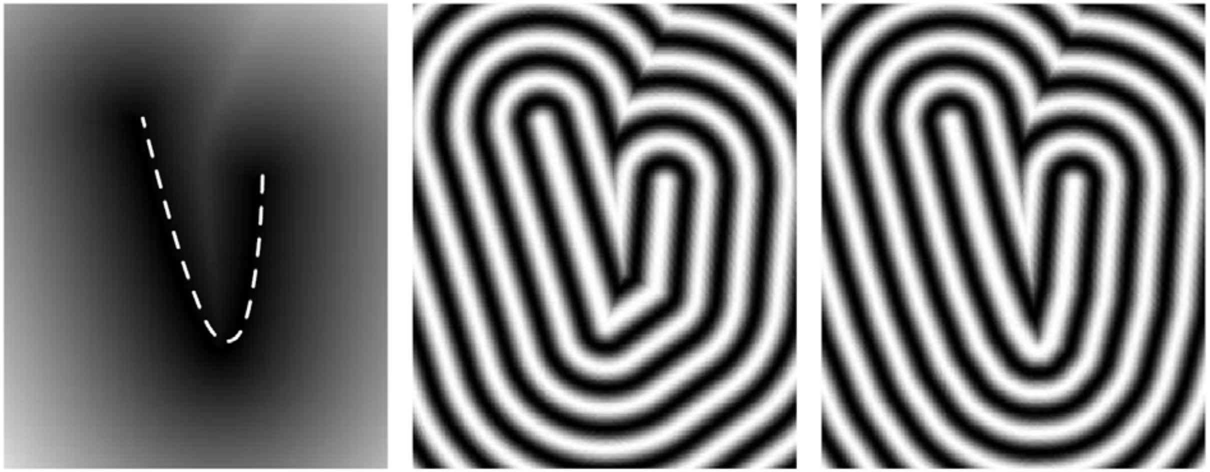
- Definition

$$f(S,p) = \max_{s \in S} \exp\left(-\frac{\|s-p\|^2}{2} \right)$$

  - S is the skeleton set
  - s is the point on one of the skeleton
  - $\max$: maximum field value, nearest point on the skeleton
  - Field function can be specified as before, not restricted as exponential function

About Union of Skeletons

- Distance surface is rounded when skeleton is convex.
- Distance surface is tangent discontinuous and exhibits a crease when the skeleton is concave



Left: curve shown as dashed and distance shown as greyscale intensity
middle and right: curve approximated by three and nine segments

Union of Distance Surfaces

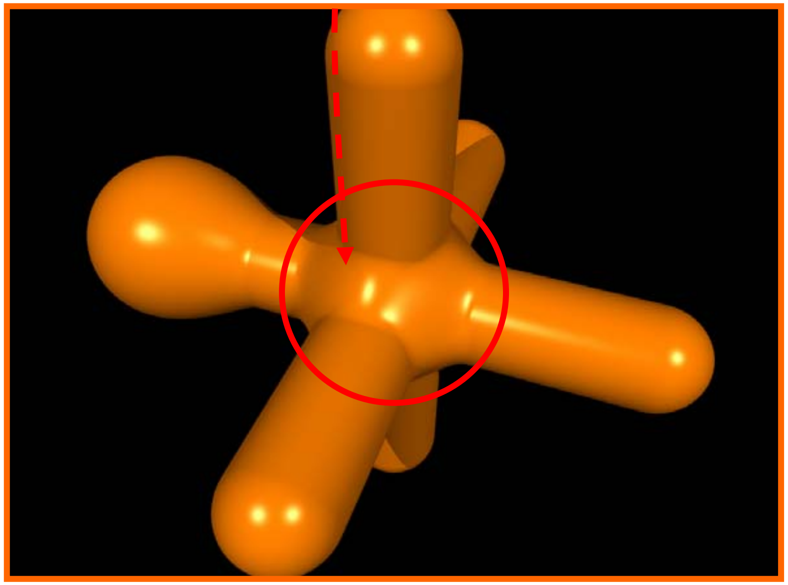
- Union of distance surface is defined as weighted union of each primitive surface
- Definition

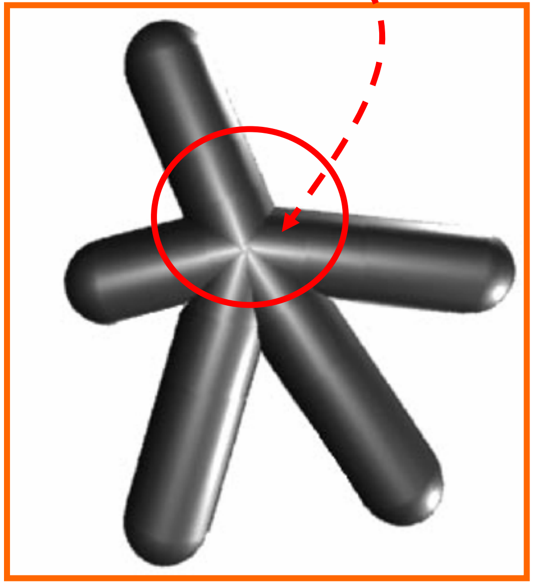
$$F_{total}(p) = \sum_i c_i F_i(r_i)$$

  - c_i: weight, positive or negative
  - F_i: field function as before
  - r_i: the distance from p to the nearest point on the i-th skeleton.

About Union of Distance Surfaces

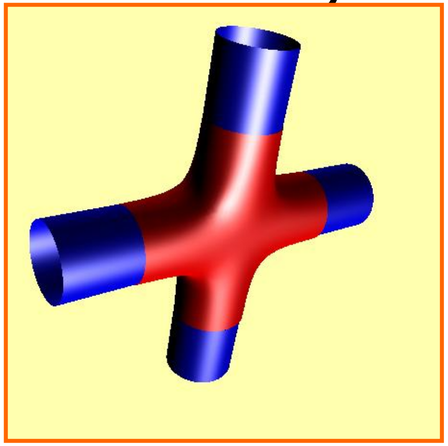
- Simple union of distance surfaces will produce bulges and creases may area where the skeletons meet.

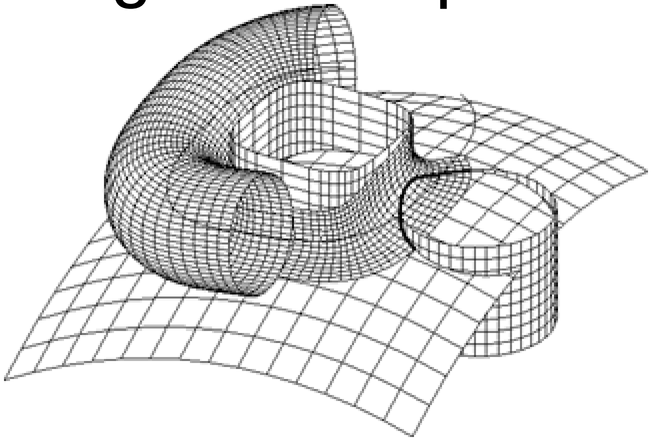




Algebraic Blends of Primitives

- To eliminate creases or bulge , the primitive volumes (“metaball ”) must form a *blend*, rather than a union.
- What is blending: smooth transition along common boundary among several primitives





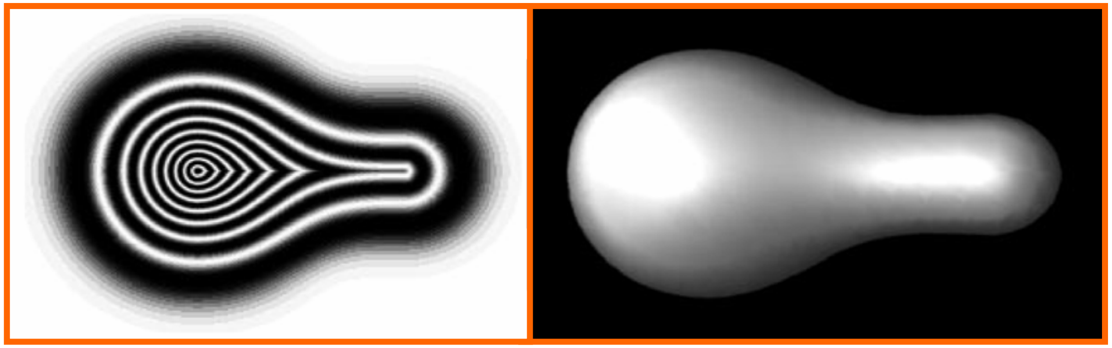
Blending Functions (1)

- Super-elliptical Blending

$$f(\mathbf{p}) = B(P_1, P_2) = 1 - \left[1 - \frac{P_1(\mathbf{p})}{r_1}\right]_+^t - \left[1 - \frac{P_2(\mathbf{p})}{r_2}\right]_+^t$$

  - P_1, P_2 are algebraic distances to skeletal elements 1 and 2, usually C^1 continuous
  - r_1 and r_2 are the ranges of influence for primitives P_1 and P_2
  - $[x]_+$ is *max* (0, *x*), and t is the 'thumbweight.'

Example of Blending Functions (1)

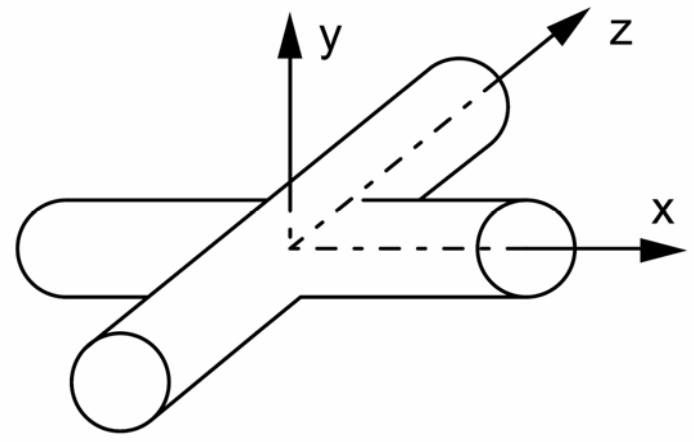


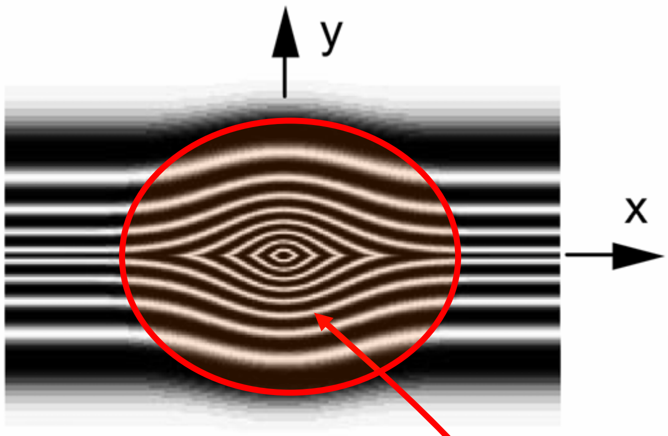
Super-elliptical Blending of sphere and cylinder primitives (t=3)

sphere: $P_1(\boldsymbol{p}) = (\boldsymbol{p}_x^2+\boldsymbol{p}_y^2+\boldsymbol{p}_z^2)^{1/2}-1$

$$\text{cylinder: } P_2(\boldsymbol{p}) = \begin{cases} (\boldsymbol{p}_x^2+\boldsymbol{p}_y^2+\boldsymbol{p}_z^2)^{1/2}-.4, & \boldsymbol{p}_x < 0 \\ (\boldsymbol{p}_y^2+\boldsymbol{p}_z^2)^{1/2}-.4, & 0 < \boldsymbol{p}_x < 2 \\ ((\boldsymbol{p}_x-2)^2+\boldsymbol{p}_y^2+\boldsymbol{p}_z^2)^{1/2}-.4, & \boldsymbol{p}_x > 2 \end{cases}$$

Example of Blending Functions (1)





Super-Elliptic Blend of Two Cylinders

There is a bulge where the cylinders intersect.

Blending Functions (2)

- Improved Super-elliptical Blending

$$B(P_1, P_2) = 1 - \left[1 - \frac{P_1(\mathbf{p})}{r_1(1 - \cos\theta))}\right]_+^t - \left[1 - \frac{P_2(\mathbf{p})}{r_2(1 - \cos\theta))}\right]_+^t$$

  - θ: the angle between the gradients of the two primitives at a point $\boldsymbol{p}$:
  - The influence range is diminished according to θ

Blending Functions (2)

- $\theta = 0$ the range is fully diminished and the simple union of the primitives results
- $\theta = 90°$(concave condition), the range is undiminished, and a blend occurs.
- To avoid enlarging the primitive ranges, $cos(\theta)$ must be nonnegative

simple union bulging blend use of $\cos$ use of nonnegative $\cos$

Blending Functions (3)

- Super-elliptical blend is extended to *k* primitives:

$$B_k = 1 - \sum_{i=1}^{k} \left[1 - \frac{P_i(\boldsymbol{p})}{r_i}\right]_+^t$$

Convolution Surface

- A bulge-free implicit blend technique

- The convolution surface treats a skeleton S as a set of points, each of which contributes to the implicit surface function according to its distance to $\boldsymbol{p}$.

Convolution Surface

- Blobby systems by Blinn

$$f(\boldsymbol{p}) = c - \sum_i e^{-\|\boldsymbol{p}-\boldsymbol{s}_i\|^2/2}$$

where $\boldsymbol{s}_i$ is a point on the skeleton, c is threshold (constant)

Convolution Surface

- If $S = \{\boldsymbol{s}_i\}$ is a set of infinitesimally spaced points, f can be expressed as an integral:

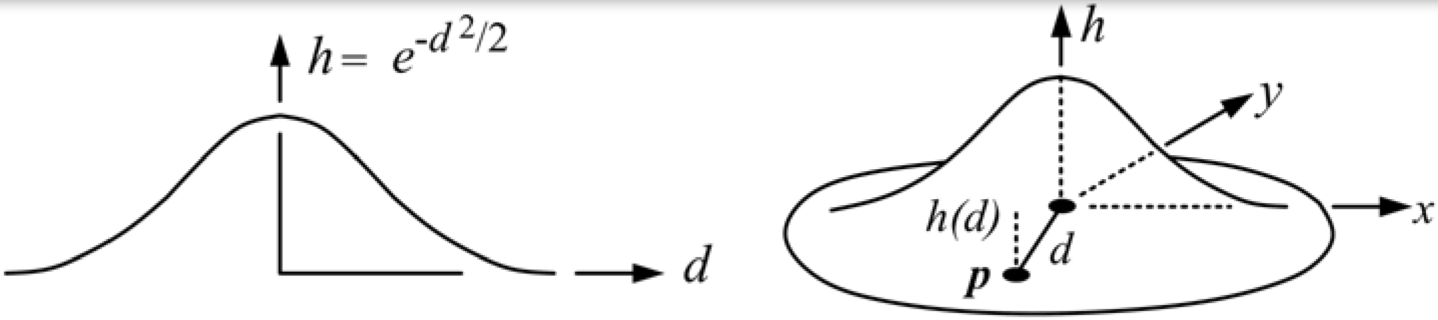
$$f(\boldsymbol{p}) = c - \int_S e^{-\|\boldsymbol{p}-\boldsymbol{u}\|^2/2}\, d\boldsymbol{u}$$

  - where $\boldsymbol{u}$ ranges over all points on the skeleton. It defines a convolution surface
  - Gaussian function is a 3D filter/kernel
- $S = \{\boldsymbol{s}_i\}$ can be any skeletons: point, line, curve, surface, volumes etc.

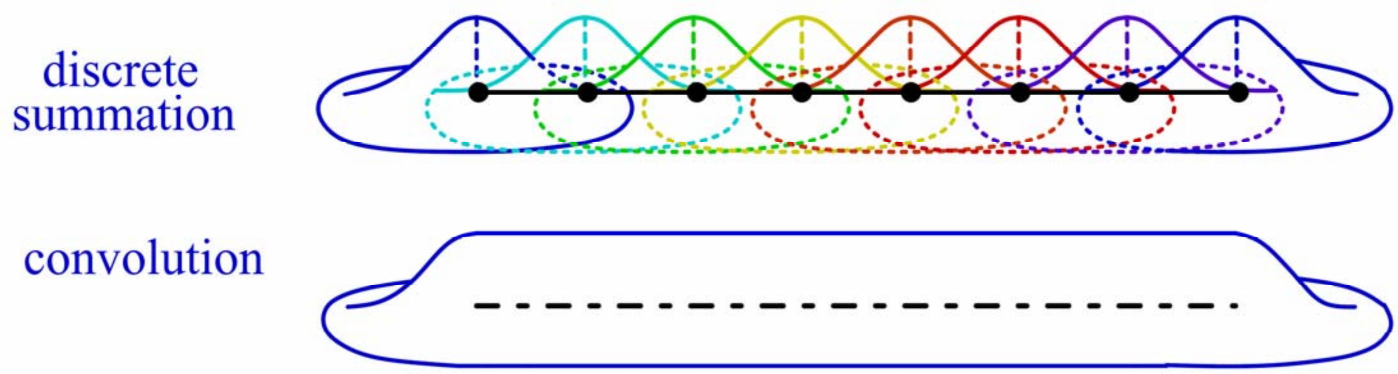
About Convolution Surface

- The convolution has been applied extensively in the processing of 1D audio signal, 2D image, 3D volume image.
- The convolution scaled the frequency components of the signal via filter(kernel)
  - Low-pass filter: smoothing effect, e.g., Gaussian kernel
  - High-pass filter: enhance detail effect

Gaussian Kernels



Gaussian Kernels: left: one-dimensional, right: two-dimensional



Infinite Sum is a Convolution

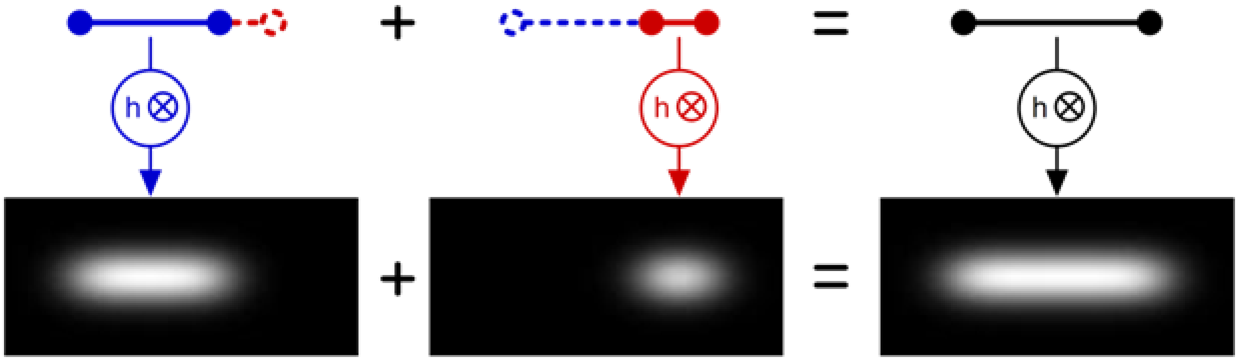
About Convolution Surface

- The convolution is a linear operator:

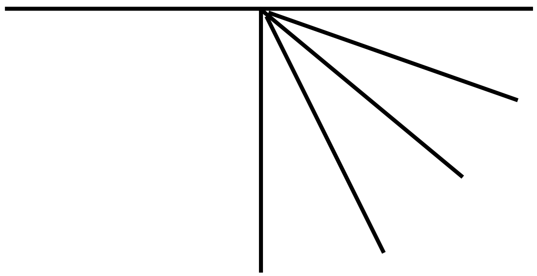
$$h \otimes (s_1 + s_2) = h \otimes s_1 + h \otimes s_2$$

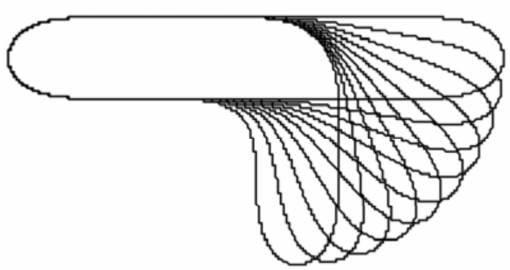
  - It is property of *superposition*
  - The sum of the convolutions of any division of a skeleton is identically equal to the single convolution of the entire skeleton
- The convolution surface is a smooth shape without introducing bulges.

Illustration of Superposition



The superposition Property: The two line segment are brought together (left); sum of convolutions of the two segments (right), convolution of single segment (bottom)





Two Segments Convolved with the Gaussian Kernel

About Convolution Surface

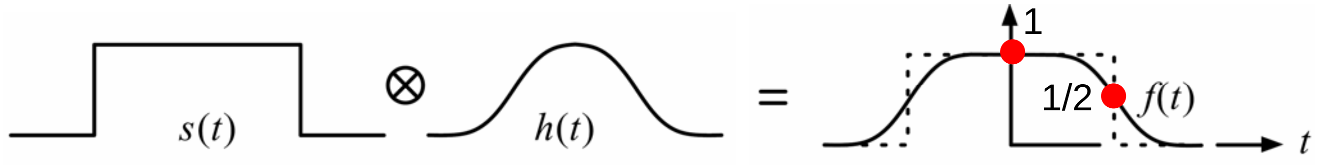
- Along convex portions of the skeleton the surface mimics the union operator
- Along concave portions, the surface yields a blend.
- For isolated convex skeletons, such as triangles or segments, convolution produces surfaces of similar shape to distance surfaces.
- For complex skeletons, however, convolution yields crease-free surfaces with adjacent primitives blending without seam or bulge.

Gaussian kernel

- For standard Gaussian kernel:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx = 1$$

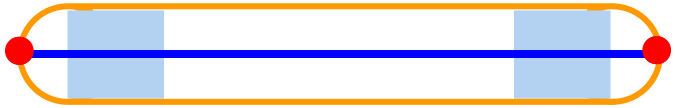
  - A signal modified by such a kernel will maintain its original energy!



Convolution of a box with unit-integral kernel

Energy Discussion in Convolution

- Because the Gaussian is symmetric, the convolution equals ½ where the box function undergoes transitions
  - For an iso-surface contour level of ½, the convolution surface will pass through the endpoints of skeletal segments and through the edges of skeletal polygons.



- A kernel with integral less (greater) than one would attenuate (amplify) a signal

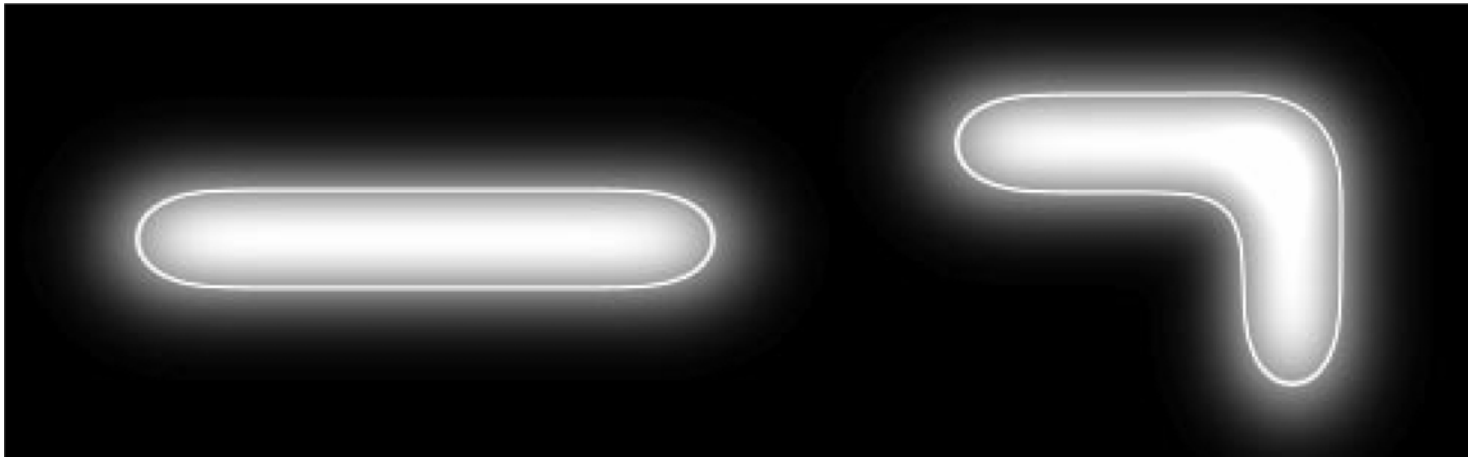
Gaussian kernel

- Advatnages of Gaussian Kernel
  - C^∞ Smooth
  - Unit integral
  - Separable

$$h(\boldsymbol{d}) = e^{-\|\boldsymbol{d}\|^2/2} = e^{-(\boldsymbol{d}_x{}^2+\boldsymbol{d}_y{}^2+\boldsymbol{d}_z{}^2)/2} = e^{-\boldsymbol{d}_x{}^2/2}\, e^{-\boldsymbol{d}_y{}^2/2}\, e^{-\boldsymbol{d}_z{}^2/2}.$$

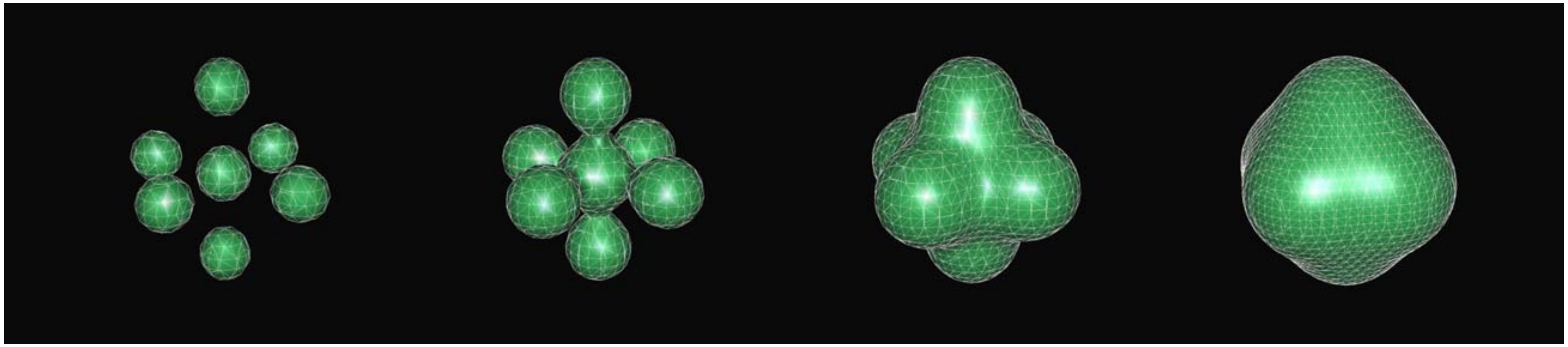
- Disadvantages of Gaussian Kernel
  - Globally Defined on the $\mathbf{R}^{1,2,3}$
  - Cannot be analytical integrated, only numerically approximated

Examples of Convolution Surfaces

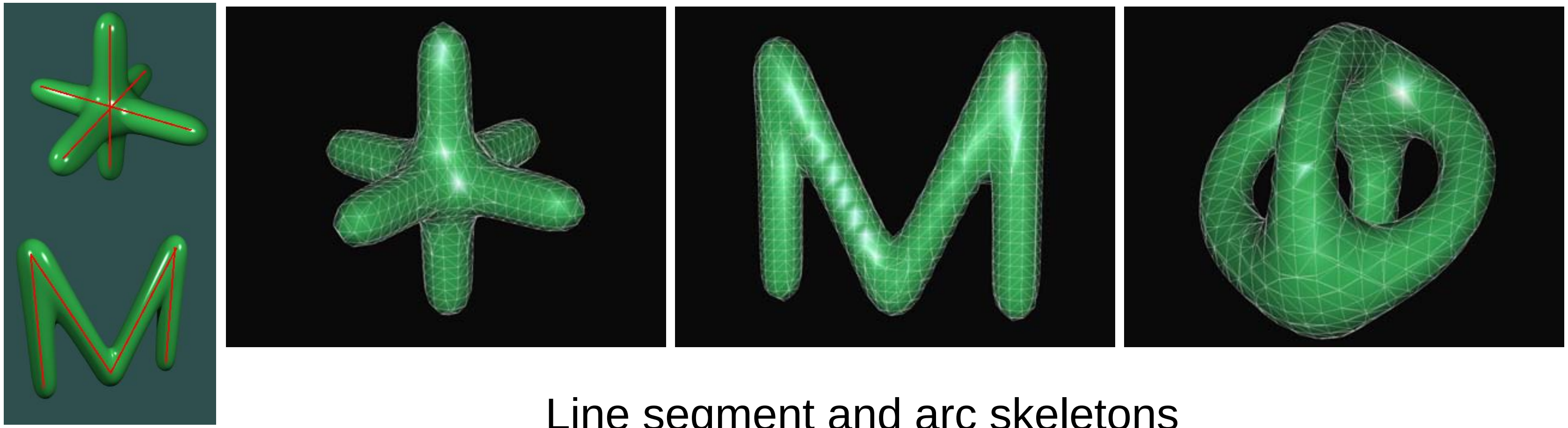


Two Segments Convolved with the Gaussian Kernel

Examples of Convolution Surfaces



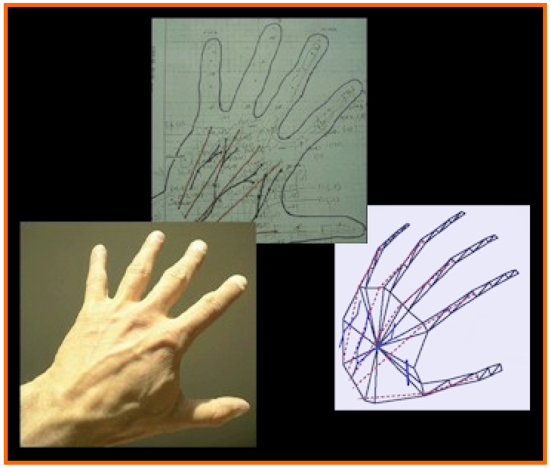
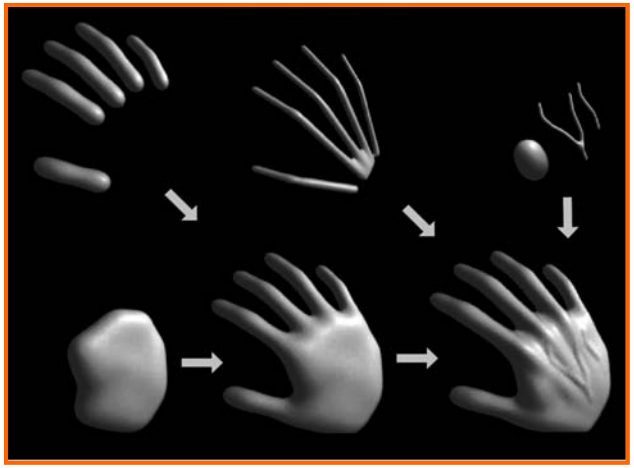
Threshold=1.0, 0.75, 0.5, 0.35



Line segment and arc skeletons

Examples of Convolution Surfaces

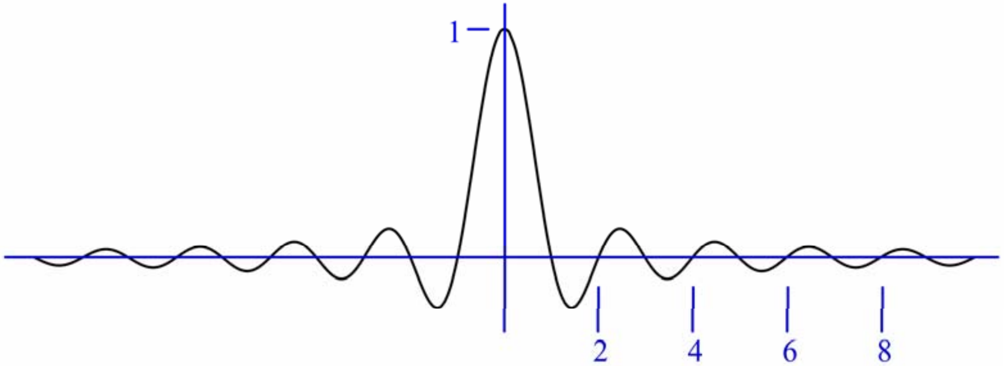

Convolution Surface


Convolution Surface: Hand Crafted by Jules Bloomenthal

Other kernels: Sinc Kernel

- Sinc kernel is : $\sin(\pi x)/\pi x$.

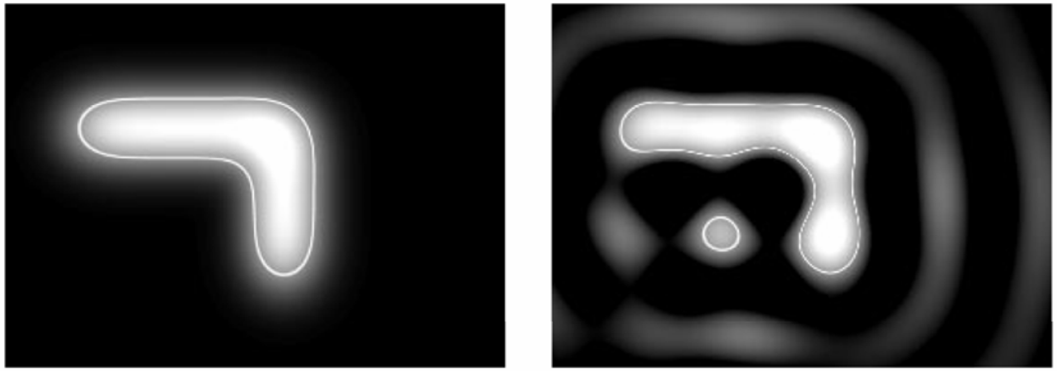


The Sinc Kernel

- The sinc is the ideal low-pass filter: its Fourier transform is the box function.
- Cutoff frequency effect

Other kernels: Sinc Kernel

- Problem with sinc kernel: none monotonically



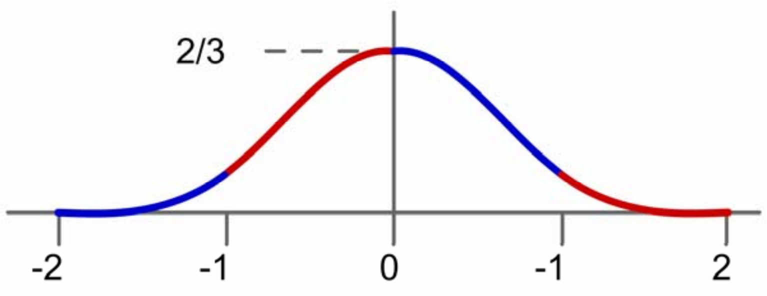
Convolution with Gaussian (left) and Sinc (right)

- Monotonicity of the kernel is a necessary property for a satisfactory convolution surface
  - Ringing of the filter may introduce a spurious contour

Other kernels: B-spline Kernels

- B-spline kernel has similar shape with Gaussian

$$h_{B\text{-}spline} = \begin{cases} (x+2)^3/6 & x \in (-2,-1) \\ (-3x^3-6x^2+4)/6 & x \in (-1,0) \\ (3x^3-6x^2+4)/6 & x \in (0,1) \\ (2-x)^3/6 & x \in (1,2) \end{cases}$$



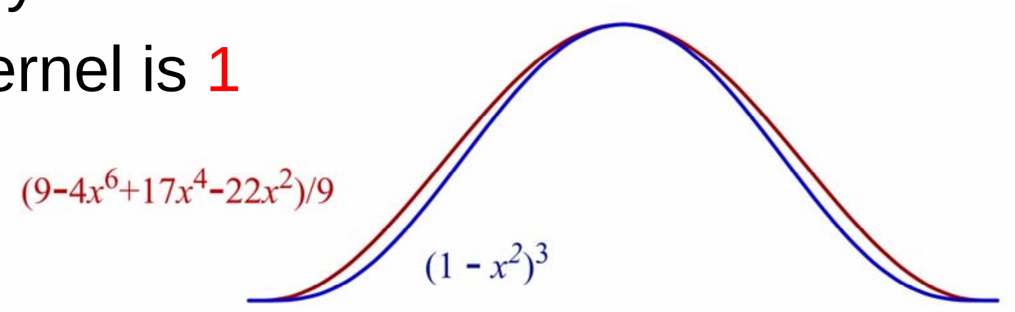
- $h_{B\text{-}spline}(0)=2/3,\ \ h_{B\text{-}spline}(0.72235)\approx 1/2$
- The support of kernel is 2
- The integral of B-Spline kernel is 1, not separable

Other kernels: Wyvill Kernels

- Wyvill kernel also has similar shape with Gaussian

$$h_{Wyvill}(x) = (9-4x^6+17x^4-22x^2)/9.$$

  - After scale: $h_{Wyvill}(0)=1, \; h_{Wyvill}(1)=1/2$
  - The integral of Wyvill kernel is 1
  - The support of kernel is 1
  - Not separable



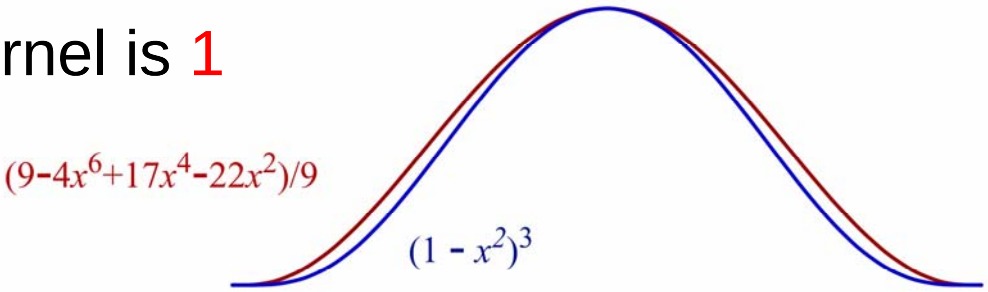
Wyvill kernel (red)

Other kernels: Modified Wyvill Kernels

- Modified Wyvill kernel also has similar shape with Gaussian

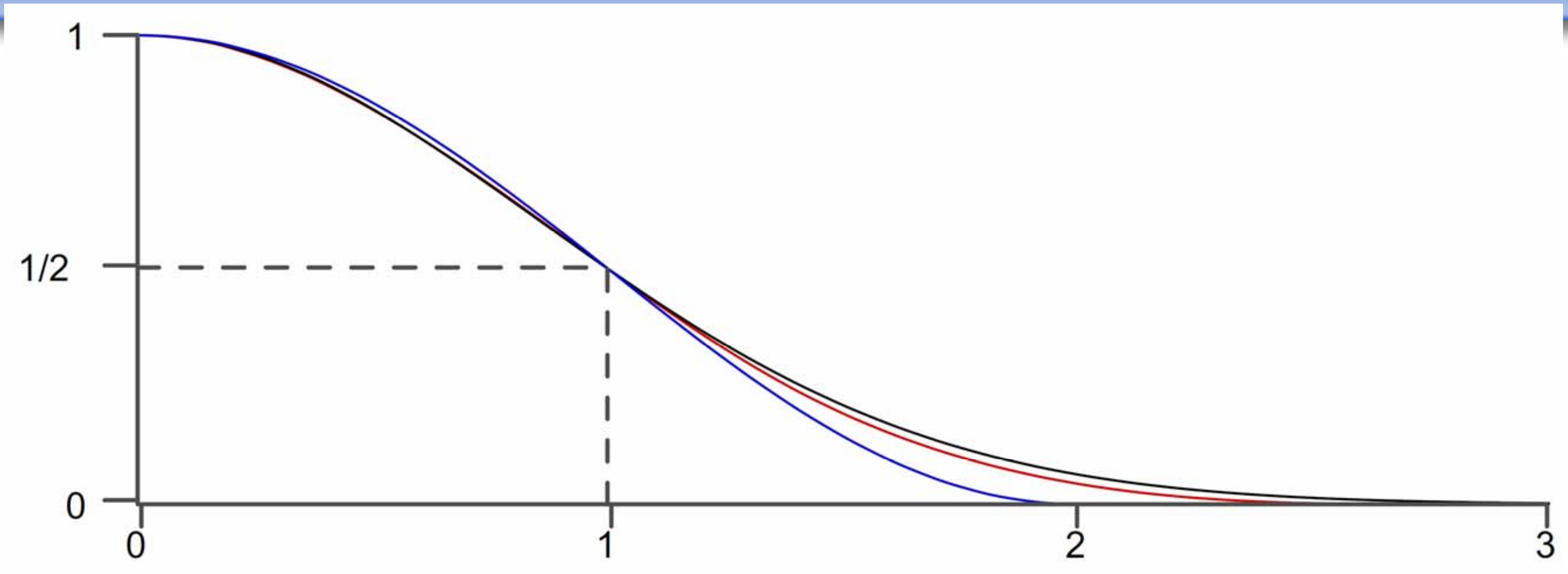
$$h_{NewWyvill}(x) = (1\text{-}x^2)^3.$$

  - After scale: $h_{Wyvill}(0)=1$
  - The integral of Modified Wyvill kernel is 1
  - The support of kernel is 1
  - Not separable



Modified Wyvill kernel (blue)

Compare the different kernels



Comparison of Filter Kernels after scale

upper (in black): the Gaussian kernel, $e^{-0.69314715x2}$

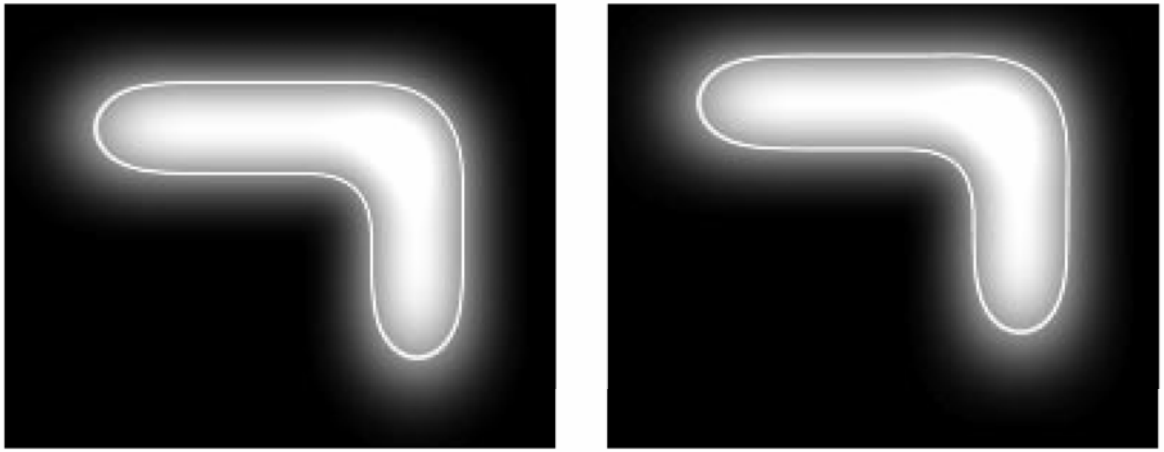
middle (in red): the B-spline kernel, $(1.5)h_{\mathrm{Bspline}}(0.72235x)$

lower (in blue): the Wyvill kernel, $h_{Wyvill}(0.5x)$

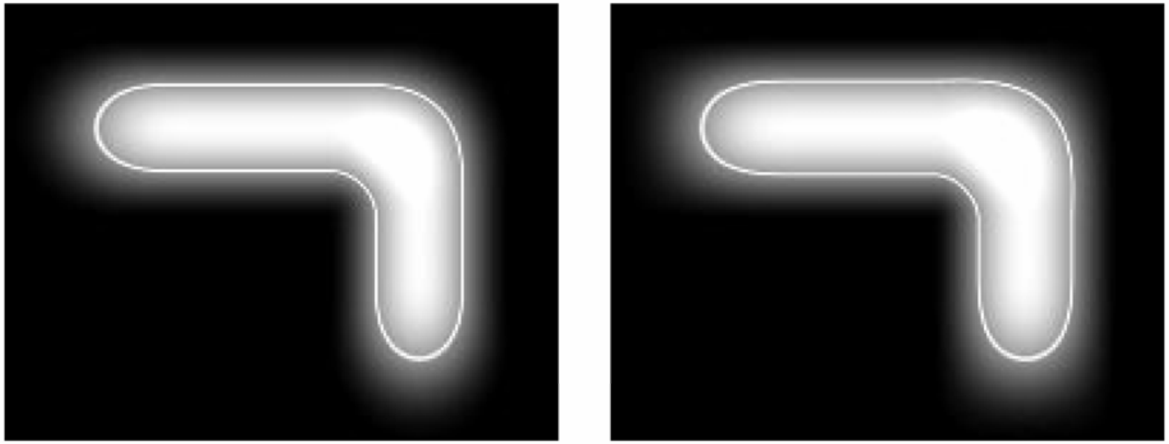
Computation of Convolution

- The Gaussian is separable and spherically symmetric
  - Integration by part!
- B-spline and Wyvill kernel is not separable
  - summation of point source terms
  - the product of integration and distance filters.

Computation of Convolution

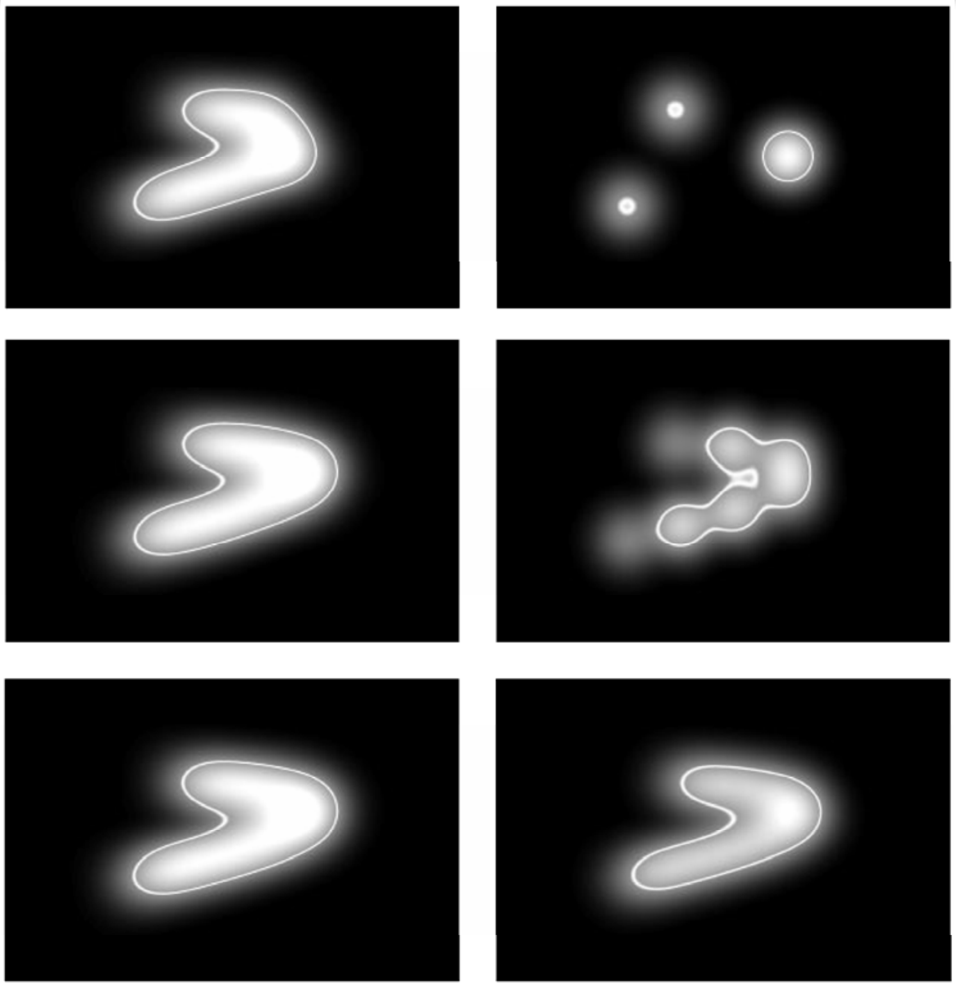


Approximations to B-Spline Convolution
left: summation of point sources
right: product of filters



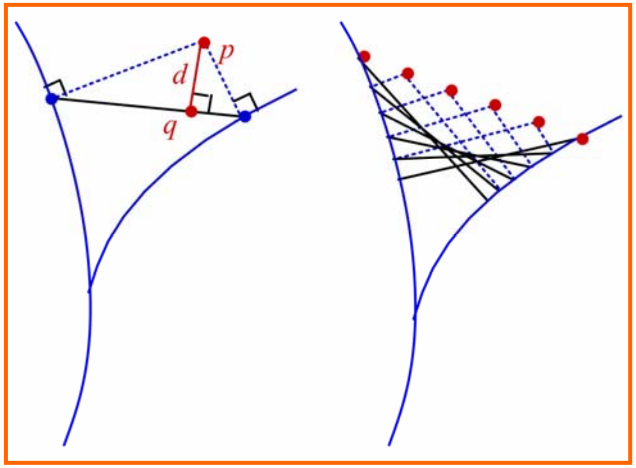
Approximations to Wyvill Convolution
left: summation of point sources
right: product of filters

Computation of Convolution

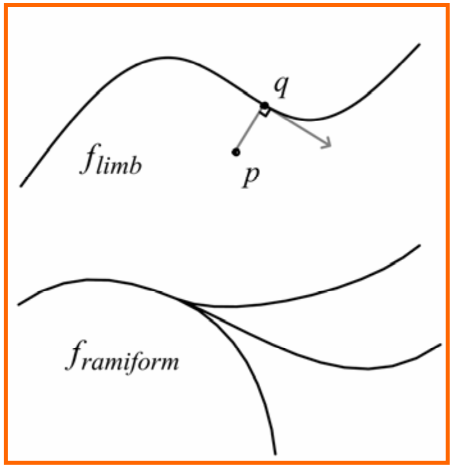


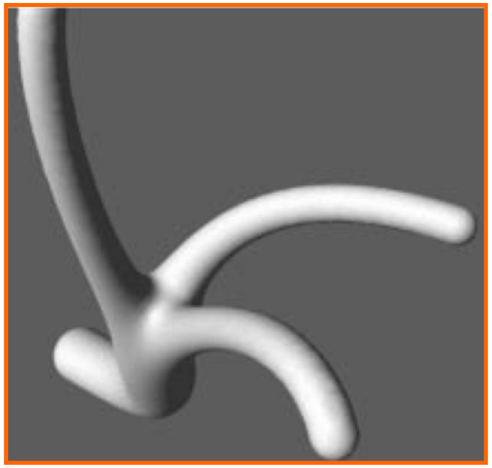
Gaussian Convolution approximation for 3, 7, and 15 Segments (left) or Points (right)

Ramification Modeling



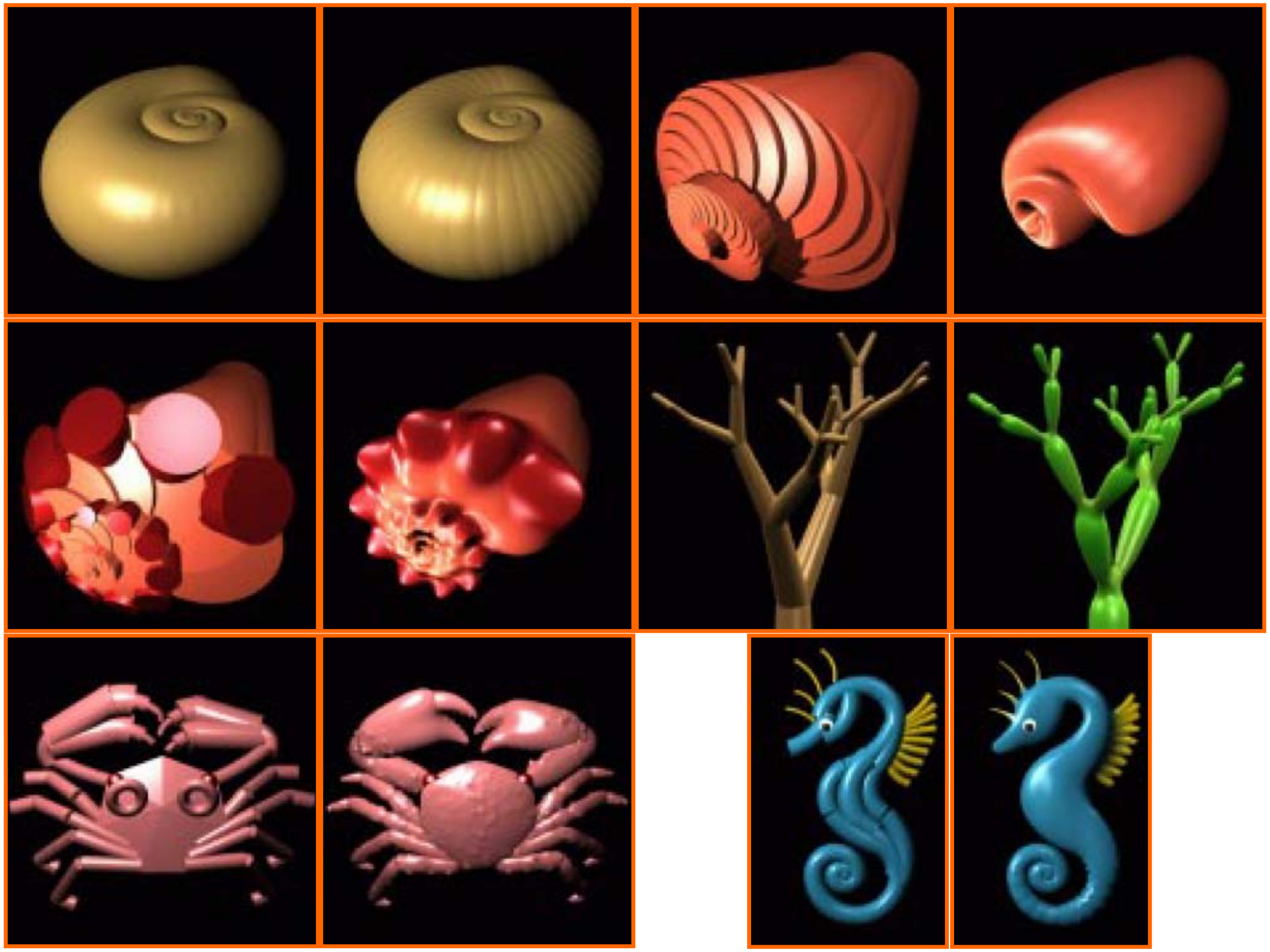
A Two-Ramiform with Constant Radii



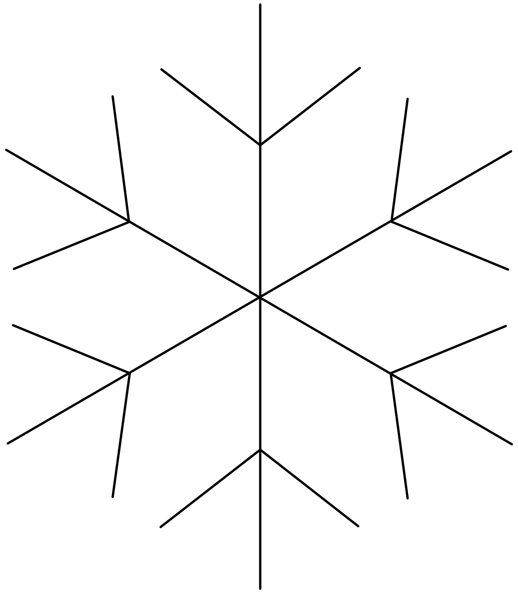


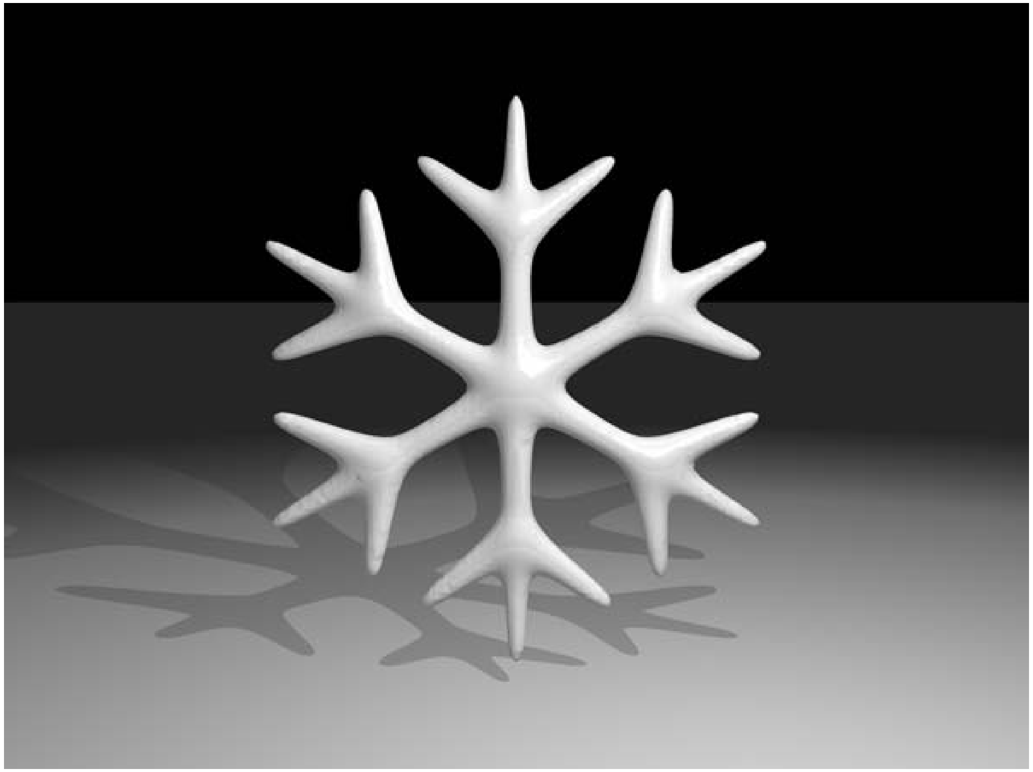
A Trifurcated Ramiform

Please Enjoy More Examples of Convolution Surface



Please Enjoy More Examples of Convolution Surface



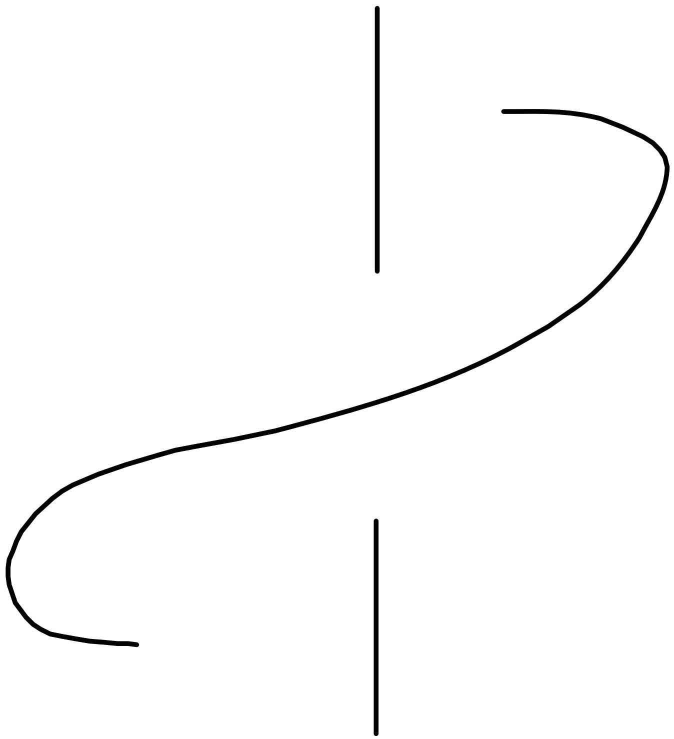


Snowflake

Potted plant

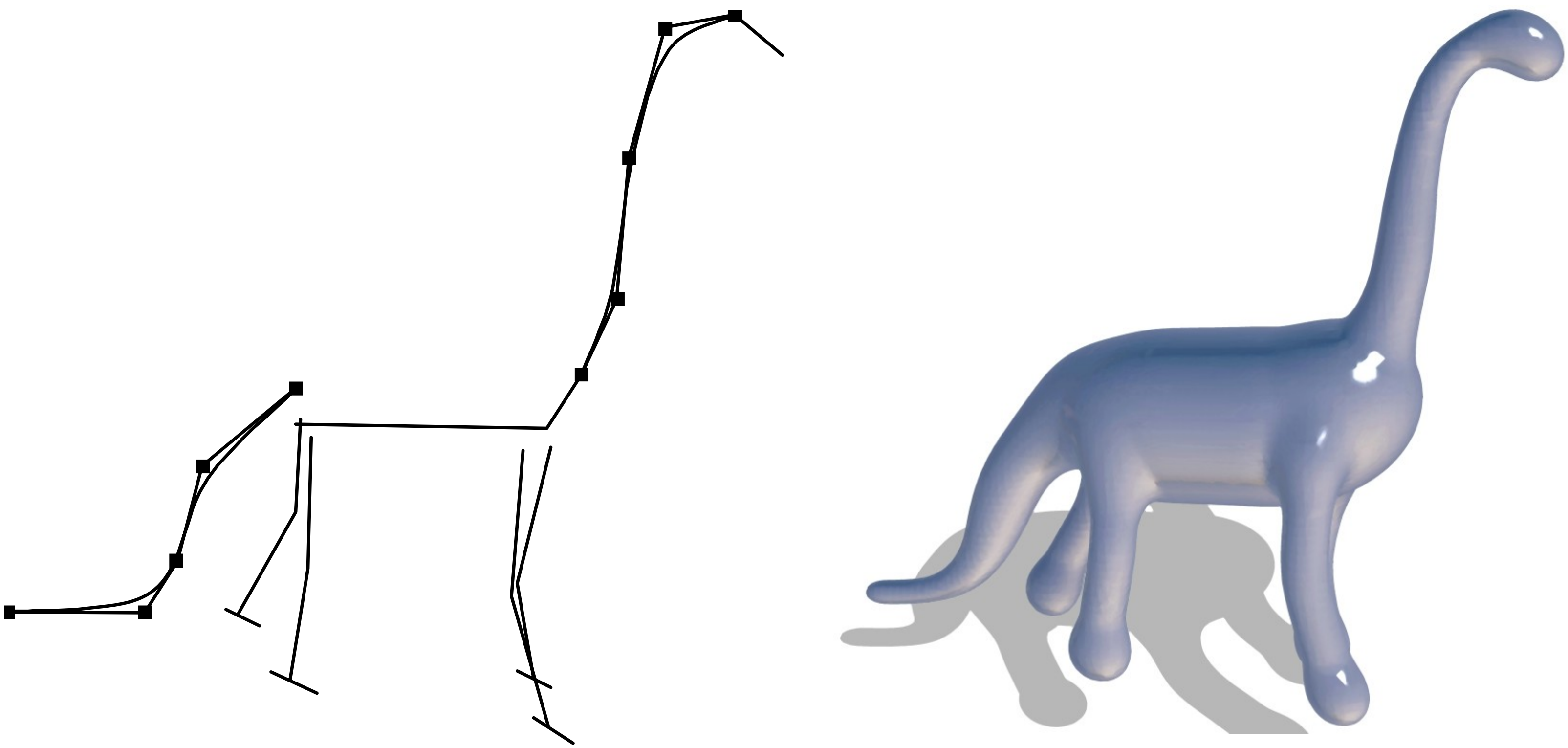
Maple tree

Please Enjoy More Examples of Convolution Surface



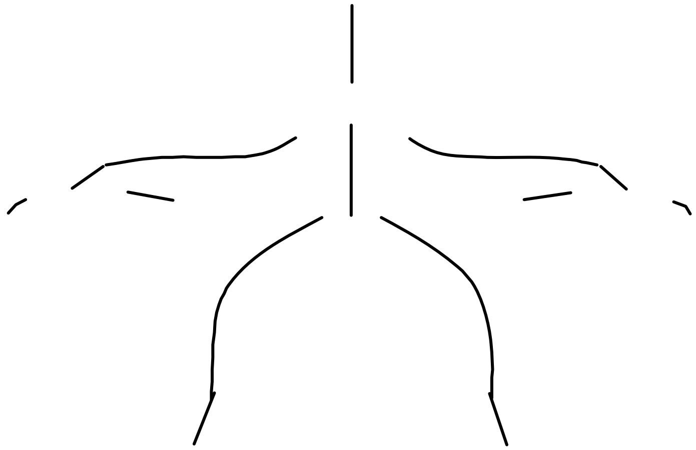
A gourd shape

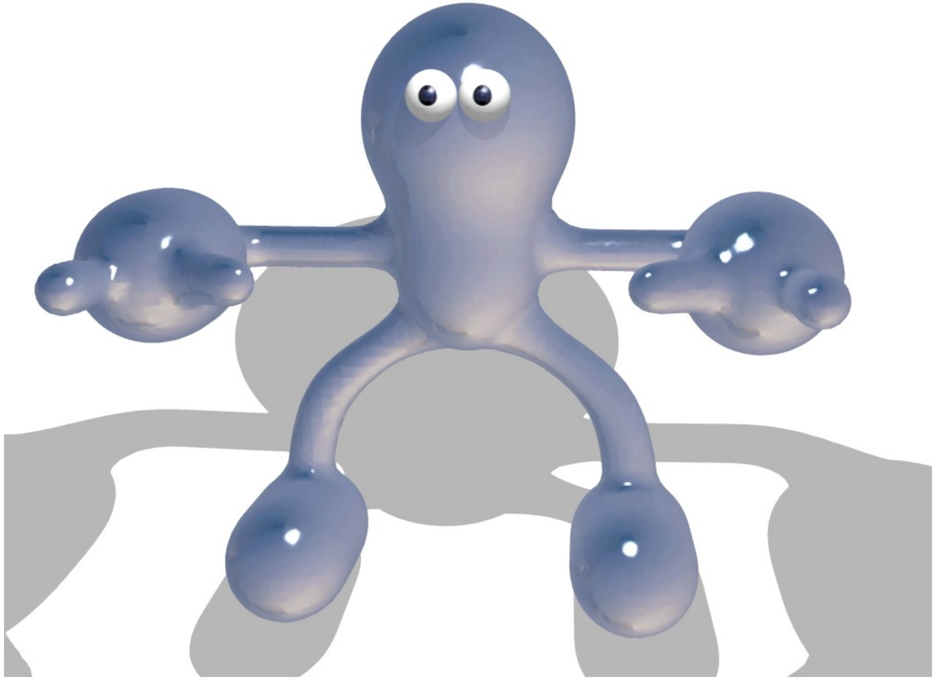
Please Enjoy More Examples of Convolution Surface



A dinosaur

Please Enjoy More Examples of Convolution Surface





An cartoon enforcer

Quadric Surfaces

- Basics of Quadric Surfaces
- Properties of Quadric Surfaces
- Representation of Quadric Surfaces
- Applications of Quadric Surfaces

Basics of Quadric Surfaces

- A *general* quadric surface is

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz + 2Gx + 2Hy + 2Jz + K = 0$$

- Its matrix representation

$$\{x \mid F(x) = x^T Q x = 0\}$$

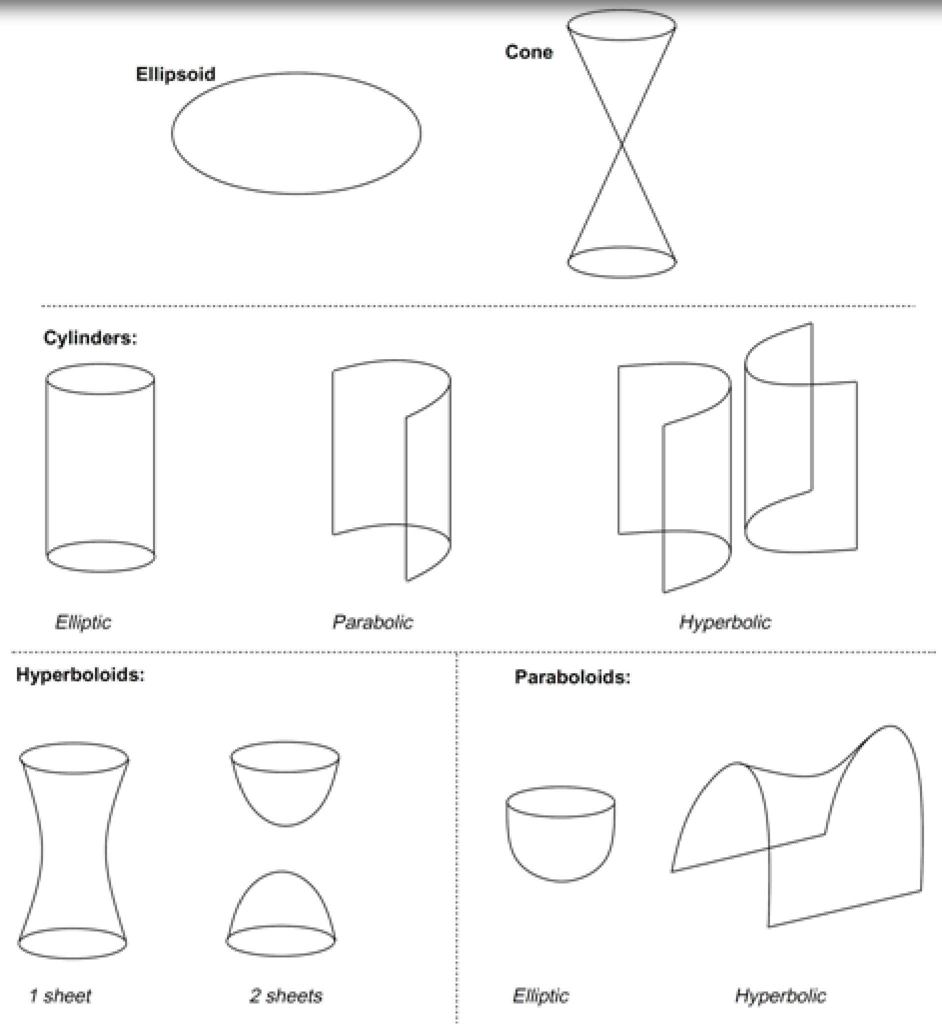
where

$$Q = \begin{bmatrix} A & D & E & G \\ D & B & F & H \\ E & F & C & J \\ G & H & J & K \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Basics of Quadric Surfaces

- There are nine types of general quadrics according coefficients of this algebraic equation
  - Ellipsoid
  - Elliptic cone
  - Cylinder (elliptic, parabolic, hyperbolic)
  - Hyperboloid (of 1 sheet, of 2 sheets)
  - Paraboloid (elliptic, hyperbolic)
- Degenerated cases
  - point, plane, parallel planes,
- Invalid shapes
  - imaginary quadric, intersecting imaginary planes, imaginary parallel planes

Sketches of General Quadric Surfaces

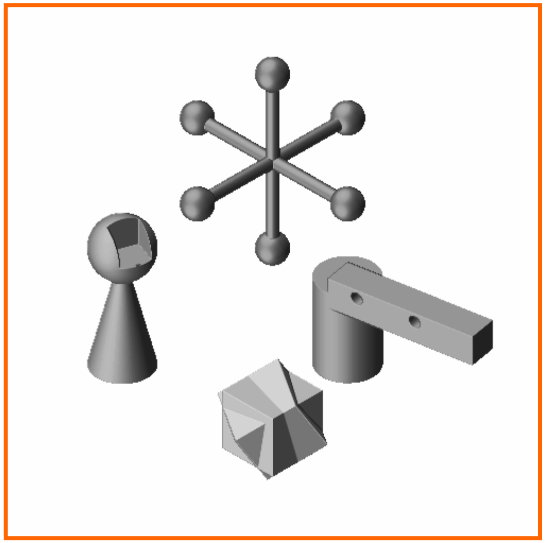


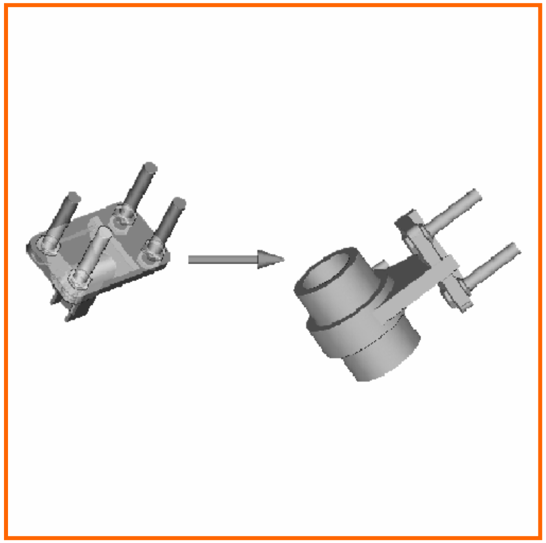
Natural Quadrics

- Natural Quadrics: subset of general
  - sphere (special case of ellipsoid)
  - right circular cone (special case of elliptic cone)
  - right circular cylinder (special case of elliptic cylinder)
  - Planes

Natural Quadrics

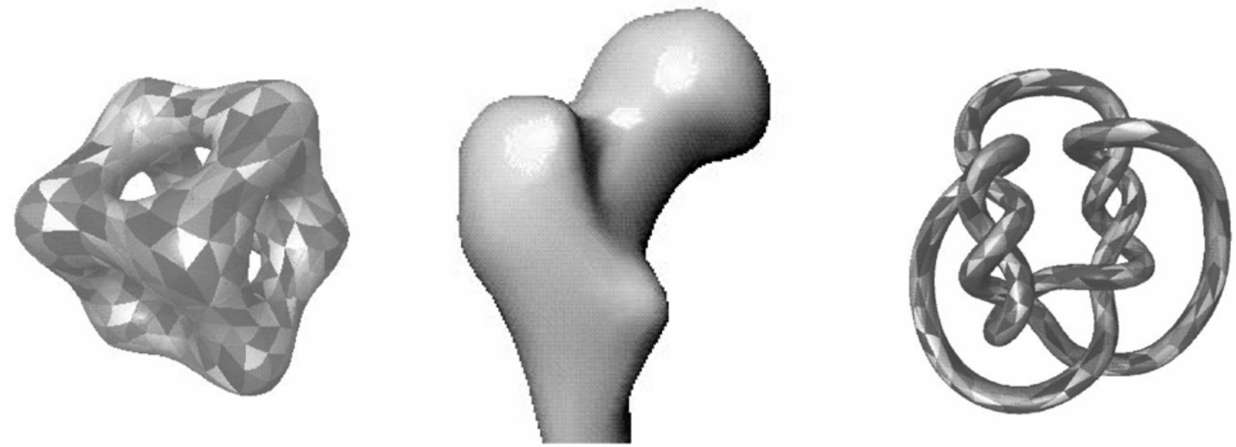
- Natural quadrics are by far the most popular set of quadric: Prismatic Solid
  - Mechanical CAD domains





Quadric Surface Patches

- Pieced together with tangent plane (G^1) continuity to create free-form shapes



Some complex free-form solids modeled with quadric algebraic patches: (a) a genus five solid with 2,400 quadratic algebraic patches, (b) a bone head, the entire bone was modeled with 5,696 quadratic algebraic patches, and (c) a knot modeled with 6,912 quadratic algebraic patches.

Quadric Surface Patches

(a) A vase modeled with 336 quadratic algebraic patches, rendered with transparency to show some of its trunctets. (b) The same vase in an environment with multiple light sources and participating media.

Properties of Quadric Surfaces

- Let Q_A (Q_B) denotes the matrix representation of quadric surface in frame A(B), M denotes an affine transformation from frame A to from B

  Then the transformation of quadrics is

$$Q_B = M^T Q_A M$$

Properties of Quadric Surfaces

- Let Q_u denotes upper-left 3×3 sub-matrix of Q

$$Q_u = \begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix}$$

  - The rank of both Q and Q_u are invariant under affine transformation
    - The type of quadric surface is invariant under affine transformations
  - The quadric surfaces are invariant under rigid motions

Characteristic Function

- Characteristic functions are

$$\mathrm{Det}(Q\text{-}\lambda I) \quad \mathrm{Det}(Q_u\text{-}\lambda I)$$

  - Invariant under rigid motion
  - Used for classifying the type of quadrics according to its eignvalues

Characteristic Function (1)

$$\mathrm{Det}(Q - \lambda I) = \lambda^4 + D_1\lambda^3 + D_2\lambda^2 + D_3\lambda + D_4$$

$$\begin{aligned}
D_1 &= A + B + C + K \\
D_2 &= AB + BC + CK + AK + AC + BK \\
&\quad - D^2 - E^2 - F^2 - G^2 - H^2 - J^2 \\
D_3 &= ABC + ABK + ACK + BCK + 2(DEF + FGJ + DGH + EHJ) \\
&\quad - (C + K)D^2 - (A + K)E^2 - (B + K)F^2 \\
&\quad - (B + C)G^2 - (A + C)H^2 - (A + B)J^2 \\
D_4 &= \mathrm{Det}(Q)
\end{aligned}$$

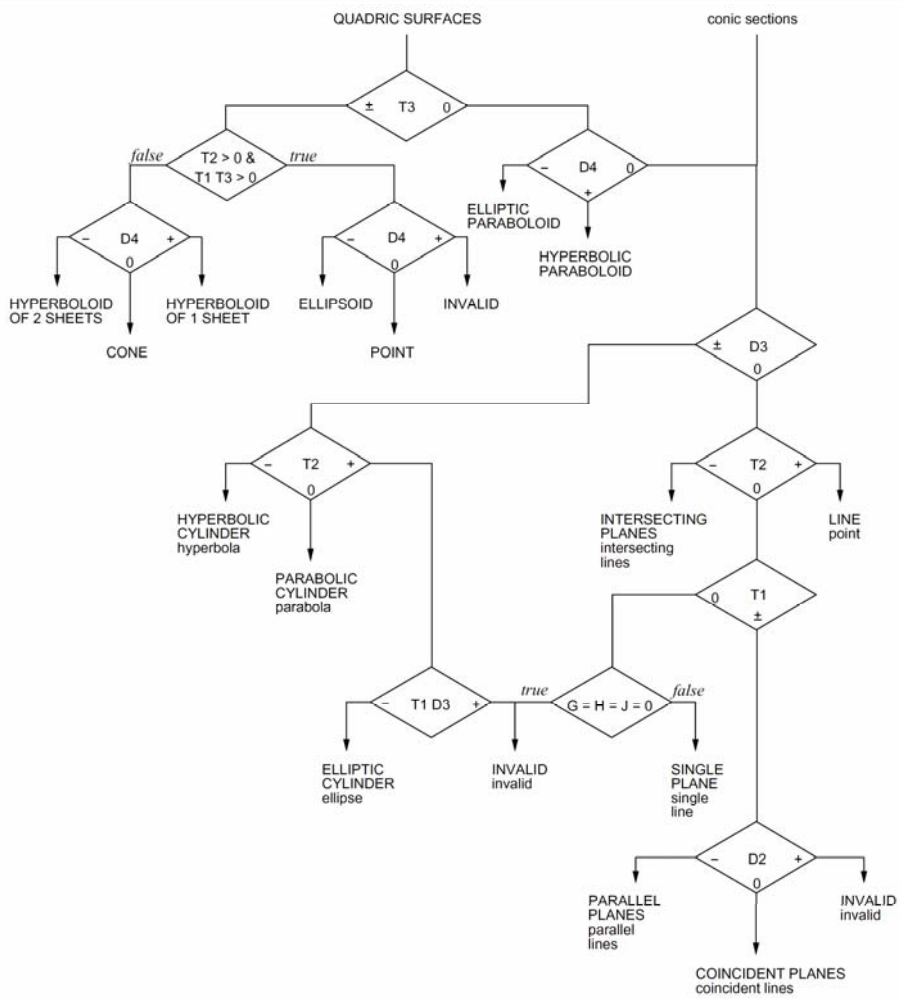
Characteristic Function (2)

$$\mathrm{Det}(Q_u - \lambda I) = \lambda^3 + T_1\lambda^2 + T_2\lambda + T_3$$

$$\begin{aligned} T_1 &= A + B + C \\ T_2 &= AB + BC + AC - D^2 - E^2 - F^2 \\ T_3 &= \mathrm{Det}(Q_u) \end{aligned}$$

Types of Quadric Surfaces

- Types are decided by using a decision tree



Ruled Quadric Surfaces

- Ruled Quadric Surfaces
  - A family of straight lines can be found which lie entirely on the quadric surface
  - cylinders, cones, hyperbolic paraboloid, hyperboloid of one sheet

Pencils of two Quadric Surfaces

- $F_1(x,y,z)$ and $F_2(x,y,z)$ are two quadric surfaces, then the pencil of them is

  $$F_1(x,y,z)+\alpha F_2(x,y,z)$$

  where α is real scalar

  - Every quadric surface in the pencil of $F_1(x,y,z)$ and $F_2(x,y,z)$ pass the intersection curves of $F_1(x,y,z)$ and $F_2(x,y,z)$

    $$F_1(x,y,z)=0 \text{ and } F_2(x,y,z)=0$$

    Thus

    $$F_1(x,y,z)+\alpha F_2(x,y,z)=0$$

Pencils of two Quadric Surface

- At least one ruled quadric in the pencil of any two general quadrics
  - Ruled quadrics are easily parameterized
  - The quadric surface intersection curves can be expressed parametrically using the parameterization scheme for the ruled quadric in the pencil

Representation of Quadric Surfaces

- Two representation techniques:
  - Algebraic
  - Geometric

Algebraic Representation of Quadric Surfaces

- Represented by ten coefficients of quadric implicit polynomial equation $F(x,y,z)$:
  - Advantage: a common set of routines can be written for handling all types of general quadric surfaces
  - Disadvantage: lack of computational robustness
    - Floating point data of imperfect accuracy
    - Internal inconsistency

Geometric Representation of Quadric Surfaces

- Represented by

  {1 point, 2 orthogonal unit vectors, 3 scalars}

  - The point fixes the position of the surface
  - The vectors define its orientation or axes
  - The scalars determine its dimensions
- Example: Ellipsoid
  - Its center (a point)
  - Two of its three orthogonal axes (two orthogonal unit vectors)
  - Three lengths (radii) along its three axes (three scalars)

Geometric Representation of Quadric Surfaces

- Advantages: its robustness and internal consistency
- Disadvantage: a large number of routines are needed to handle all the special cases that arise as a result of each type of surface being treated as a separate entity (e.g. a problem of combinatorial explosion in the number of intersection routines)

Applications of Quadric Surfaces

- A surface/surface or curve/surface intersection: computationally and topologically tractable as compared to parametric patches.
  - For example,
    a quadric surface $F(x,y,z)=0$
    a line $L:(x(t),y(t),z(t))$,
    line and surface intersection $F(x(t),y(t),z(t))=0$ is a uni-variate quadratic function.
- Quadratic algebraic patches with smooth continuity for modeling complex free-form shapes

Download Course

http://www.cad.zju.edu.cn/home/jqfeng/GM/GM07.zip