Subdivision Surface (2)

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Subdivision Methods for Quadratic B-Spline Surfaces

- Overview
- The Matrix Equation for a Uniform Biquadratic B-Spline Surface
- Subdividing the Surface
- Generating the Refinement Procedure
- Summary

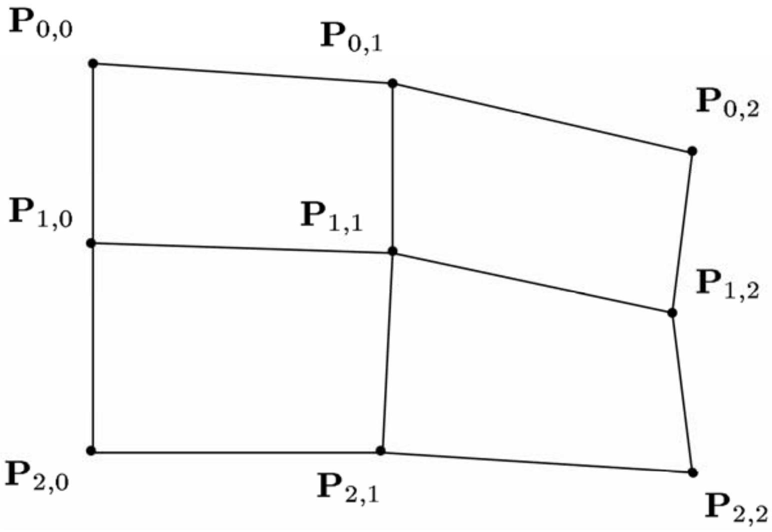
Overview

- Subdivision surfaces are based upon the binary subdivision of the uniform B-spline surface.
  - a initial polygonal mesh + a subdivision (or *refinement*) operation = a new polygonal mesh that has a greater number of polygonal elements
  - it is hopefully "closer" to some resulting surface.
  - By repetitively applying the subdivision procedure to the initial mesh, we generate a sequence of meshes that (hopefully) converges to a resulting surface.

The Matrix Equation for a Uniform Biquadratic B-Spline Surface

Consider the biquadratic uniform B-spline surface $\mathbf{P}(u,v)$ defined by the 3×3 array of control points

$$P = \begin{bmatrix} \mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} \\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} \\ \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2} \end{bmatrix}$$



The Matrix Equation for a Uniform Biquadratic B-Spline Surface

Biquadratic B-spline surface can be written as the following equation in matrix form

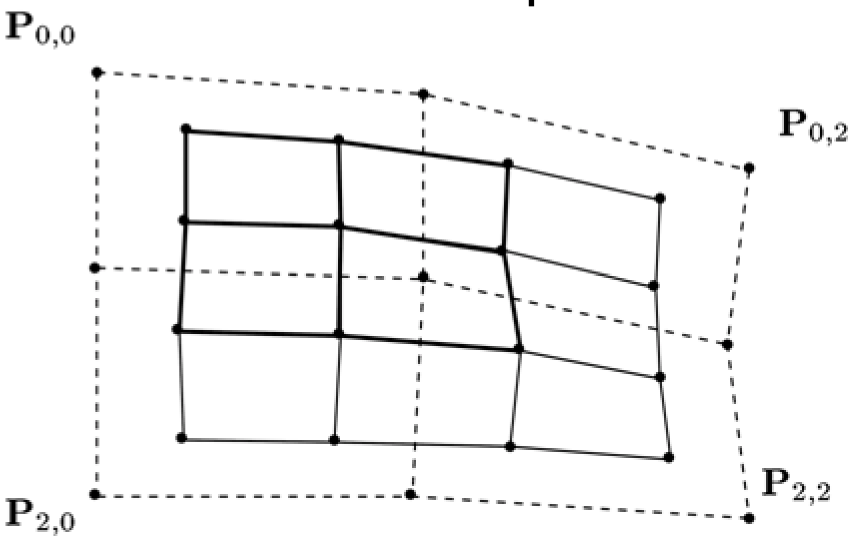
$$\mathbf{P}(u,v) = \begin{bmatrix} 1 & u & u^2 \end{bmatrix} MPM^T \begin{bmatrix} 1 \\ v \\ v^2 \end{bmatrix}$$

where M is the 3×3 matrix

$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

Subdividing the Surface

- This patch can be subdivided into four subpatches, 16 sub-control points.
  - the subpatch corresponding to $0 \le u, v \le 1/2$,
  - the four "interior" control points



Subdividing the Surface

To subdivide the surface, we reparameterizing of the surface by $u'=u/2$ and $v'= v/2$ and define new the surface $\mathbf{P}'(u,v)$

$$
\begin{aligned}
\mathbf{P}'(u,v) &= \mathbf{P}(\frac{u}{2},\frac{v}{2}) \\
&= \begin{bmatrix} 1 & \frac{u}{2} & (\frac{u}{2})^2 \end{bmatrix} MPM^T \begin{bmatrix} 1 \\ \frac{v}{2} \\ (\frac{v}{2})^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & u & u^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} MPM^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}^T \begin{bmatrix} 1 \\ v \\ v^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & u & u^2 \end{bmatrix} MM^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} MPM^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}^T (M^{-1})^T M^T \begin{bmatrix} 1 \\ v \\ v^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & u & u^2 \end{bmatrix} M \left(M^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} M \right) P \left(M^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}^T (M^{-1})^T \right) M^T \begin{bmatrix} 1 \\ v \\ v^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & u & u^2 \end{bmatrix} M \left(M^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} M \right) P \left(M^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} M \right)^T M^T \begin{bmatrix} 1 \\ v \\ v^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & u & u^2 \end{bmatrix} MP'M^T \begin{bmatrix} 1 \\ v \\ v^2 \end{bmatrix}
\end{aligned}
$$

Subdividing the Surface

where $P' = SPS^T$, and

$$S = M^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} M$$

The surface $\mathbf{P}'(u,v)$ is written as

$$\mathbf{P}'(u,v) = \begin{bmatrix} 1 & u & u^2 \end{bmatrix} MP'M^T \begin{bmatrix} 1 \\ v \\ v^2 \end{bmatrix}$$

The $\mathbf{P}'(u,v)$ is a uniform biquadratic B-spline patch

Subdividing the Surface

The matrix S is called the "splitting matrix", and is given by

$$
\begin{aligned}
S &= \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix}
\end{aligned}
$$

Subdividing the Surface

The new control point mesh $P' = SPS^T$ corresponding to the subdivided patch

$$S = \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

Subdividing the Surface

The $\mathbf{P}'_{i,j}$ can be written as

$$
\begin{aligned}
\mathbf{P}'_{0,0} &= \frac{1}{16}(3(3\mathbf{P}_{0,0}+\mathbf{P}_{1,0})+(3\mathbf{P}_{0,1}+\mathbf{P}_{1,1})) \\
\mathbf{P}'_{0,1} &= \frac{1}{16}((3\mathbf{P}_{0,0}+\mathbf{P}_{1,0})+3(3\mathbf{P}_{0,1}+\mathbf{P}_{1,1})) \\
\mathbf{P}'_{0,2} &= \frac{1}{16}(3(3\mathbf{P}_{0,1}+\mathbf{P}_{1,1})+(3\mathbf{P}_{0,2}+\mathbf{P}_{1,2})) \\
\mathbf{P}'_{1,0} &= \frac{1}{16}(3(\mathbf{P}_{0,0}+3\mathbf{P}_{1,0})+(\mathbf{P}_{0,1}+3\mathbf{P}_{1,1})) \\
\mathbf{P}'_{1,1} &= \frac{1}{16}((\mathbf{P}_{0,0}+3\mathbf{P}_{1,0})+3(\mathbf{P}_{0,1}+3\mathbf{P}_{1,1})) \\
\mathbf{P}'_{1,2} &= \frac{1}{16}(3(\mathbf{P}_{0,1}+3\mathbf{P}_{1,1})+(\mathbf{P}_{0,2}+3\mathbf{P}_{1,2})) \\
\mathbf{P}'_{2,0} &= \frac{1}{16}(3(3\mathbf{P}_{1,0}+\mathbf{P}_{2,0})+(3\mathbf{P}_{1,1}+\mathbf{P}_{2,1})) \\
\mathbf{P}'_{2,1} &= \frac{1}{16}((3\mathbf{P}_{1,0}+\mathbf{P}_{2,0})+3(3\mathbf{P}_{1,1}+\mathbf{P}_{2,1})) \\
\mathbf{P}'_{2,2} &= \frac{1}{16}(3(3\mathbf{P}_{1,1}+\mathbf{P}_{2,1})+(3\mathbf{P}_{1,2}+\mathbf{P}_{2,2}))
\end{aligned}
$$

Subdividing the Surface

- These equations can be looked at in two ways
  - *Subdivision masks*: weighing the four points in the ratio of 9-3-3-1

$$\begin{array}{ccc} 9 & — & 3 \\ | & & | \\ 3 & — & 1 \end{array} \qquad \begin{array}{ccc} 3 & — & 1 \\ | & & | \\ 9 & — & 3 \end{array} \qquad \begin{array}{ccc} 1 & — & 3 \\ | & & | \\ 3 & — & 9 \end{array} \qquad \begin{array}{ccc} 3 & — & 9 \\ | & & | \\ 1 & — & 3 \end{array}$$

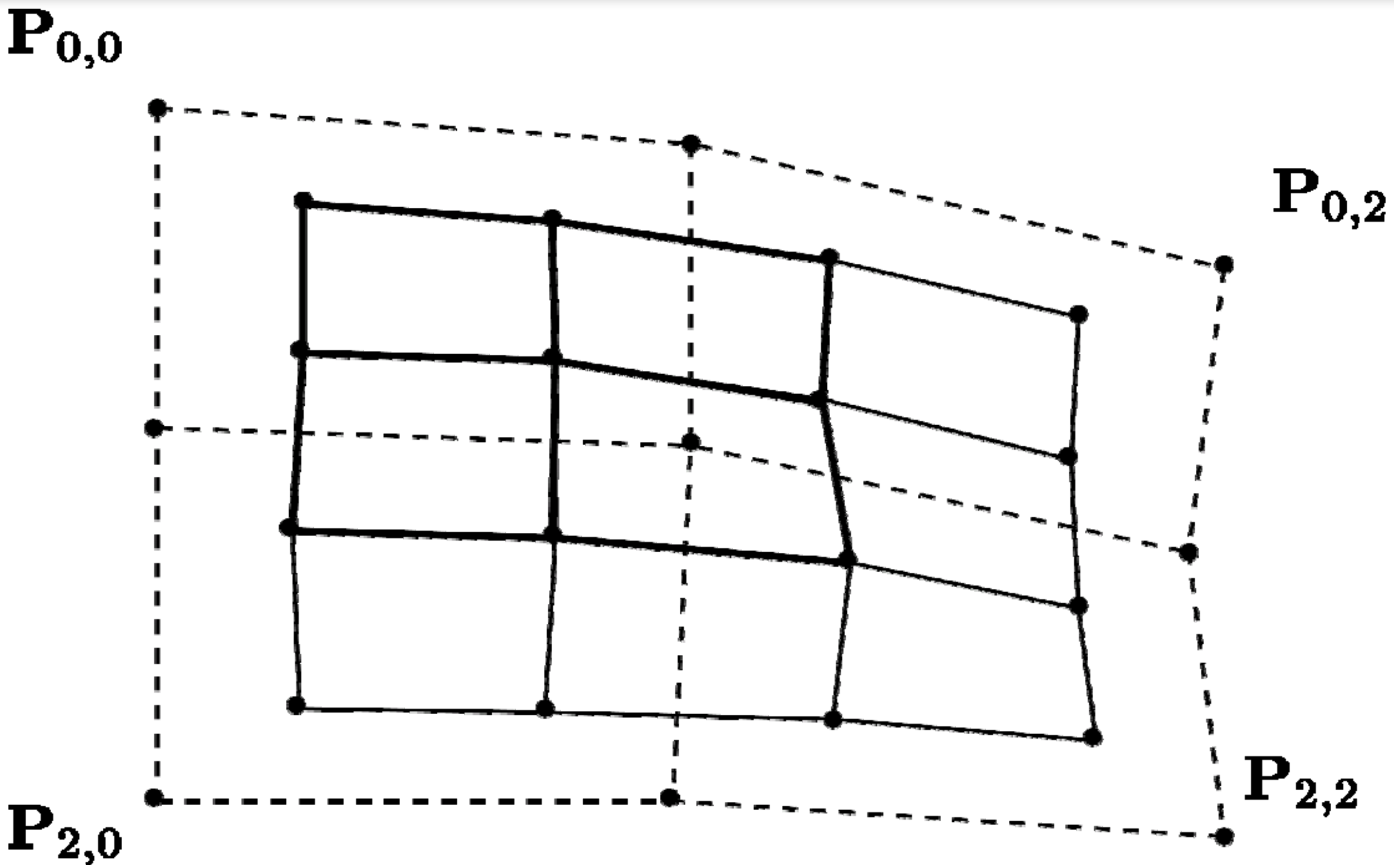
  - *Extension of Chaikin's curve to surface*: each of these equations is built from weighing the points on an edge in the ratio of 3-1. and then weighing the resulting points in the ratio 3-1

Generating the Refinement Procedure

- All 16 of the possible points generated through the binary subdivision of the quadratic patch can be defined by the same subdivision masks

$$
\begin{array}{ccc} 9 & - & 3 \\ | & & | \\ 3 & - & 1 \end{array}
\qquad
\begin{array}{ccc} 3 & - & 1 \\ | & & | \\ 9 & - & 3 \end{array}
\qquad
\begin{array}{ccc} 1 & - & 3 \\ | & & | \\ 3 & - & 9 \end{array}
\qquad
\begin{array}{ccc} 3 & - & 9 \\ | & & | \\ 1 & - & 3 \end{array}
$$

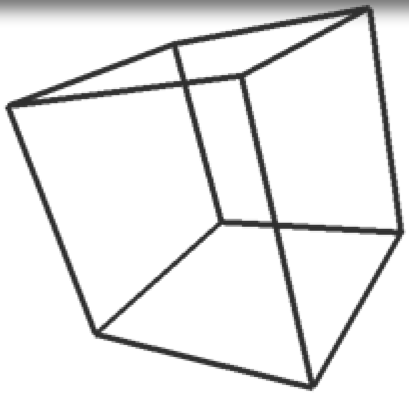
Generating the Refinement Procedure



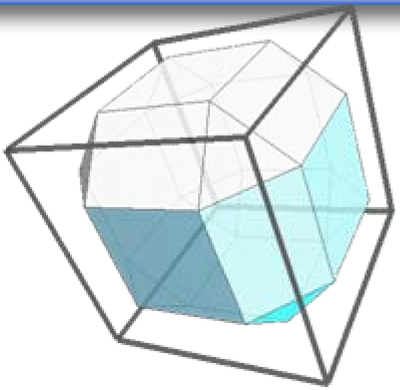
Summary

- The extension of Chaikin's Curve to surfaces
- Non-rectangular topological structure
  - Doo-Sabin surfaces by Donald Doo and Malcolm Sabin

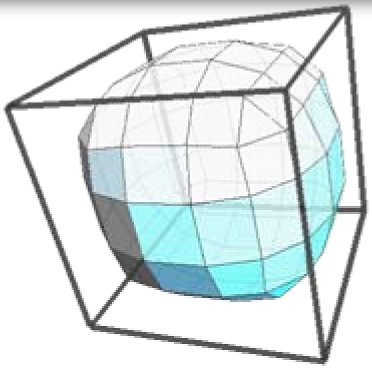
Doo-Sabin Surfaces



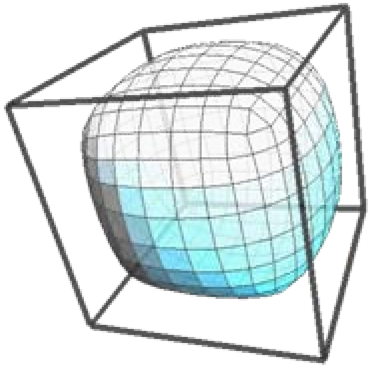
Original Cube



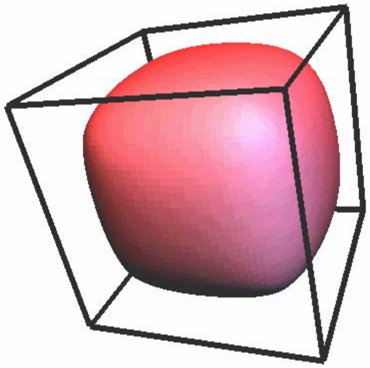
The 1st subdivision



The 2nd subdivision



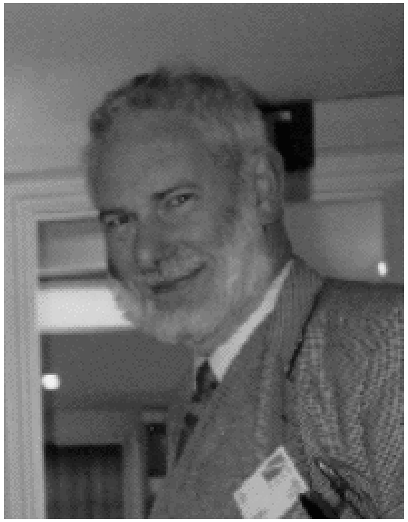
The 3rd subdivision



The 5th subdivision

Doo-Sabin Surfaces

- The Procedure for the Biquadratic Uniform B-Spline Patch
- The Procedure for Meshes of Arbitrary Topology
- Illustrations
- Summary



Malcolm Sabin

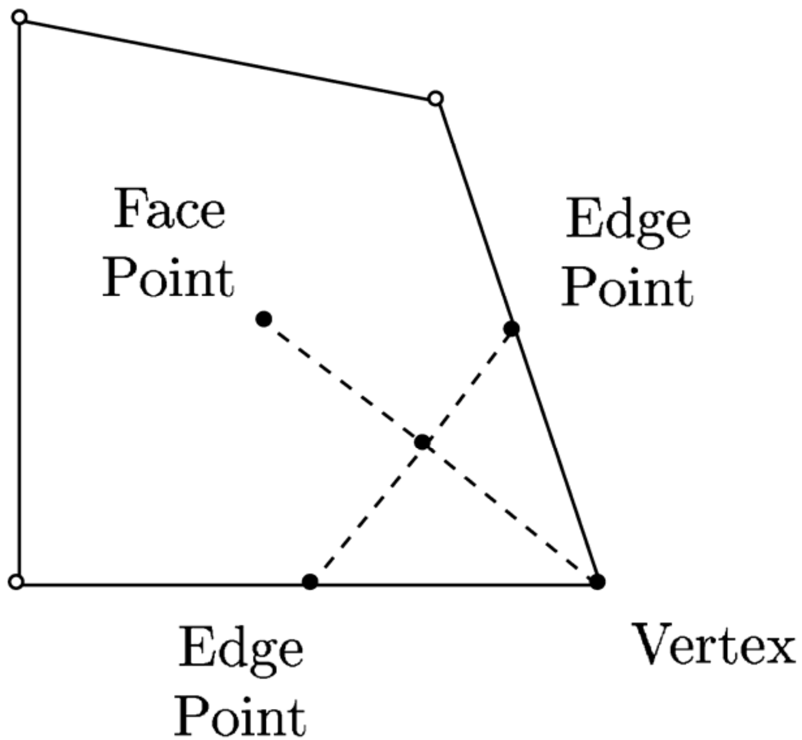
The Procedure for the Biquadratic Uniform B-Spline Patch

- The subdivision masks for the biquadratic uniform B-spline patches

$$
\begin{array}{ccc} 9 & — & 3 \\ | & & | \\ 3 & — & 1 \end{array}
\qquad
\begin{array}{ccc} 3 & — & 1 \\ | & & | \\ 9 & — & 3 \end{array}
\qquad
\begin{array}{ccc} 1 & — & 3 \\ | & & | \\ 3 & — & 9 \end{array}
\qquad
\begin{array}{ccc} 3 & — & 9 \\ | & & | \\ 1 & — & 3 \end{array}
$$

- Each new points are simply the average of four particular points taken in a polygon
  - the corresponding original point
  - the two *edge points* (the midpoints of the edges that are adjacent to this vertex in the polygon)
  - the *face point* (average of the vertices of the polygon)

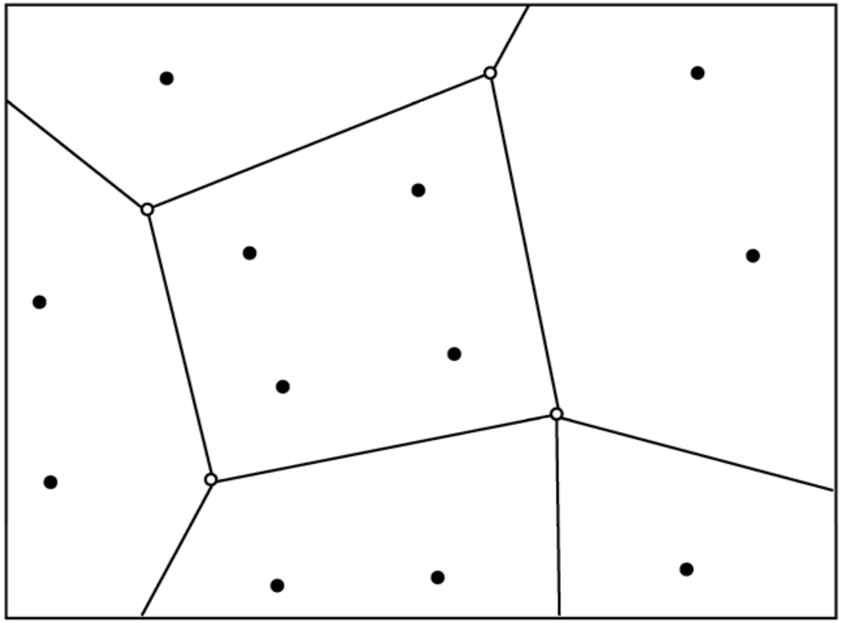
The Procedure for the Biquadratic Uniform B-Spline Patch



Each new face generated has four vertices

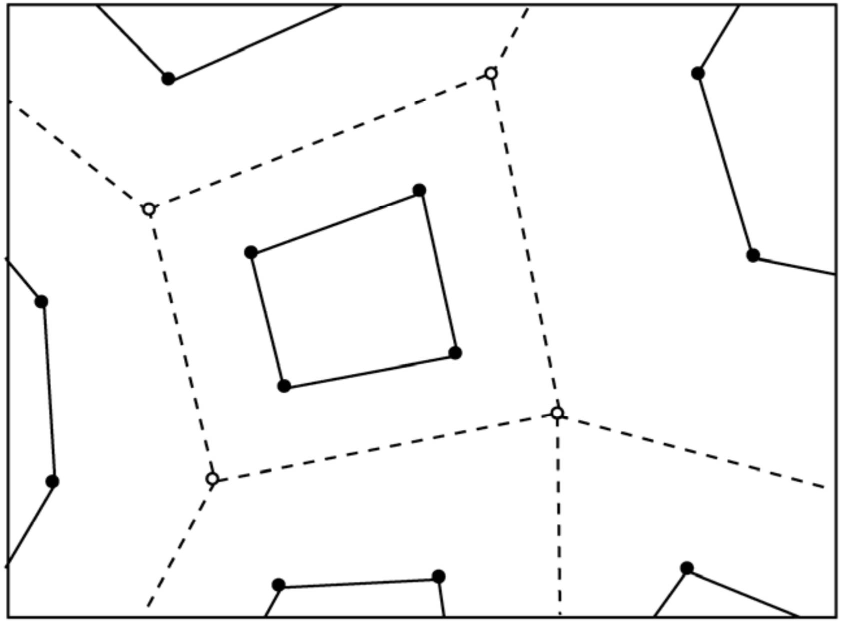
The Procedure for Meshes of Arbitrary Topology

- For each vertex $\mathbf{P}_i$ of each face of the object, generate a new point $\mathbf{P}'_i$ as average of the vertex, the two edge points and the face point of the face



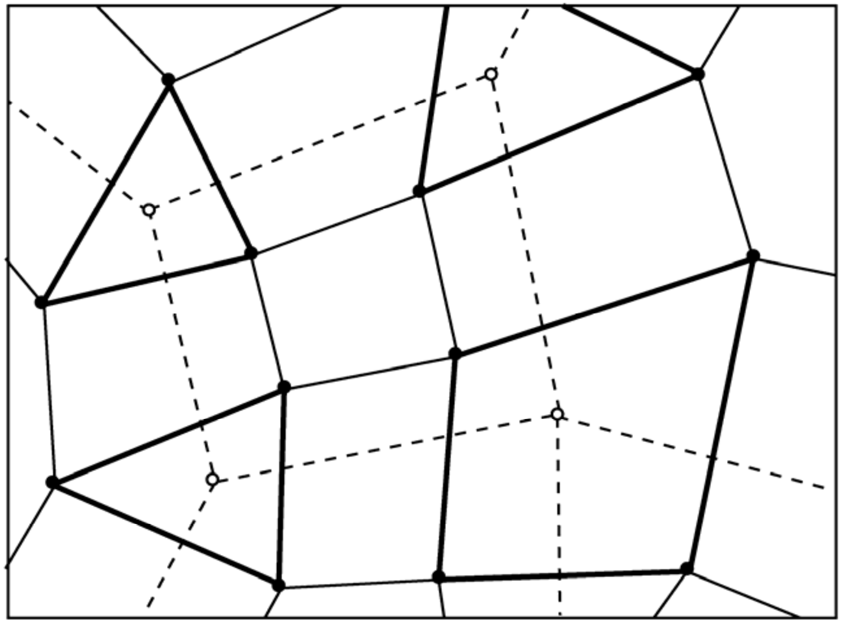
The Procedure for Meshes of Arbitrary Topology

- For each face, connect the new points that have been generated for each vertex of the face.



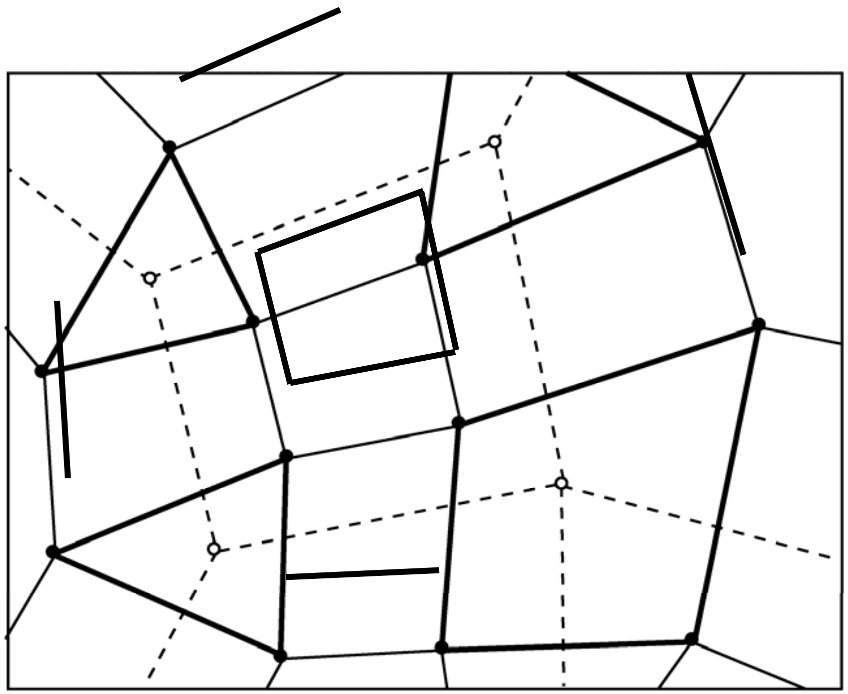
The Procedure for Meshes of Arbitrary Topology

- For each vertex, connect the new points that have been generated for the faces that are adjacent to this vertex.



The Procedure for Meshes of Arbitrary Topology

- For each edge, connect the new points that have been generated for the faces that are adjacent to this edge.

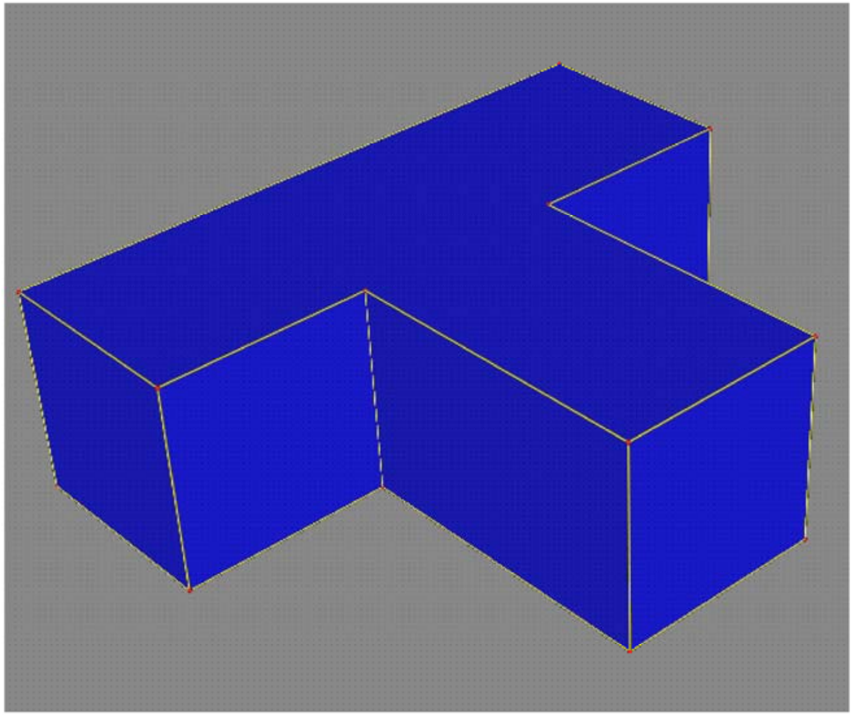


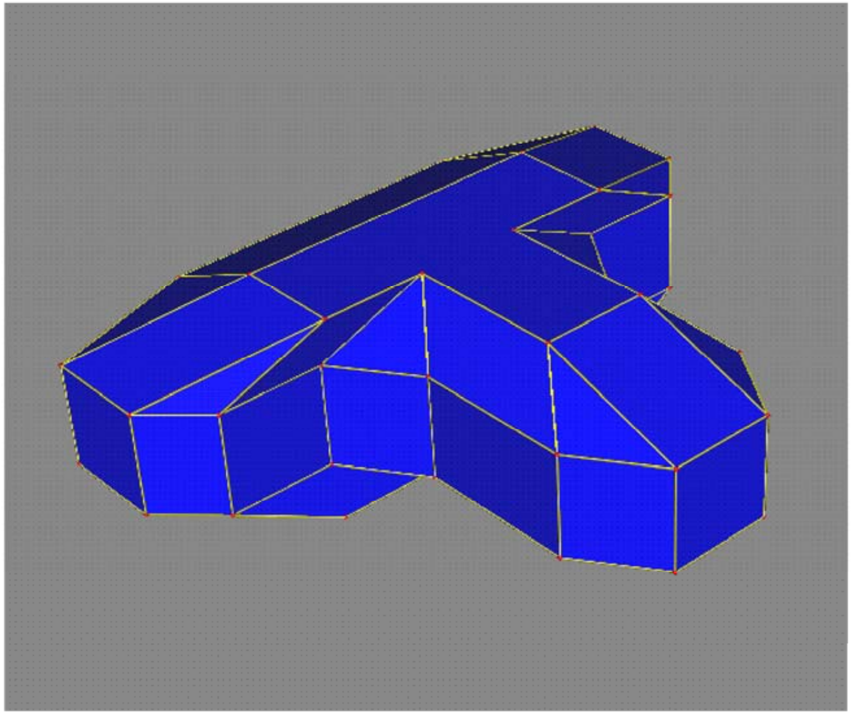
The Procedure for Meshes of Arbitrary Topology

- The polygons generated through this refinement step become the set of polygons for the next step.

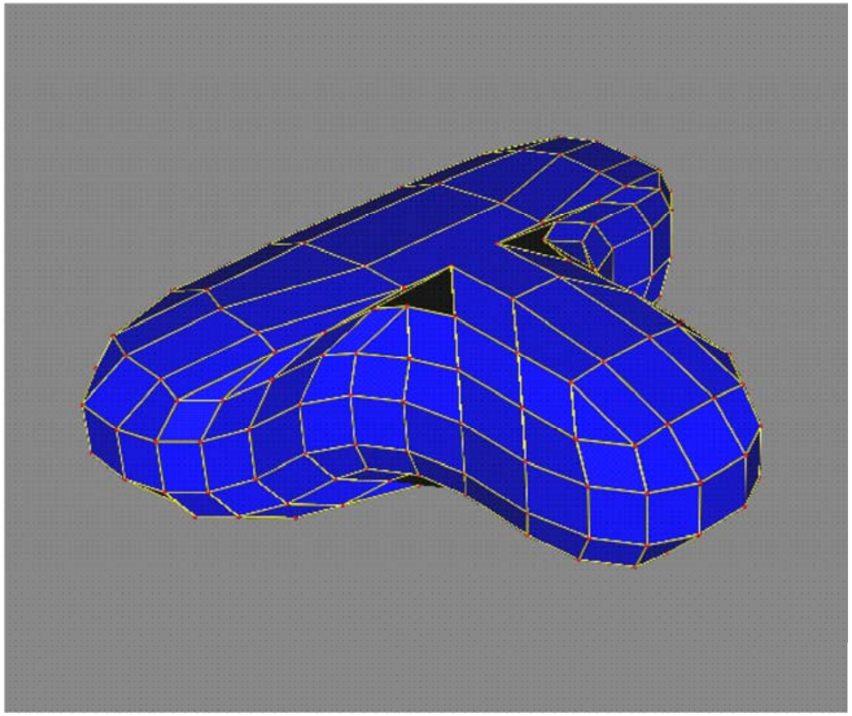
Note: the “cutting off the corners” of the polygonal mesh

Illustrations



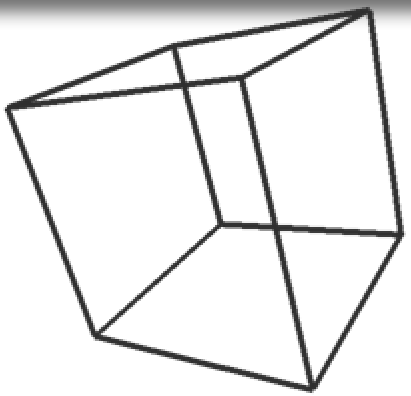


Illustrations

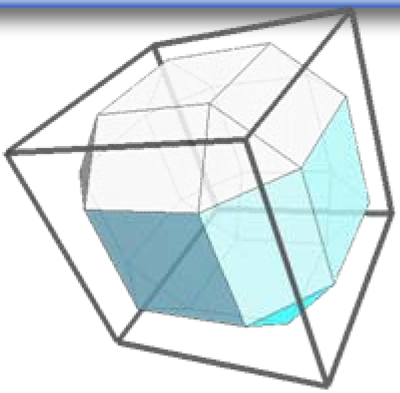


- *extraordinary points*
  - Not 4 edges emitted from the point
- after the first subdivision, all vertices have *valence* four

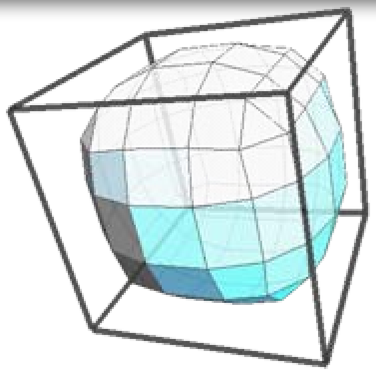
Illustrations



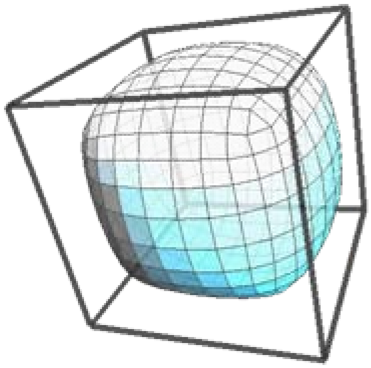
Original Cube



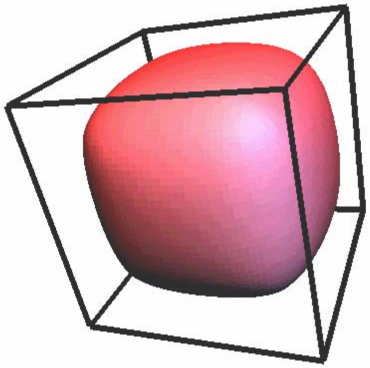
The 1st subdivision



The 2nd subdivision

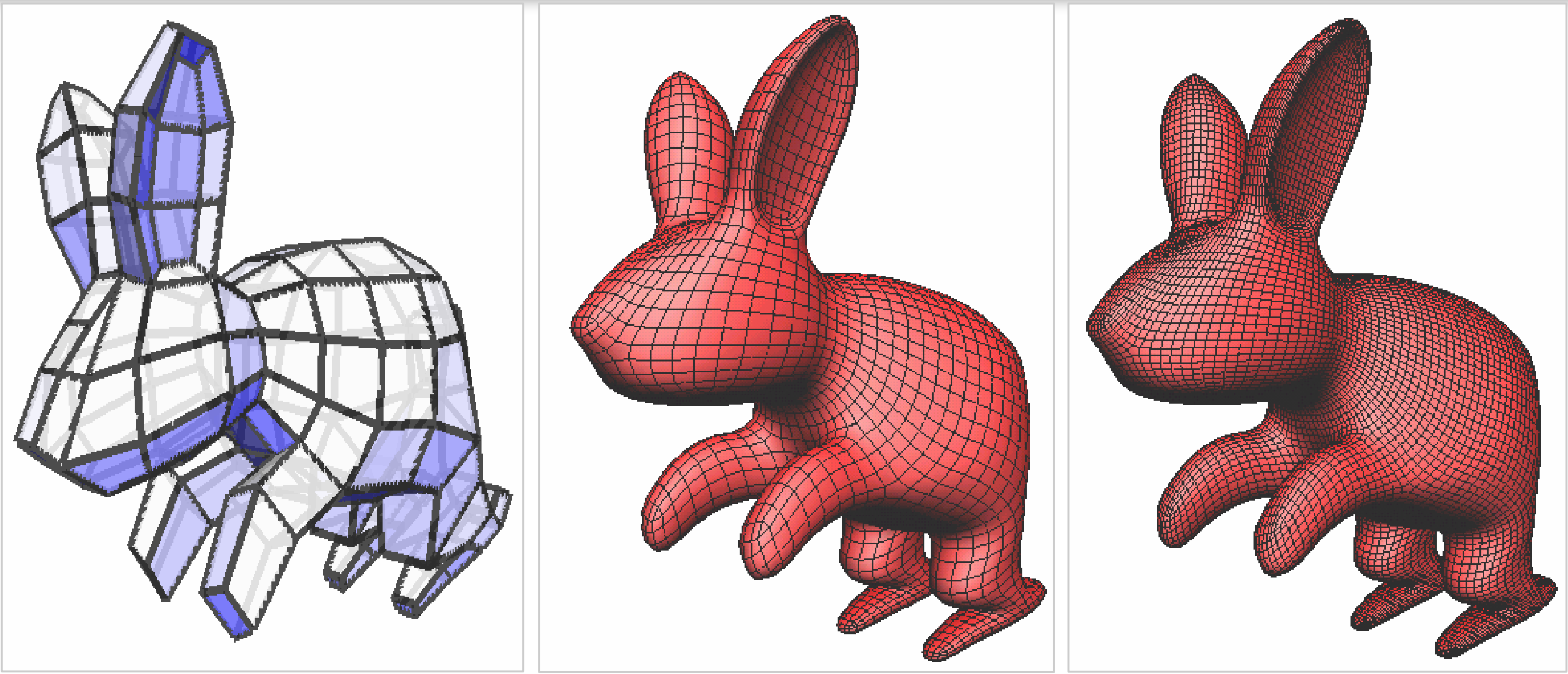


The 3rd subdivision



The 5th subdivision

Illustrations



Doo-Sabin Subdivision Surface

Summary

- The algorithm defines new points on each face in the refinement process, these new points only depend on
  - the face point
  - the vertex on a face
  - two edge midpoints for edges adjacent to the vertex

- The Doo-Sabin surface is locally a bi-quadratic B-spline surface, except at a finite number of extraordinary control points.

Subdivision Methods for Cubic B-Spline Surfaces

- Overview
- A Matrix Equation for the Bicubic Uniform Spline Surfaces
- Subdividing the Bicubic Patch
- Extending this Subdivision Procedure to the Entire Patch
- Summary

Overview

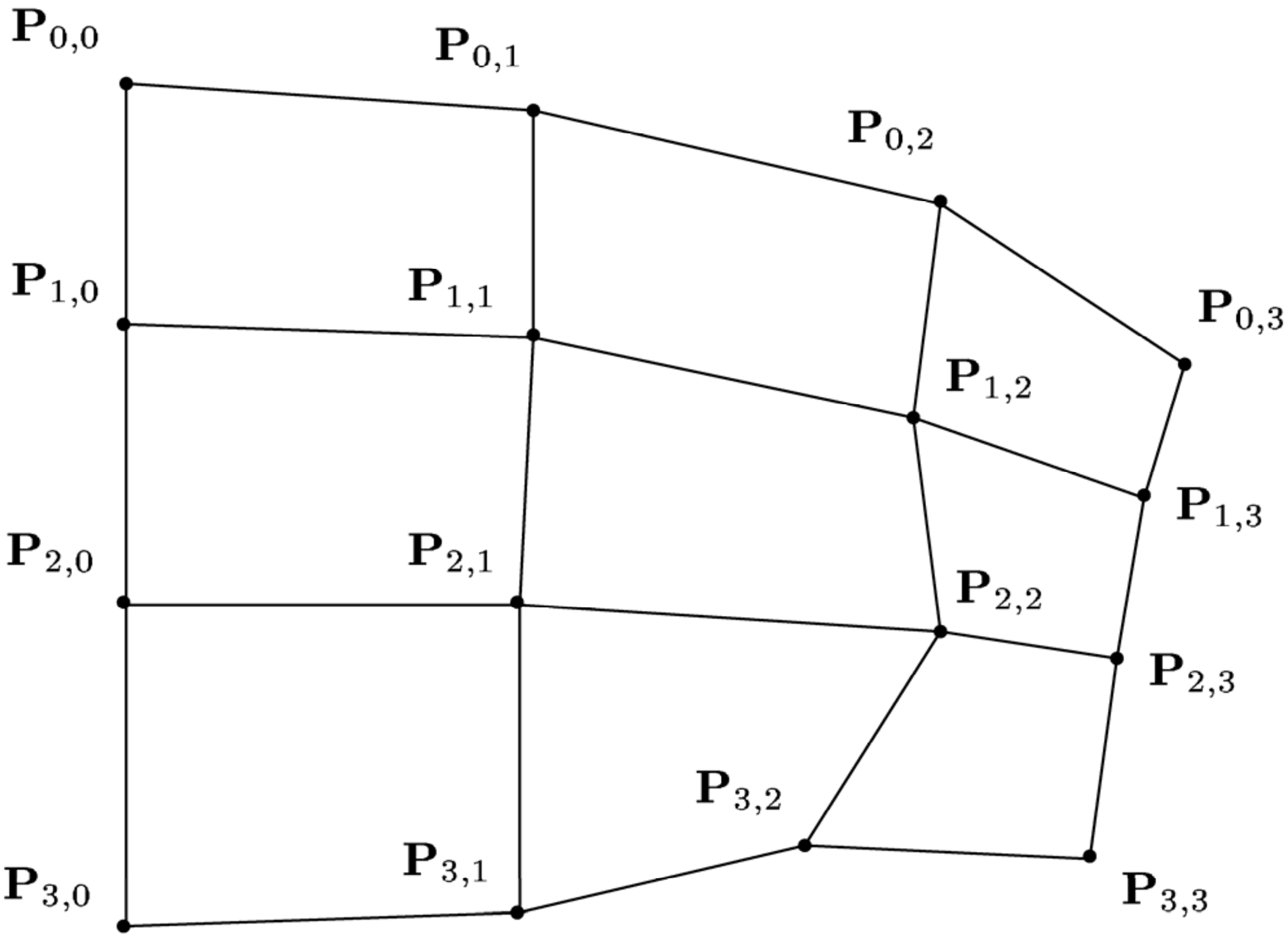
- Ed Catmull and Jim Clark: extend Doo and Sabin's algorithm to bicubic surfaces.
  - The methods of Doo-Sabin algorithm: binary subdivision of the uniform bi-quadratic B-spline patches.
  - Study of the cubic case would lead to a better subdivision surface generation scheme.

A Matrix Equation for the Bicubic Uniform Spline Surfaces

Consider the biqcubic uniform B-spline surface $\mathbf{P}(u,v)$ defined by the 4×4 array of control points

$$P = \begin{bmatrix} \mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} & \mathbf{P}_{0,3} \\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \mathbf{P}_{1,3} \\ \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \mathbf{P}_{2,3} \\ \mathbf{P}_{3,0} & \mathbf{P}_{3,1} & \mathbf{P}_{3,2} & \mathbf{P}_{3,3} \end{bmatrix}$$

A Matrix Equation for the Bicubic Uniform Spline Surfaces



A Matrix Equation for the Bicubic Uniform Spline Surfaces

where

$$\mathbf{P}(u,v) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} M P M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}$$

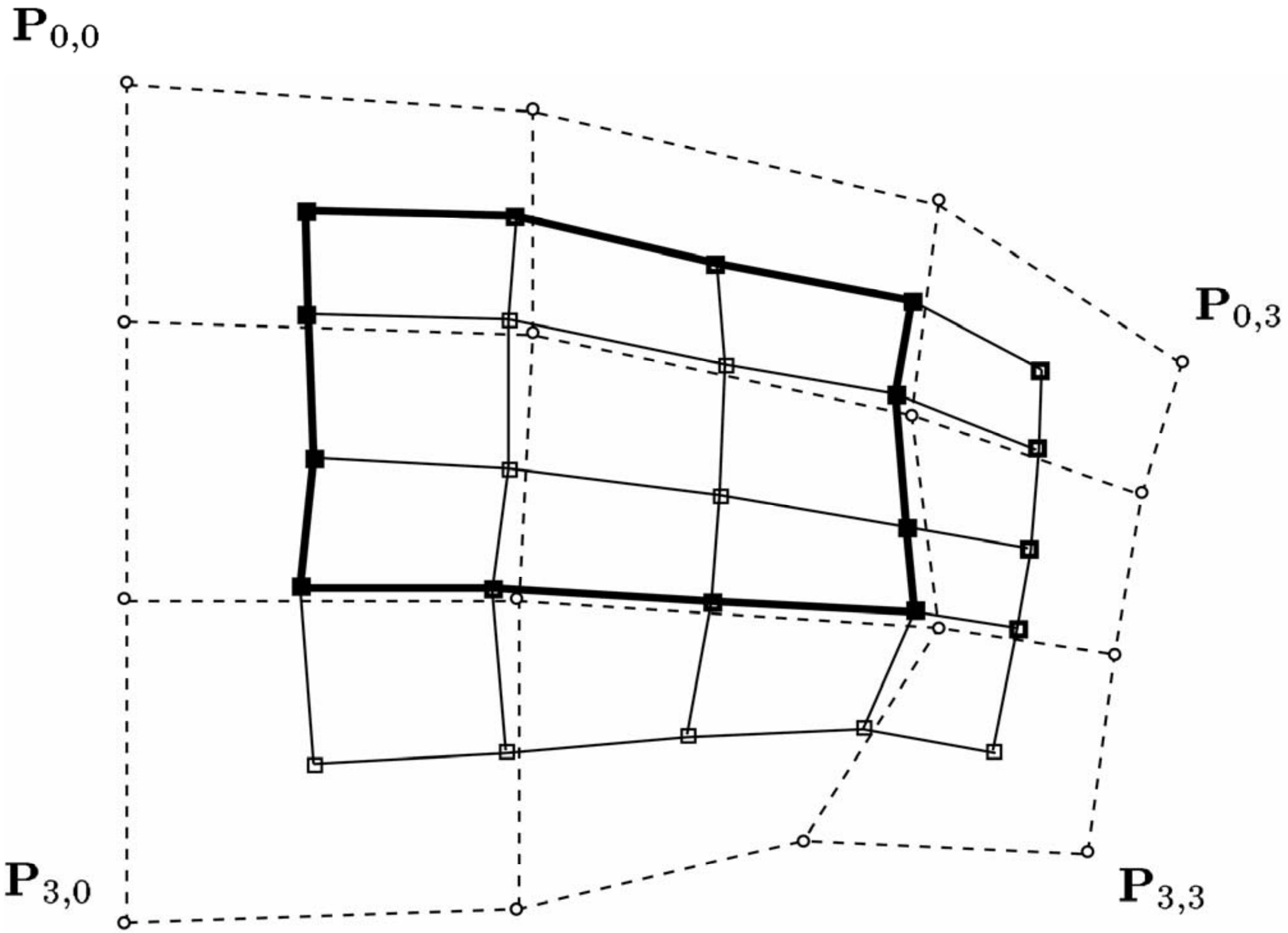
where M is the 4×4 matrix

$$M = \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Subdividing the Bicubic Patch

- The bicubic uniform B-spline patch can be subdivided into
  - four subpatches ($1 \rightarrow 4$)
  - 25 unique sub-control points ($16 \rightarrow 25$)
- Focus on the subpatch corresponding to $0 \le u,v \le 1/2$
  - the others will follow by symmetry

Subdividing the Bicubic Patch



Subdividing the Bicubic Patch

To subdivide the surface, we reparameterizing of the surface by $u'=u/2$ and $v'= v/2$ and define new the surface $\mathbf{P}'(u,v)$

$$
\begin{aligned}
\mathbf{P}'(u,v) &= \mathbf{P}\left(\frac{u}{2},\frac{v}{2}\right) \\
&= \begin{bmatrix} 1 & \frac{u}{2} & \left(\frac{u}{2}\right)^2 & \left(\frac{u}{2}\right)^3 \end{bmatrix} MPM^T \begin{bmatrix} 1 \\ \frac{v}{2} \\ \left(\frac{v}{2}\right)^2 \\ \left(\frac{v}{2}\right)^3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} MPM^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix}^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} MM^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} MPM^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix}^T (M^{-1})^T M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} MSPS^T M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} MP'M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}
\end{aligned}
$$

Subdividing the Bicubic Patch

where $P' = SPS^T$, and

$$S = M^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} M$$

The surface $\mathbf{P}'(u,v)$ is written as

$$\mathbf{P}'(u,v) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} M P' M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}$$

Subdividing the Bicubic Patch

The $\mathbf{P}'(u,v)$ is a uniform bicubic B-spline patch

The matrix S is called the "splitting matrix", and is given by

$$S = \frac{1}{8}\begin{bmatrix} 4 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \end{bmatrix}$$

Subdividing the Bicubic Patch

The control point array P'

$$
\begin{aligned}
P' &= \frac{1}{8}\begin{bmatrix} 4 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \end{bmatrix}
\begin{bmatrix} \mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} & \mathbf{P}_{0,3} \\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \mathbf{P}_{1,3} \\ \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \mathbf{P}_{2,3} \\ \mathbf{P}_{3,0} & \mathbf{P}_{3,1} & \mathbf{P}_{3,2} & \mathbf{P}_{3,3} \end{bmatrix}
\frac{1}{8}\begin{bmatrix} 4 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \end{bmatrix}^T \\
&= \frac{1}{8}\begin{bmatrix}
4\mathbf{P}_{0,0}+4\mathbf{P}_{1,0} & 4\mathbf{P}_{0,1}+4\mathbf{P}_{1,1} & 4\mathbf{P}_{0,2}+4\mathbf{P}_{1,2} & 4\mathbf{P}_{0,3}+4\mathbf{P}_{1,3} \\
\mathbf{P}_{0,0}+6\mathbf{P}_{1,0}+\mathbf{P}_{2,0} & \mathbf{P}_{0,1}+6\mathbf{P}_{1,1}+\mathbf{P}_{2,1} & \mathbf{P}_{0,2}+6\mathbf{P}_{1,2}+\mathbf{P}_{2,2} & \mathbf{P}_{0,3}+6\mathbf{P}_{1,3}+\mathbf{P}_{2,3} \\
4\mathbf{P}_{1,0}+4\mathbf{P}_{2,0} & 4\mathbf{P}_{1,1}+4\mathbf{P}_{2,1} & 4\mathbf{P}_{1,2}+4\mathbf{P}_{2,2} & 4\mathbf{P}_{1,3}+4\mathbf{P}_{2,3} \\
\mathbf{P}_{1,0}+6\mathbf{P}_{2,0}+\mathbf{P}_{3,0} & \mathbf{P}_{1,1}+6\mathbf{P}_{2,1}+\mathbf{P}_{3,1} & \mathbf{P}_{1,2}+6\mathbf{P}_{2,2}+\mathbf{P}_{3,2} & \mathbf{P}_{1,3}+6\mathbf{P}_{2,3}+\mathbf{P}_{3,3}
\end{bmatrix}
\frac{1}{8}\begin{bmatrix} 4 & 1 & 0 & 0 \\ 4 & 6 & 4 & 1 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 6 & 1 \end{bmatrix}
\end{aligned}
$$

Calculating above matrix multiplication, we can obtain

Subdividing the Bicubic Patch

$$
\begin{aligned}
\mathbf{P}'_{0,0} &= \frac{\mathbf{P}_{0,0}+\mathbf{P}_{1,0}+\mathbf{P}_{0,1}+\mathbf{P}_{1,1}}{4} \\
\mathbf{P}'_{0,1} &= \frac{\mathbf{P}_{0,0}+\mathbf{P}_{1,0}+6(\mathbf{P}_{0,1}+\mathbf{P}_{1,1})+\mathbf{P}_{0,2}+\mathbf{P}_{1,2}}{16} \\
\mathbf{P}'_{0,2} &= \frac{\mathbf{P}_{0,1}+\mathbf{P}_{1,1}+\mathbf{P}_{0,2}+\mathbf{P}_{1,2}}{4} \\
\mathbf{P}'_{0,3} &= \frac{\mathbf{P}_{0,1}+\mathbf{P}_{1,1}+6(\mathbf{P}_{0,2}+\mathbf{P}_{1,2})+\mathbf{P}_{0,3}+\mathbf{P}_{1,3}}{16} \\
\mathbf{P}'_{1,0} &= \frac{\mathbf{P}_{0,0}+\mathbf{P}_{0,1}+6(\mathbf{P}_{1,0}+\mathbf{P}_{1,1})+\mathbf{P}_{2,0}+\mathbf{P}_{2,1}}{16} \\
\mathbf{P}'_{1,1} &= \frac{\mathbf{P}_{0,0}+6\mathbf{P}_{1,0}+\mathbf{P}_{2,0}+6(\mathbf{P}_{0,1}+6\mathbf{P}_{1,1}+\mathbf{P}_{2,1})+\mathbf{P}_{0,2}+6\mathbf{P}_{1,2}+\mathbf{P}_{2,2}}{64} \\
\mathbf{P}'_{1,2} &= \frac{\mathbf{P}_{0,1}+\mathbf{P}_{0,2}+6(\mathbf{P}_{1,1}+\mathbf{P}_{1,2})+\mathbf{P}_{2,1}+\mathbf{P}_{2,2}}{16} \\
\mathbf{P}'_{1,3} &= \frac{\mathbf{P}_{0,1}+6\mathbf{P}_{1,1}+\mathbf{P}_{2,1}+6(\mathbf{P}_{0,2}+6\mathbf{P}_{1,2}+\mathbf{P}_{2,2})+\mathbf{P}_{0,3}+6\mathbf{P}_{1,3}+\mathbf{P}_{2,3}}{64}
\end{aligned}
$$

Subdividing the Bicubic Patch

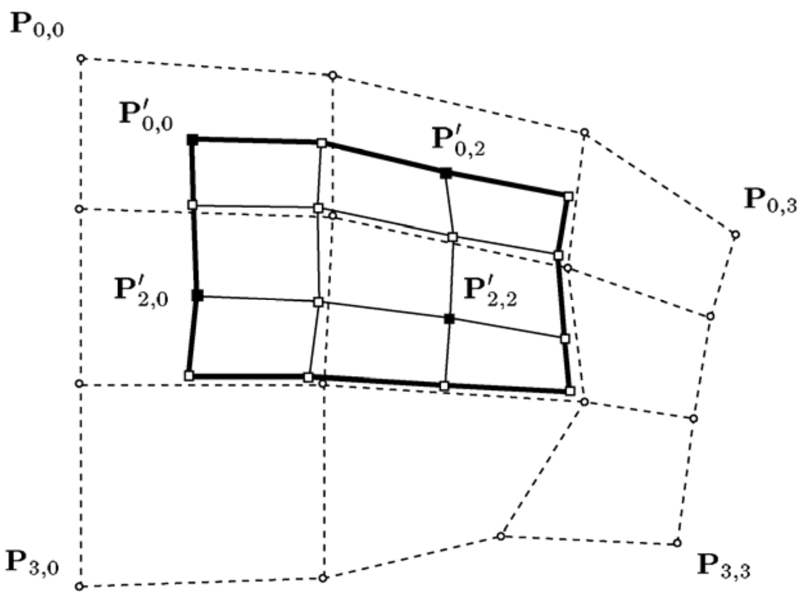
$$
\begin{aligned}
\mathbf{P}'_{2,0} &= \frac{\mathbf{P}_{1,0}+\mathbf{P}_{2,0}+\mathbf{P}_{1,1}+\mathbf{P}_{2,1}}{4} \\
\mathbf{P}'_{2,1} &= \frac{\mathbf{P}_{1,0}+\mathbf{P}_{2,0}+6(\mathbf{P}_{1,1}+\mathbf{P}_{2,1})+\mathbf{P}_{1,2}+\mathbf{P}_{2,2}}{16} \\
\mathbf{P}'_{2,2} &= \frac{\mathbf{P}_{1,1}+\mathbf{P}_{2,1}+\mathbf{P}_{1,2}+\mathbf{P}_{2,2}}{4} \\
\mathbf{P}'_{2,3} &= \frac{\mathbf{P}_{1,1}+\mathbf{P}_{2,1}+6(\mathbf{P}_{1,2}+\mathbf{P}_{2,2})+\mathbf{P}_{1,3}+\mathbf{P}_{2,3}}{16} \\
\mathbf{P}'_{3,0} &= \frac{\mathbf{P}_{1,0}+\mathbf{P}_{1,1}+6(\mathbf{P}_{2,0}+\mathbf{P}_{2,1})+\mathbf{P}_{3,0}+\mathbf{P}_{3,1}}{16} \\
\mathbf{P}'_{3,1} &= \frac{\mathbf{P}_{1,0}+6\mathbf{P}_{2,0}+\mathbf{P}_{3,0}+6(\mathbf{P}_{1,1}+6\mathbf{P}_{2,1}+\mathbf{P}_{3,1})+\mathbf{P}_{1,2}+6\mathbf{P}_{2,2}+\mathbf{P}_{3,2}}{64} \\
\mathbf{P}'_{3,2} &= \frac{\mathbf{P}_{1,1}+\mathbf{P}_{1,2}+6(\mathbf{P}_{2,1}+\mathbf{P}_{2,2})+\mathbf{P}_{3,1}+\mathbf{P}_{3,2}}{16} \\
\mathbf{P}'_{3,3} &= \frac{\mathbf{P}_{1,1}+6\mathbf{P}_{2,1}+\mathbf{P}_{3,1}+6(\mathbf{P}_{1,2}+6\mathbf{P}_{2,2}+\mathbf{P}_{3,2})+\mathbf{P}_{1,3}+6\mathbf{P}_{2,3}+\mathbf{P}_{3,3}}{64}
\end{aligned}
$$

Subdividing the Bicubic Patch

- Each of these refined points can be classified into three categories
  - face points
  - edge points
  - vertex points

Subdividing the Bicubic Patch

- Face point: the average of the four points that bound the respective face ($\mathbf{P}_{00}'$,$\mathbf{P}_{02}'$, $\mathbf{P}_{20}'$ and $\mathbf{P}_{22}'$)



$$\mathbf{P}'_{0,0} = \frac{\mathbf{P}_{0,0} + \mathbf{P}_{1,0} + \mathbf{P}_{0,1} + \mathbf{P}_{1,1}}{4}$$

$$\mathbf{P}'_{0,2} = \frac{\mathbf{P}_{0,1} + \mathbf{P}_{1,1} + \mathbf{P}_{0,2} + \mathbf{P}_{1,2}}{4}$$

$$\mathbf{P}'_{2,0} = \frac{\mathbf{P}_{1,0} + \mathbf{P}_{2,0} + \mathbf{P}_{1,1} + \mathbf{P}_{2,1}}{4}$$

$$\mathbf{P}'_{2,2} = \frac{\mathbf{P}_{1,1} + \mathbf{P}_{2,1} + \mathbf{P}_{1,2} + \mathbf{P}_{2,2}}{4}$$

Subdividing the Bicubic Patch

- In general , the face point $\mathbf{F}_{i,j}$

$$\mathbf{F}_{i,j} = (\mathbf{P}_{i,j} + \mathbf{P}_{i+1,j} + \mathbf{P}_{i,j+1} + \mathbf{P}_{i+1,j+1})/4$$

- we can rewrite the above equations with these face points substituted on the right-hand side, and obtain

Subdividing the Bicubic Patch

$$
\begin{aligned}
\mathbf{P}'_{0,0} &= \mathbf{F}_{0,0} \\
\mathbf{P}'_{0,1} &= \frac{4\mathbf{F}_{0,0} + 4\mathbf{F}_{0,1} + 4\mathbf{P}_{0,1} + 4\mathbf{P}_{1,1}}{16} \\
\mathbf{P}'_{0,2} &= \mathbf{F}_{0,1} \\
\mathbf{P}'_{0,3} &= \frac{4\mathbf{F}_{0,1} + 4\mathbf{F}_{0,2} + 4\mathbf{P}_{0,2} + 4\mathbf{P}_{1,2}}{16} \\
\mathbf{P}'_{1,0} &= \frac{4\mathbf{F}_{0,0} + 4\mathbf{F}_{1,0} + 4\mathbf{P}_{1,0} + 4\mathbf{P}_{1,1}}{16} \\
\mathbf{P}'_{1,1} &= \frac{4\mathbf{F}_{0,0} + 4\mathbf{F}_{0,1} + 4\mathbf{F}_{1,0} + 4\mathbf{F}_{1,1} + 4\mathbf{P}_{1,0} + 4\mathbf{P}_{0,1} + 32\mathbf{P}_{1,1} + 4\mathbf{P}_{2,1} + 4\mathbf{P}_{1,2}}{64} \\
\mathbf{P}'_{1,2} &= \frac{4\mathbf{F}_{0,1} + 4\mathbf{F}_{1,1} + 4\mathbf{P}_{1,1} + 4\mathbf{P}_{1,2}}{16} \\
\mathbf{P}'_{1,3} &= \frac{4\mathbf{F}_{0,1} + 4\mathbf{F}_{0,2} + 4\mathbf{F}_{1,1} + 4\mathbf{F}_{1,2} + 4\mathbf{P}_{1,1} + 4\mathbf{P}_{0,2} + 32\mathbf{P}_{1,2} + 4\mathbf{P}_{2,2} + 4\mathbf{P}_{1,3}}{64}
\end{aligned}
$$

Subdividing the Bicubic Patch

$$
\begin{aligned}
\mathbf{P}'_{2,0} &= \mathbf{F}_{1,0} \\
\mathbf{P}'_{2,1} &= \frac{4\mathbf{F}_{1,0} + 4\mathbf{F}_{1,1} + 4\mathbf{P}_{1,1} + 4\mathbf{P}_{2,1}}{16} \\
\mathbf{P}'_{2,2} &= \mathbf{F}_{1,1} \\
\mathbf{P}'_{2,3} &= \frac{4\mathbf{F}_{1,1} + 4\mathbf{F}_{1,2} + 4\mathbf{P}_{1,2} + 4\mathbf{P}_{2,2}}{16} \\
\mathbf{P}'_{3,0} &= \frac{4\mathbf{F}_{1,0} + 4\mathbf{F}_{2,0} + 4\mathbf{P}_{2,0} + 4\mathbf{P}_{2,1}}{16} \\
\mathbf{P}'_{3,1} &= \frac{4\mathbf{F}_{1,0} + 4\mathbf{F}_{2,0} + 4\mathbf{F}_{1,1} + 4\mathbf{F}_{2,1} + 4\mathbf{P}_{2,0} + 4\mathbf{P}_{1,1} + 32\mathbf{P}_{2,1} + 4\mathbf{P}_{3,1} + 4\mathbf{P}_{2,2}}{64} \\
\mathbf{P}'_{3,2} &= \frac{4\mathbf{F}_{1,1} + 4\mathbf{F}_{2,1} + 4\mathbf{P}_{2,1} + 4\mathbf{P}_{2,2}}{16} \\
\mathbf{P}'_{3,3} &= \frac{4\mathbf{F}_{1,1} + 4\mathbf{F}_{2,1} + 4\mathbf{F}_{1,2} + 4\mathbf{F}_{2,2} + 4\mathbf{P}_{2,1} + 4\mathbf{P}_{1,2} + 32\mathbf{P}_{2,2} + 4\mathbf{P}_{3,2} + 4\mathbf{P}_{2,3}}{64}
\end{aligned}
$$

Simplifying these equations, we obtain

Subdividing the Bicubic Patch

$$
\begin{aligned}
\mathbf{P}'_{0,0} &= \mathbf{F}_{0,0} \\
\mathbf{P}'_{0,1} &= \frac{\mathbf{F}_{0,0}+\mathbf{F}_{0,1}+\mathbf{P}_{0,1}+\mathbf{P}_{1,1}}{4} \\
\mathbf{P}'_{0,2} &= \mathbf{F}_{0,1} \\
\mathbf{P}'_{0,3} &= \frac{\mathbf{F}_{0,1}+\mathbf{F}_{0,2}+\mathbf{P}_{0,2}+\mathbf{P}_{1,2}}{4} \\
\mathbf{P}'_{1,0} &= \frac{\mathbf{F}_{0,0}+\mathbf{F}_{1,0}+\mathbf{P}_{1,0}+\mathbf{P}_{1,1}}{4} \\
\mathbf{P}'_{1,1} &= \frac{\mathbf{F}_{0,0}+\mathbf{F}_{0,1}+\mathbf{F}_{1,0}+\mathbf{F}_{1,1}+\mathbf{P}_{1,0}+\mathbf{P}_{0,1}+8\mathbf{P}_{1,1}+\mathbf{P}_{2,1}+\mathbf{P}_{1,2}}{16} \\
\mathbf{P}'_{1,2} &= \frac{\mathbf{F}_{0,1}+\mathbf{F}_{1,1}+\mathbf{P}_{1,1}+\mathbf{P}_{1,2}}{4} \\
\mathbf{P}'_{1,3} &= \frac{\mathbf{F}_{0,1}+\mathbf{F}_{0,2}+\mathbf{F}_{1,1}+\mathbf{F}_{1,2}+\mathbf{P}_{1,1}+\mathbf{P}_{0,2}+8\mathbf{P}_{1,2}+\mathbf{P}_{2,2}+\mathbf{P}_{1,3}}{16}
\end{aligned}
$$

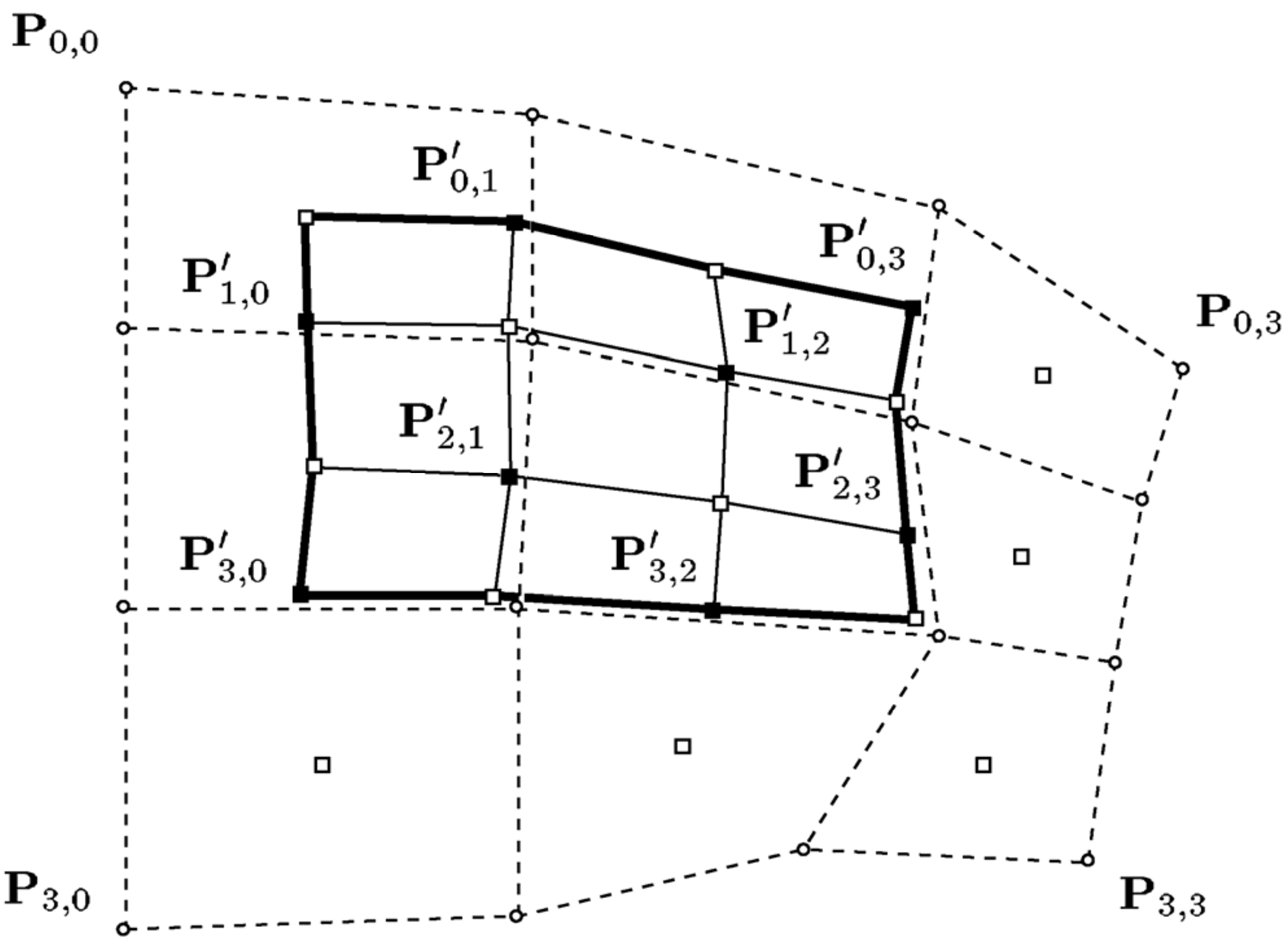
Subdividing the Bicubic Patch

$$
\begin{aligned}
\mathbf{P}'_{2,0} &= \mathbf{F}_{1,0} \\
\mathbf{P}'_{2,1} &= \frac{\mathbf{F}_{1,0}+\mathbf{F}_{1,1}+\mathbf{P}_{1,1}+\mathbf{P}_{2,1}}{4} \\
\mathbf{P}'_{2,2} &= \mathbf{F}_{1,1} \\
\mathbf{P}'_{2,3} &= \frac{\mathbf{F}_{1,1}+\mathbf{F}_{1,2}+\mathbf{P}_{1,2}+\mathbf{P}_{2,2}}{4} \\
\mathbf{P}'_{3,0} &= \frac{\mathbf{F}_{1,0}+\mathbf{F}_{2,0}+\mathbf{P}_{2,0}+\mathbf{P}_{2,1}}{4} \\
\mathbf{P}'_{3,1} &= \frac{\mathbf{F}_{1,0}+\mathbf{F}_{2,0}+\mathbf{F}_{1,1}+\mathbf{F}_{2,1}+\mathbf{P}_{2,0}+\mathbf{P}_{1,1}+8\mathbf{P}_{2,1}+\mathbf{P}_{3,1}+\mathbf{P}_{2,2}}{16} \\
\mathbf{P}'_{3,2} &= \frac{\mathbf{F}_{1,1}+\mathbf{F}_{2,1}+\mathbf{P}_{2,1}+\mathbf{P}_{2,2}}{4} \\
\mathbf{P}'_{3,3} &= \frac{\mathbf{F}_{1,1}+\mathbf{F}_{2,1}+\mathbf{F}_{1,2}+\mathbf{F}_{2,2}+\mathbf{P}_{2,1}+\mathbf{P}_{1,2}+8\mathbf{P}_{2,2}+\mathbf{P}_{3,2}+\mathbf{P}_{2,3}}{16}
\end{aligned}
$$

Subdividing the Bicubic Patch

- Edge points: average of
  - the two points that define the original edge
  - the two new face points of the faces sharing the edge
  - $\mathbf{P}'_{01}$, $\mathbf{P}'_{03}$, $\mathbf{P}'_{10}$, $\mathbf{P}'_{12}$, $\mathbf{P}'_{21}$, $\mathbf{P}'_{23}$, $\mathbf{P}'_{30}$, $\mathbf{P}'_{32}$

Subdividing the Bicubic Patch



Subdividing the Bicubic Patch

- The edge points $\mathbf{E}_{i,j}$ can be calculated

$$\mathbf{E}_{i,j} = \frac{\mathbf{F}_{i,j-1} + \mathbf{F}_{i,j} + \mathbf{P}_{i,j} + \mathbf{P}_{i+1,j}}{4}$$

$$\mathbf{E}_{i,j} = \frac{\mathbf{F}_{i-1,j} + \mathbf{F}_{i,j} + \mathbf{P}_{i,j} + \mathbf{P}_{i,j+1}}{4}$$

depending on which side of the edge point the two faces lie.

Subdividing the Bicubic Patch

Replacing edge points on the right-hand side of the above equations

$$
\begin{aligned}
\mathbf{P}'_{0,0} &= \mathbf{F}_{0,0} \\
\mathbf{P}'_{0,1} &= \mathbf{E}_{0,1} \\
\mathbf{P}'_{0,2} &= \mathbf{F}_{0,1} \\
\mathbf{P}'_{0,3} &= \mathbf{E}_{0,2} \\
\mathbf{P}'_{1,0} &= \mathbf{E}_{1,0} \\
\mathbf{P}'_{1,1} &= \frac{\mathbf{F}_{0,0}+\mathbf{F}_{0,1}+\mathbf{F}_{1,0}+\mathbf{F}_{1,1}+\mathbf{P}_{1,0}+\mathbf{P}_{0,1}+8\mathbf{P}_{1,1}+\mathbf{P}_{2,1}+\mathbf{P}_{1,2}}{16} \\
\mathbf{P}'_{1,2} &= \mathbf{E}_{1,2} \\
\mathbf{P}'_{1,3} &= \frac{\mathbf{F}_{0,1}+\mathbf{F}_{0,2}+\mathbf{F}_{1,1}+\mathbf{F}_{1,2}+\mathbf{P}_{1,1}+\mathbf{P}_{0,2}+8\mathbf{P}_{1,2}+\mathbf{P}_{2,2}+\mathbf{P}_{1,3}}{16}
\end{aligned}
$$

Subdividing the Bicubic Patch

$$
\begin{aligned}
\mathbf{P}'_{2,0} &= \mathbf{F}_{1,0} \\
\mathbf{P}'_{2,1} &= \mathbf{E}_{2,1} \\
\mathbf{P}'_{2,2} &= \mathbf{F}_{1,1} \\
\mathbf{P}'_{2,3} &= \mathbf{E}_{2,2} \\
\mathbf{P}'_{3,0} &= \mathbf{E}_{3,0} \\
\mathbf{P}'_{3,1} &= \frac{\mathbf{F}_{1,0} + \mathbf{F}_{2,0} + \mathbf{F}_{1,1} + \mathbf{F}_{2,1} + \mathbf{P}_{2,0} + \mathbf{P}_{1,1} + 8\mathbf{P}_{2,1} + \mathbf{P}_{3,1} + \mathbf{P}_{2,2}}{16} \\
\mathbf{P}'_{3,2} &= \mathbf{E}_{3,2} \\
\mathbf{P}'_{3,3} &= \frac{\mathbf{F}_{1,1} + \mathbf{F}_{2,1} + \mathbf{F}_{1,2} + \mathbf{F}_{2,2} + \mathbf{P}_{2,1} + \mathbf{P}_{1,2} + 8\mathbf{P}_{2,2} + \mathbf{P}_{3,2} + \mathbf{P}_{2,3}}{16}
\end{aligned}
$$

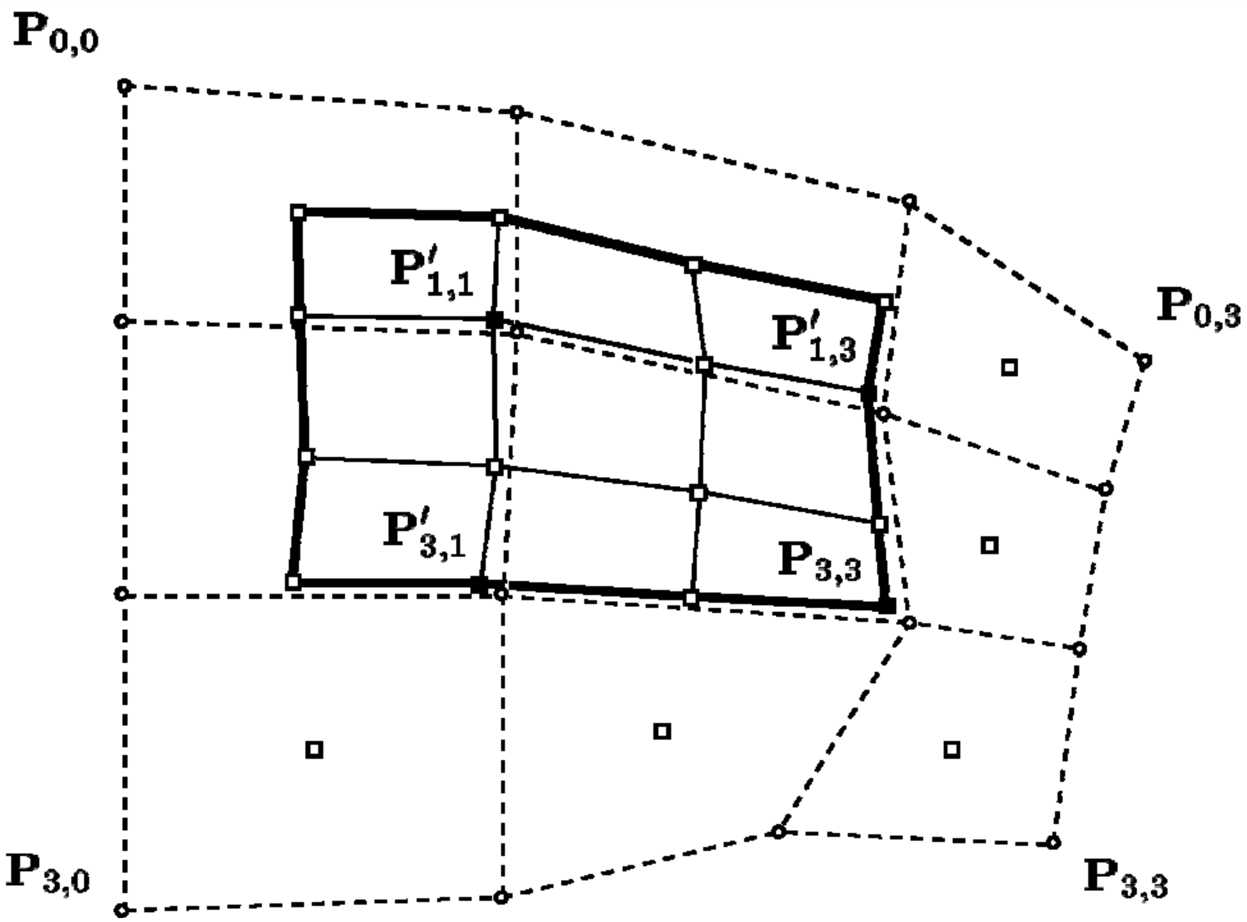
Subdividing the Bicubic Patch

- Vertex points: $\mathbf{P'}_{11}$, $\mathbf{P'}_{13}$, $\mathbf{P'}_{13}$, $\mathbf{P'}_{33}$,

$$\mathbf{P}'_{i,j} = \frac{\mathbf{Q} + 2\mathbf{R} + \mathbf{S}}{4}$$

  - **Q**: average of the face points of the faces adjacent to the vertex point
  - **R**: average of the midpoints of the edges adjacent to the vertex point
  - **S:** corresponding vertex from the original mesh

Subdividing the Bicubic Patch



Subdividing the Bicubic Patch

For example: $\mathbf{P}'_{13}$

$$
\begin{aligned}
\mathbf{Q} &= \frac{\mathbf{F}_{0,1} + \mathbf{F}_{0,2} + \mathbf{F}_{1,1} + \mathbf{F}_{1,2}}{4} \\
\mathbf{R} &= \frac{\frac{\mathbf{P}_{0,2}+\mathbf{P}_{1,2}}{2} + \frac{\mathbf{P}_{1,1}+\mathbf{P}_{1,2}}{2} + \frac{\mathbf{P}_{1,3}+\mathbf{P}_{1,2}}{2} + \frac{\mathbf{P}_{2,2}+\mathbf{P}_{1,2}}{2}}{4} \\
\mathbf{S} &= \mathbf{P}_{1,2}
\end{aligned}
$$

Subdividing the Bicubic Patch

- Summary: All sixteen new points of the subdivision have been characterized in terms of
  - face points,
  - edge points
  - vertex points
- A geometric method has been developed to calculate these points.

Extending this Subdivision Procedure to the Entire Patch

- One subdiviosn step:

  16 points → 25 points

- We call the mesh generated by the 25 points as a refinement of the original mesh
- Three types of new points

  face points, edge points, vertex points

Extending this Subdivision Procedure to the Entire Patch

- For each face in the original mesh, generate the new face points - which are the average of all the original points defining the face.
- For each internal edge of the original mesh (i.e. an edge not on the boundary), generate the new edge points - which are calculated as the average of four points: the two points which define the edge, and the two new face points for the faces that are adjacent to the edge.

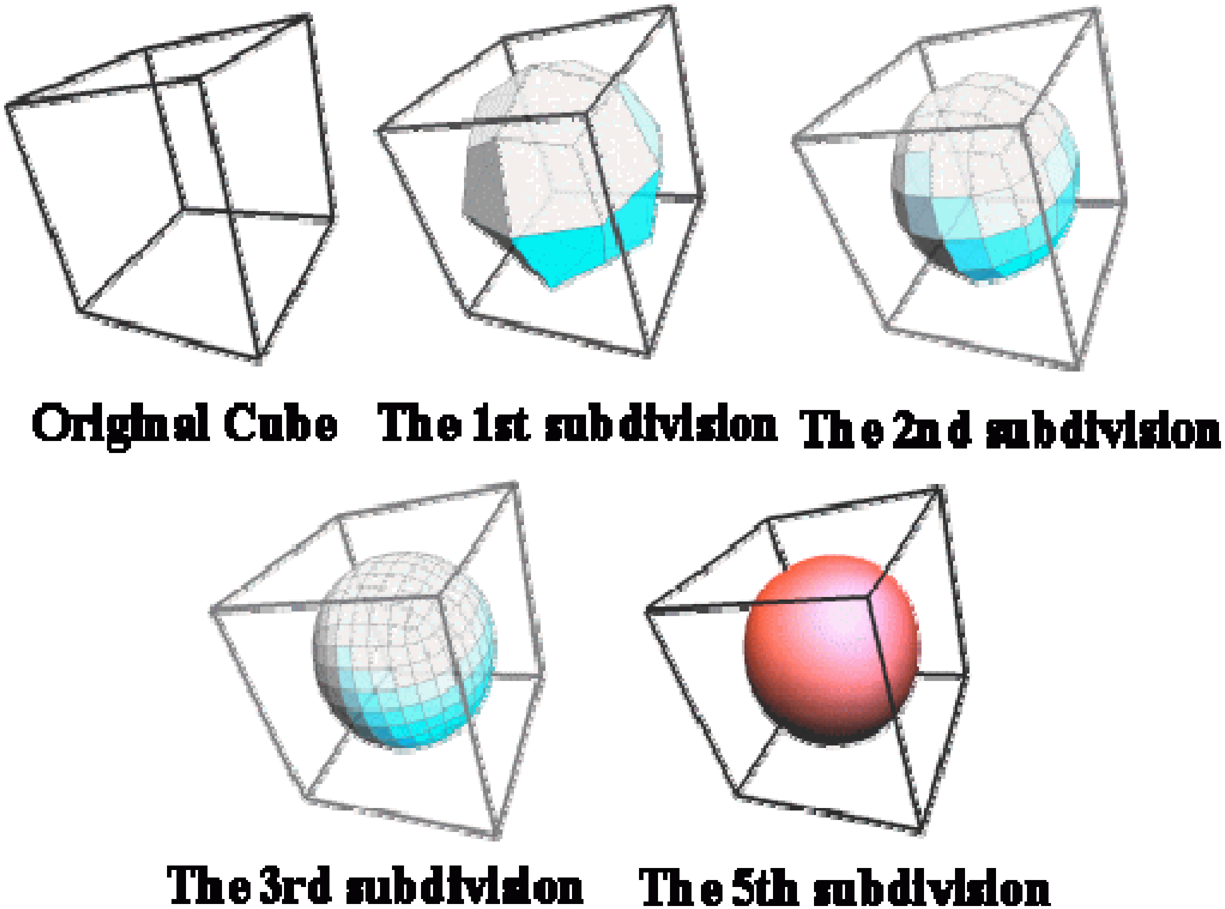
Extending this Subdivision Procedure to the Entire Patch

- For each internal vertex of the original mesh (i.e. a vertex not on the boundary of the mesh), generate the new vertex points – which are calculated as the average of **Q**, **2R** and **S**
  - **Q** is the average of the new face points of all faces adjacent to the original vertex point
  - **R** is the average of the midpoints of all original edges incident on the original vertex point
  - **S** is the original vertex point.

Summary

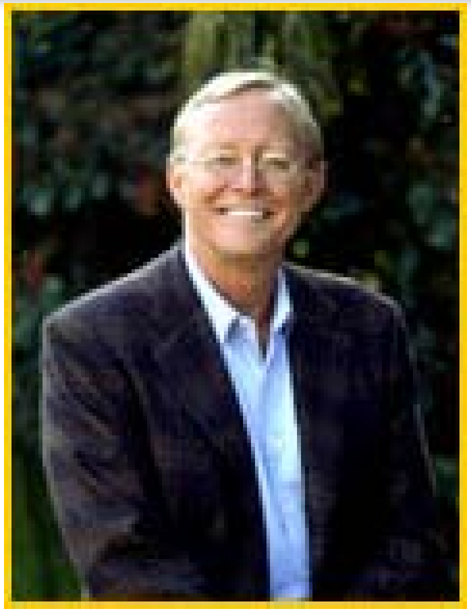
- The subdivision of the bicubic uniform B-spline surface produces a simple procedure based upon face points, edge points and vertex points
- Catmull and Clark produce a refinement strategy that works on a mesh of arbitrary topology based on above procedure

Catmull-Clark Surfaces



Edwin Catmull Jim Clark: COOL!

Edwin Catmull, President of Pixar Animation Studios



Jim Clark is the founder of Netscape Communications Corp., Healtheon/Web MD Corp., and Silicon Graphics, Inc. Currently he serves as Chairman of the Board for myCFO, which he also founded

Catmull-Clark Surfaces

- Overview
- Specifying the Refinement Procedure
- Example
- Summary

Overview

- Ed Catmull and Jim Clark
  - Following the methodology of Doo and Sabin
  - Utilizing the subdivision of bicubic uniform B-spline surfaces for
    - rectangular meshes,
    - meshes of an arbitrary topology
  - Generalizing the definition of a face point
  - Modifying the method for calculating vertex points
  - Specifying a method for reconnecting the points into a mesh

Specifying the Refinement Procedure

- Given a mesh of control points with an arbitrary topology, we can generalize the face point, edge point, vertex point specification from the uniform B-spline surface calculations

Specifying the Refinement Procedure

- For each face of the mesh, generate the new face points - which are the average of all the original points defining the face (We note that faces may have 3, 4, 5, or many points now defining them).
- Generate the new edge points - which are calculated as the average of the midpoints of the original edge with the two new face points of the faces adjacent to the edge.

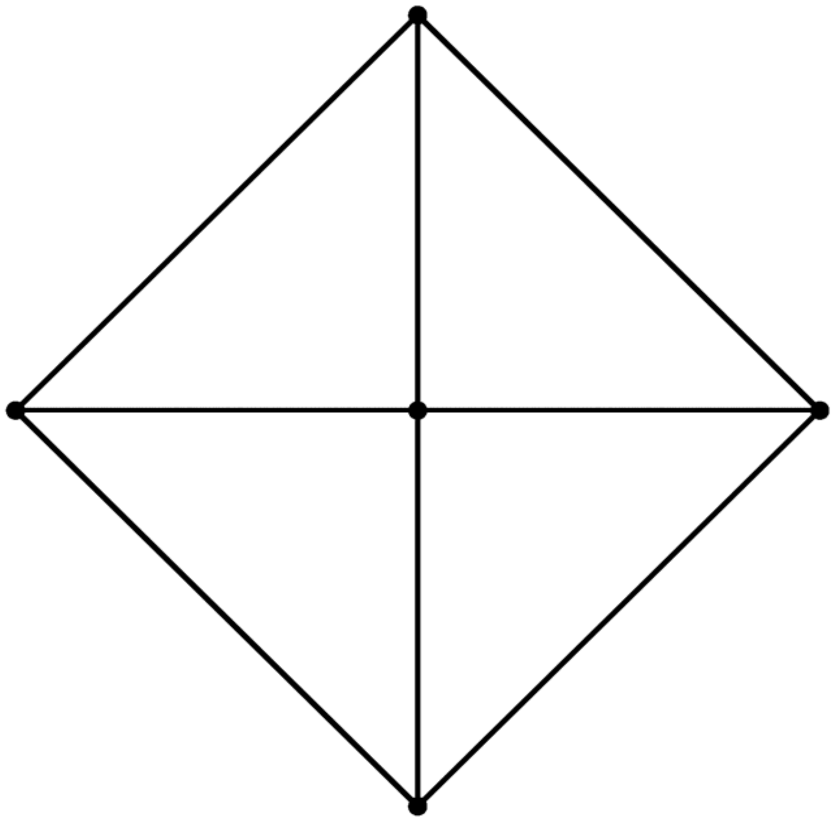
Specifying the Refinement Procedure

- Calculate the new vertex points – which are calculated as the average of **Q**, **2R** and **[(n-3)/n]S**
  - **Q** is the average of the new face points of all faces adjacent to the original face point
  - **R** is the average of the midpoints of all original edges incident on the original vertex point
  - **S** is the original vertex point

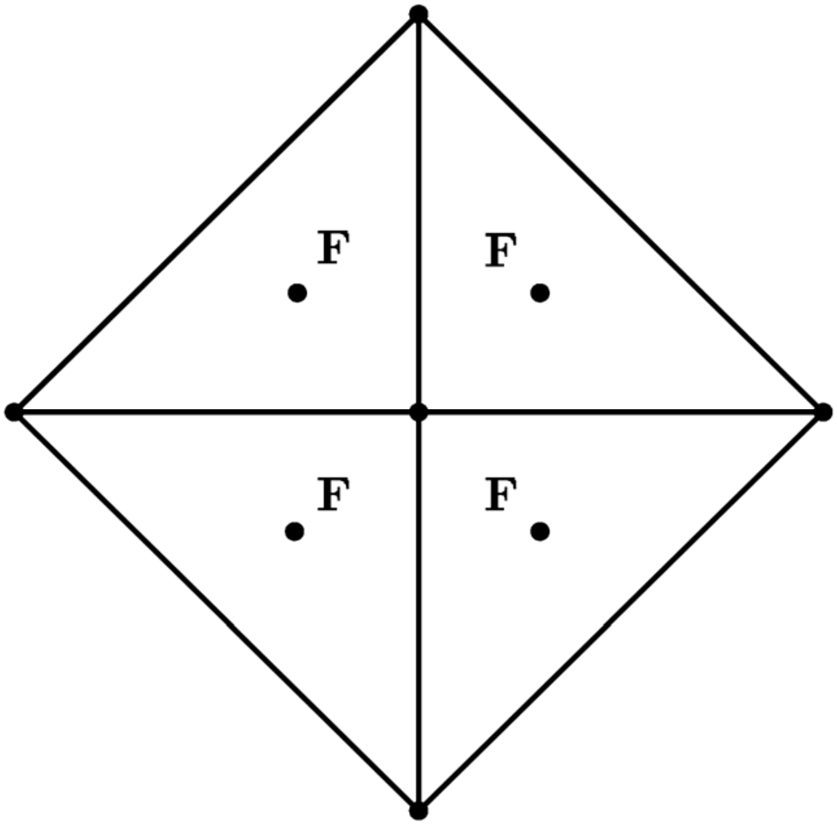
Specifying the Refinement Procedure

- The mesh is reconnected by the following method
  - Each new face point is connected to the new edge points of the edges defining the original face.
  - Each new vertex point is connected to the new edge points of all original edges incident on the original vertex point.

Example (1)

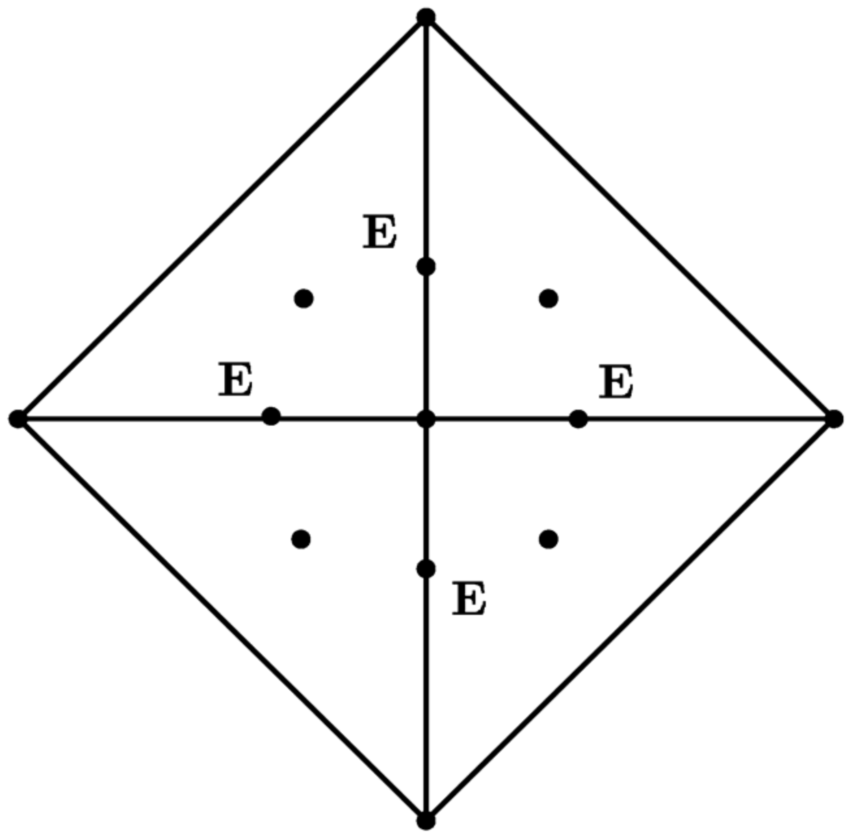


Original mesh

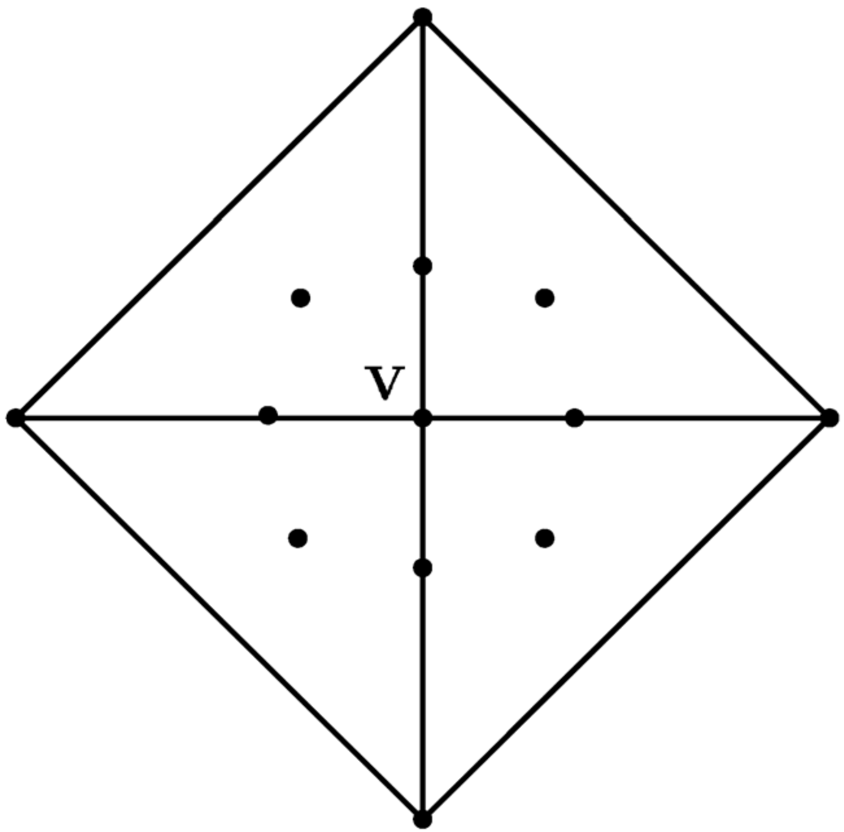


Face points

Example (1)

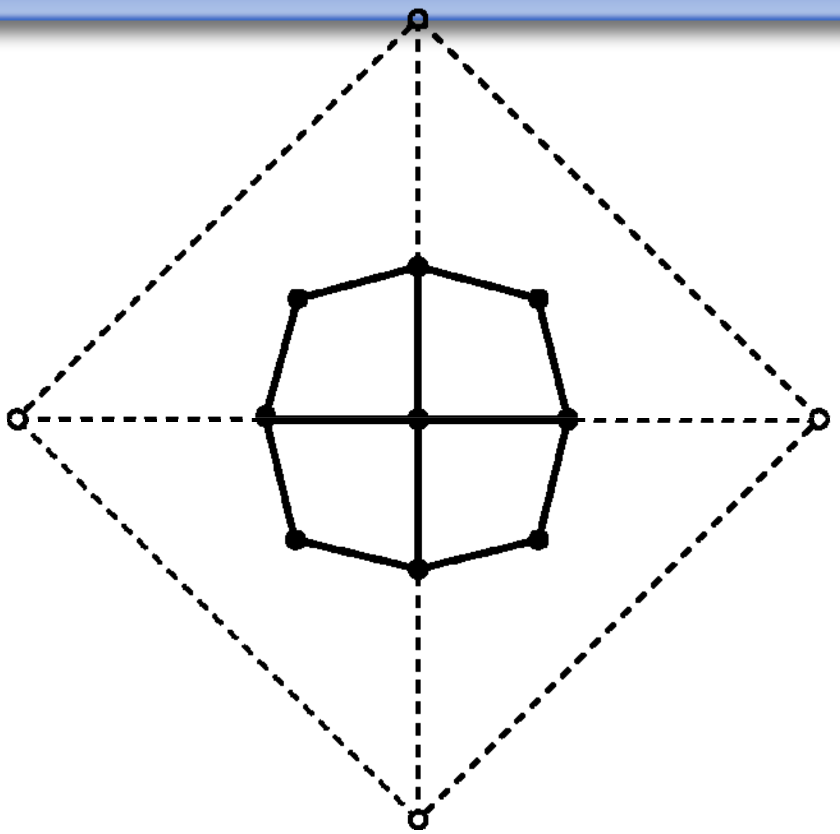


Edge points



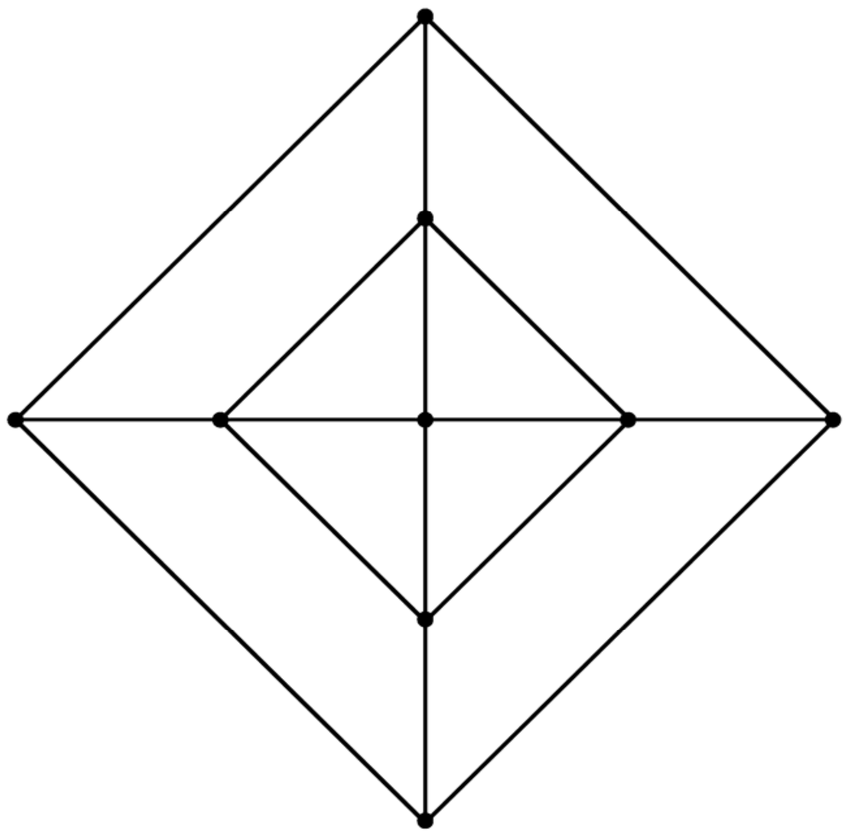
Vertex point

Example (1)

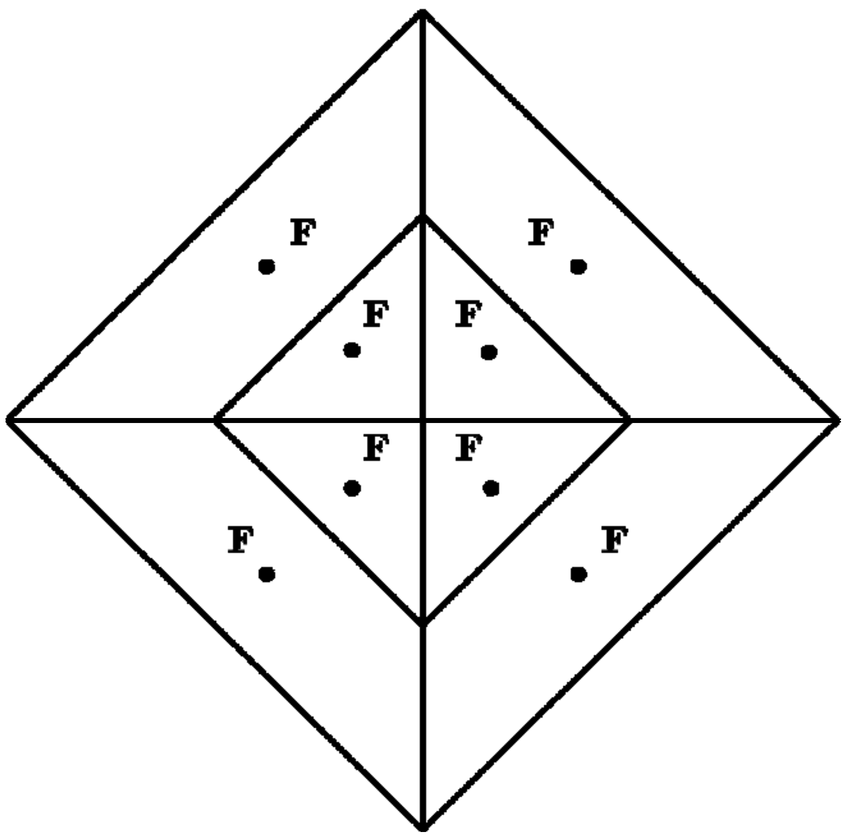


Connect new points to generate refined mesh

Example (2)

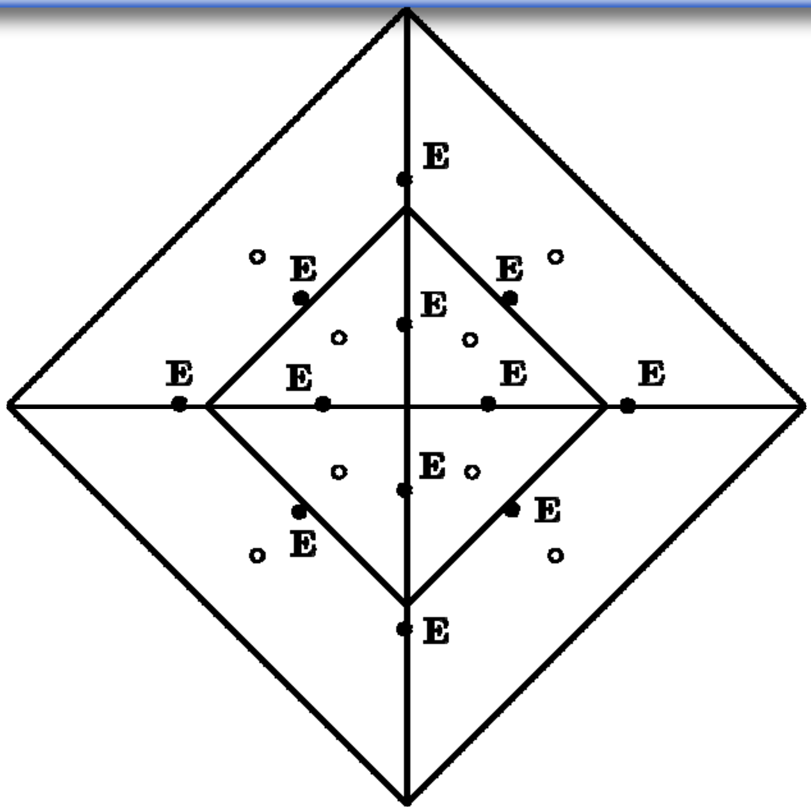


Original mesh

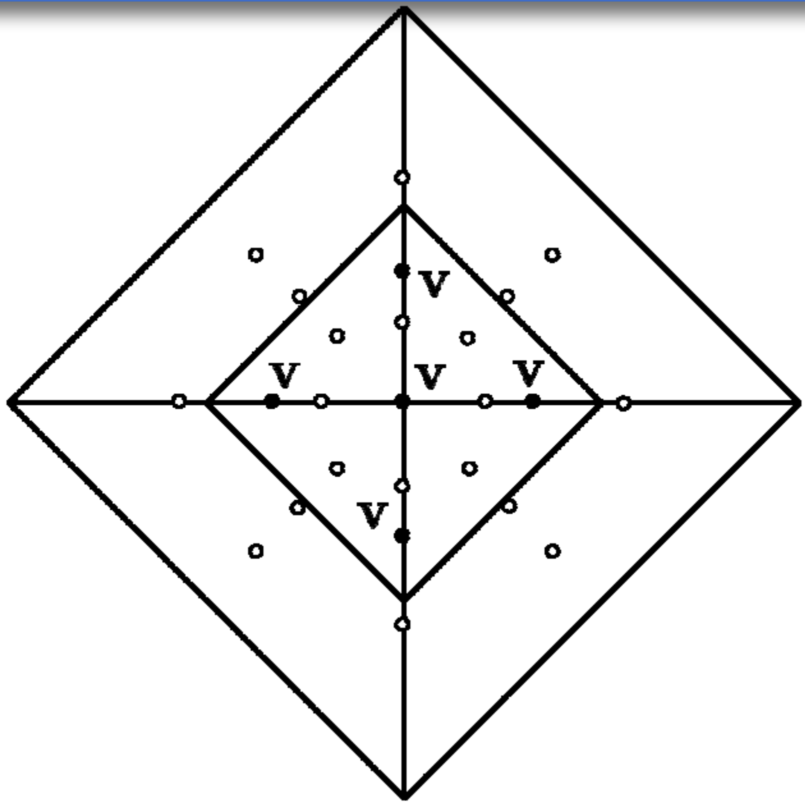


Face points

Example (2)

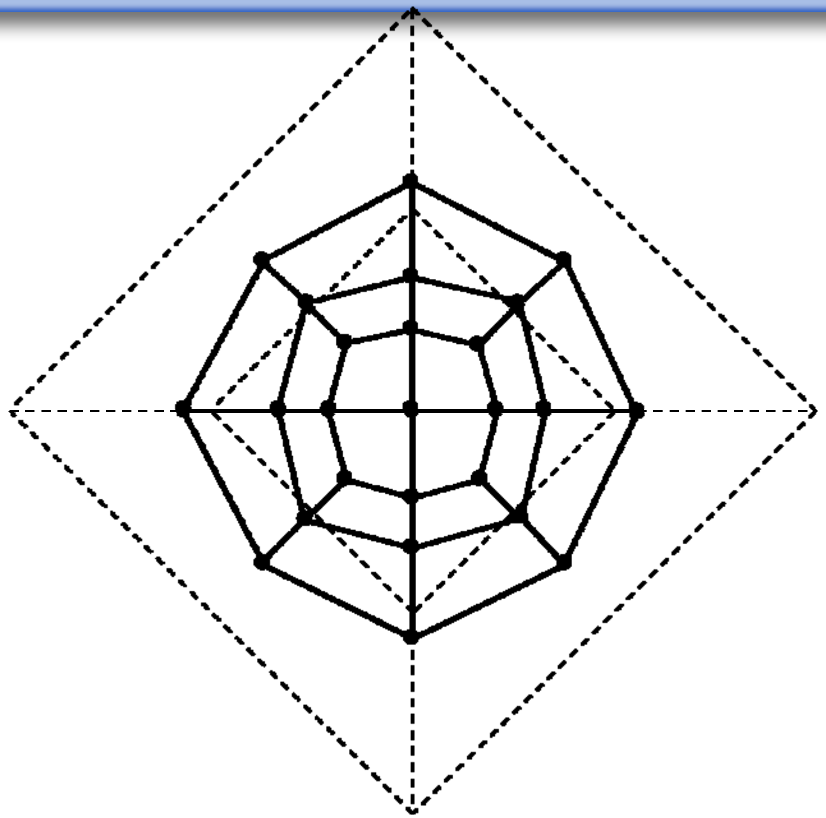


Edge points



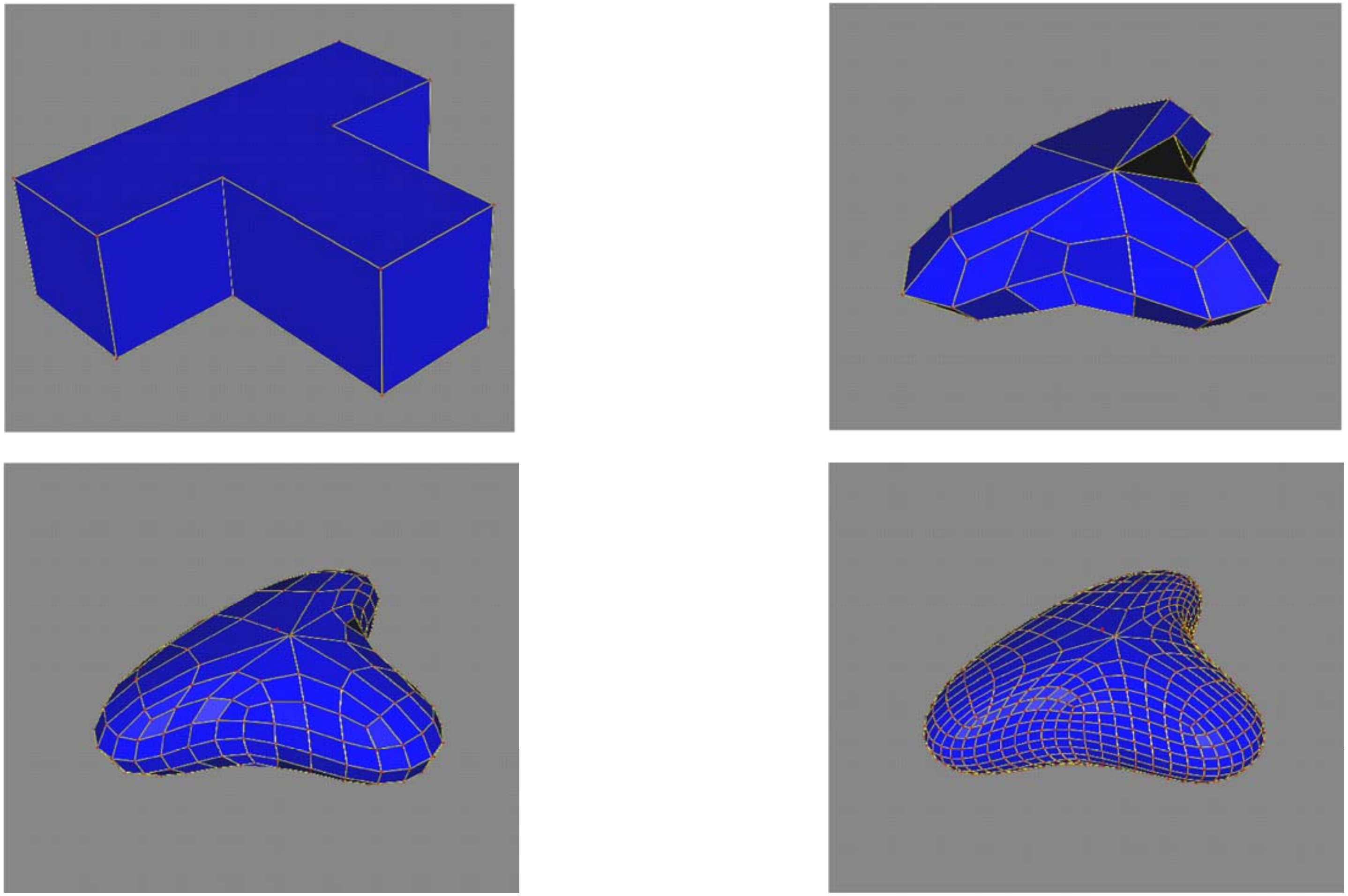
Vertex point

Example (2)

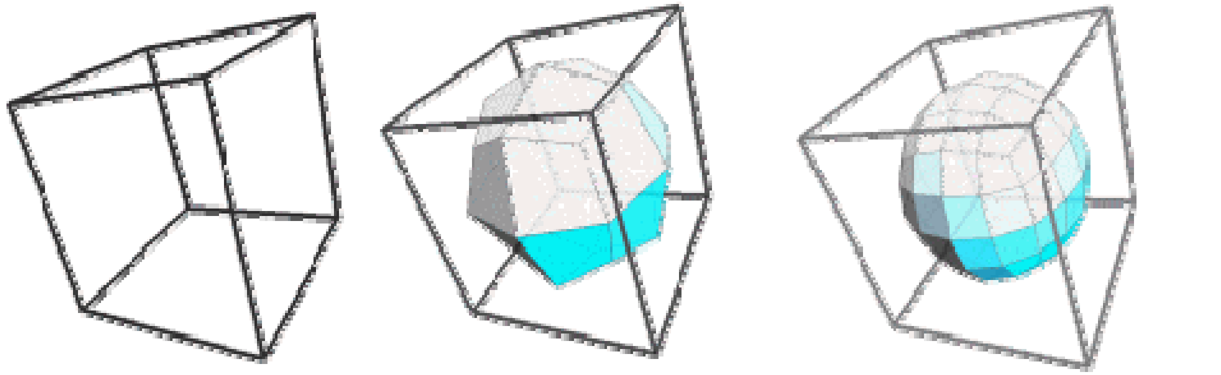


Connect new points to generate refined mesh

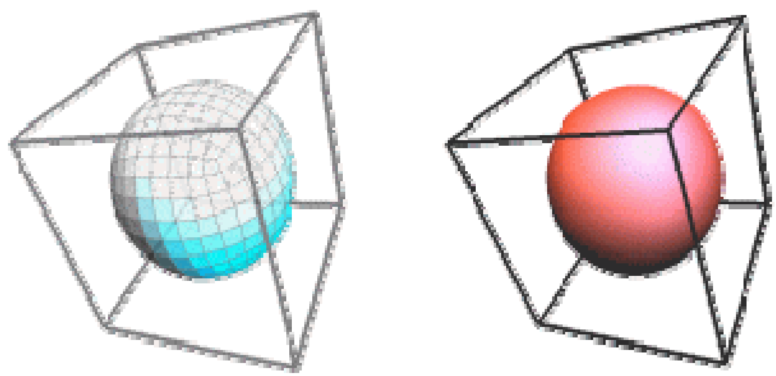
Example (3) --- 3 steps



Example (4)



Original Cube The 1st subdivision The 2nd subdivision



The 3rd subdivision The 5th subdivision

Summary

- We note that the new set of control points has the property that all faces have four sides.
- We also note that the vertices corresponding to the original control points retain the *valence* (the number of edges that are adjacent to the vertex).

Summary

- We note that any portion of the surface where we have a 4×4 array of control points in a rectangular topology, represents a bicubic uniform B-spline surface patch

Summary

- Extraordinary point: the valence is not four
- The limit of Catmull-Clark subdivision surface are piecewise uniform bicubic B-spline surface patches except for the Extraordinary point

Loop Surfaces

- Overview
- Loop Surfaces for Regular Meshes
- Specifying the New Vertex Rules for Extraordinary Points
- Examples

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Researcher
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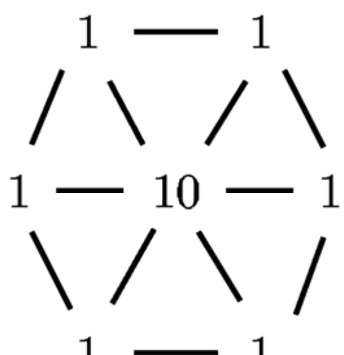
Overview

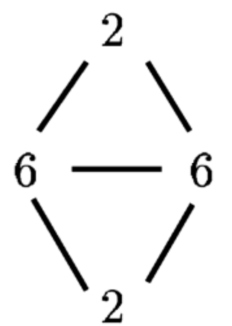
- Doo-Sabin surfaces: uniform biquadratic B-spline surface subdivision
- Catmull-Clark surfaces: bicubic uniform B-spline surface subdivision
- Loop surfaces: the quartic uniform box splines subdivision
  - a mesh of triangles

Loop Surfaces for Regular Meshes

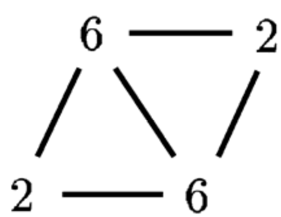
- Given a triangular mesh, the Loop refinement scheme generates
  - vertex points: vertex mask
  - edge points: edge mask



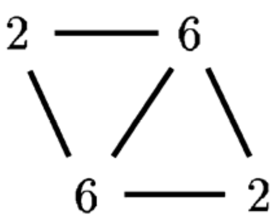
Vertex mask



Edge mask



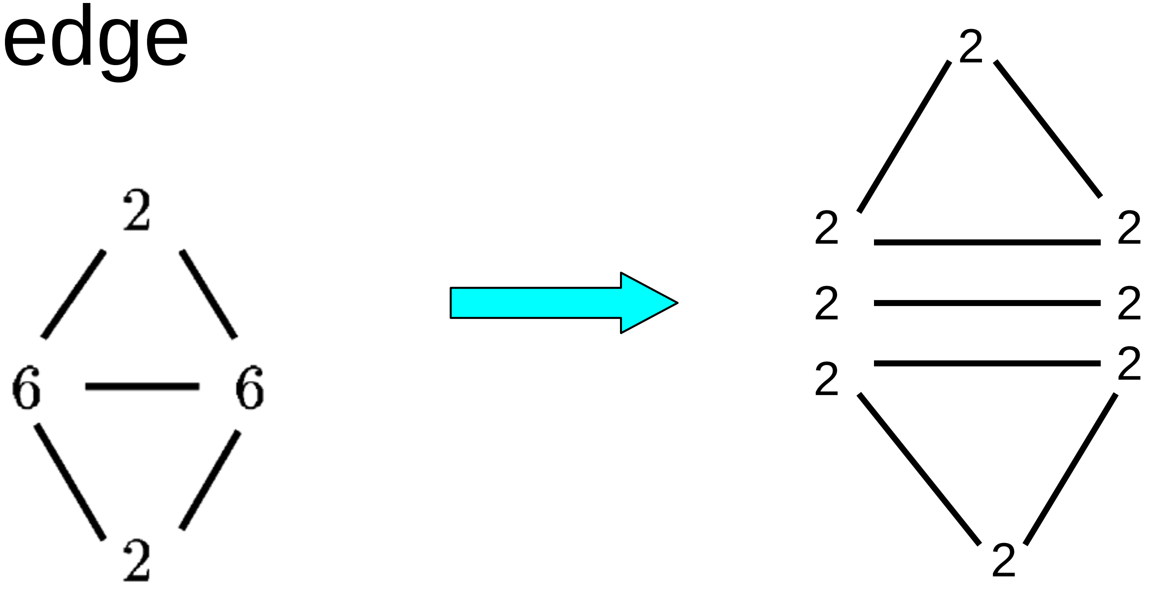
Edge mask



Edge mask

Loop Surfaces for Regular Meshes

- About edge mask: edge point is the average of the two centers of the faces that share the edge and the midpoint of the edge



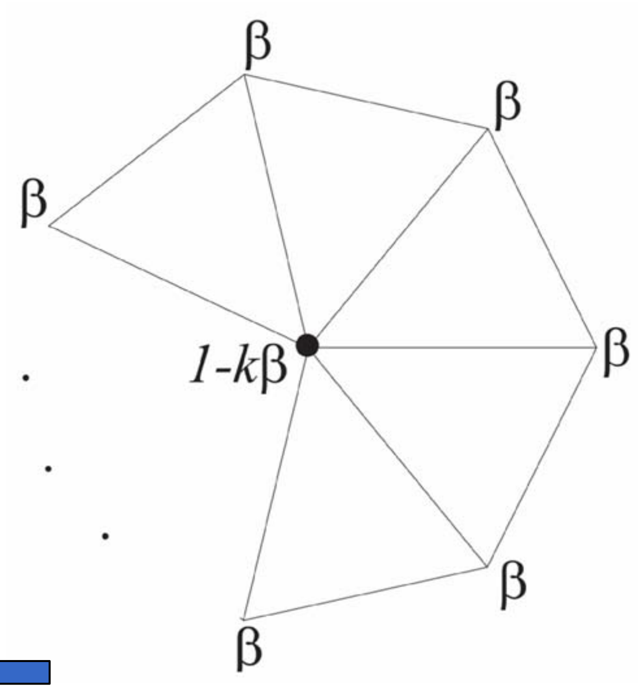
Loop Surfaces for Regular Meshes

- About the vertex mask: the vertex point is a convex combination of the points **V** (5/8), the original vertex, and **Q**(3/8) the average of the original points that share an edge with **V**

$$
\begin{aligned}
\mathbf{V}^1 &= \frac{10\mathbf{V} + \mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 + \mathbf{Q}_4 + \mathbf{Q}_5 + \mathbf{Q}_6}{16} \\
&= \frac{5}{8}\mathbf{V} + \frac{\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3 + \mathbf{Q}_4 + \mathbf{Q}_5 + \mathbf{Q}_6}{16} \\
&= \frac{5}{8}\mathbf{V} + \frac{6\mathbf{Q}}{16} \\
&= \frac{5}{8}\mathbf{V} + \frac{3}{8}\mathbf{Q}
\end{aligned}
$$

Specifying the New Vertex Rules for Extraordinary Points

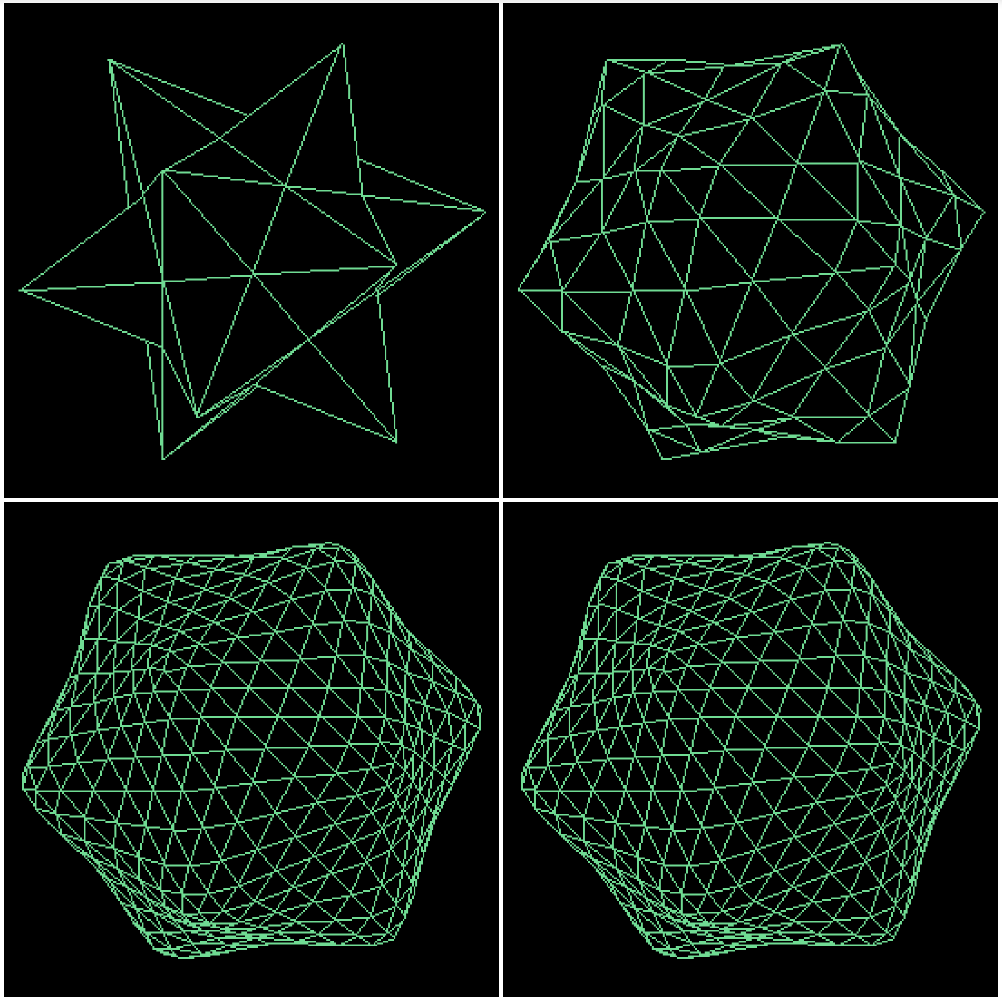
- Extraordinary point (valence $\neq 6$):



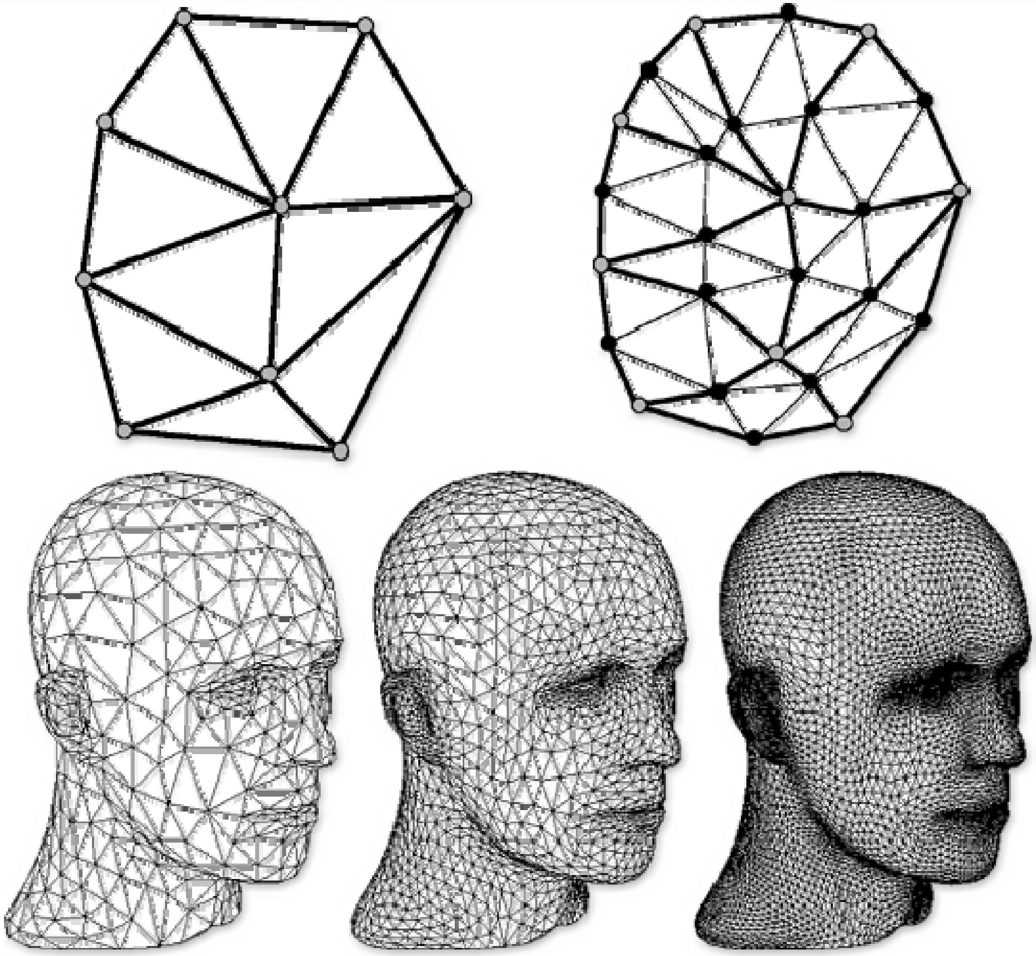
$$\beta = \frac{1}{k}\left(\left(\frac{3}{8}+\frac{1}{4}\cos\frac{2\pi}{k}\right)^2+\frac{3}{8}\right)$$

k: valence of the extraordinary points

Examples



Examples



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