B-Spline Interpolation and Approximation

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Parameter Selection and Knot Vector Generation

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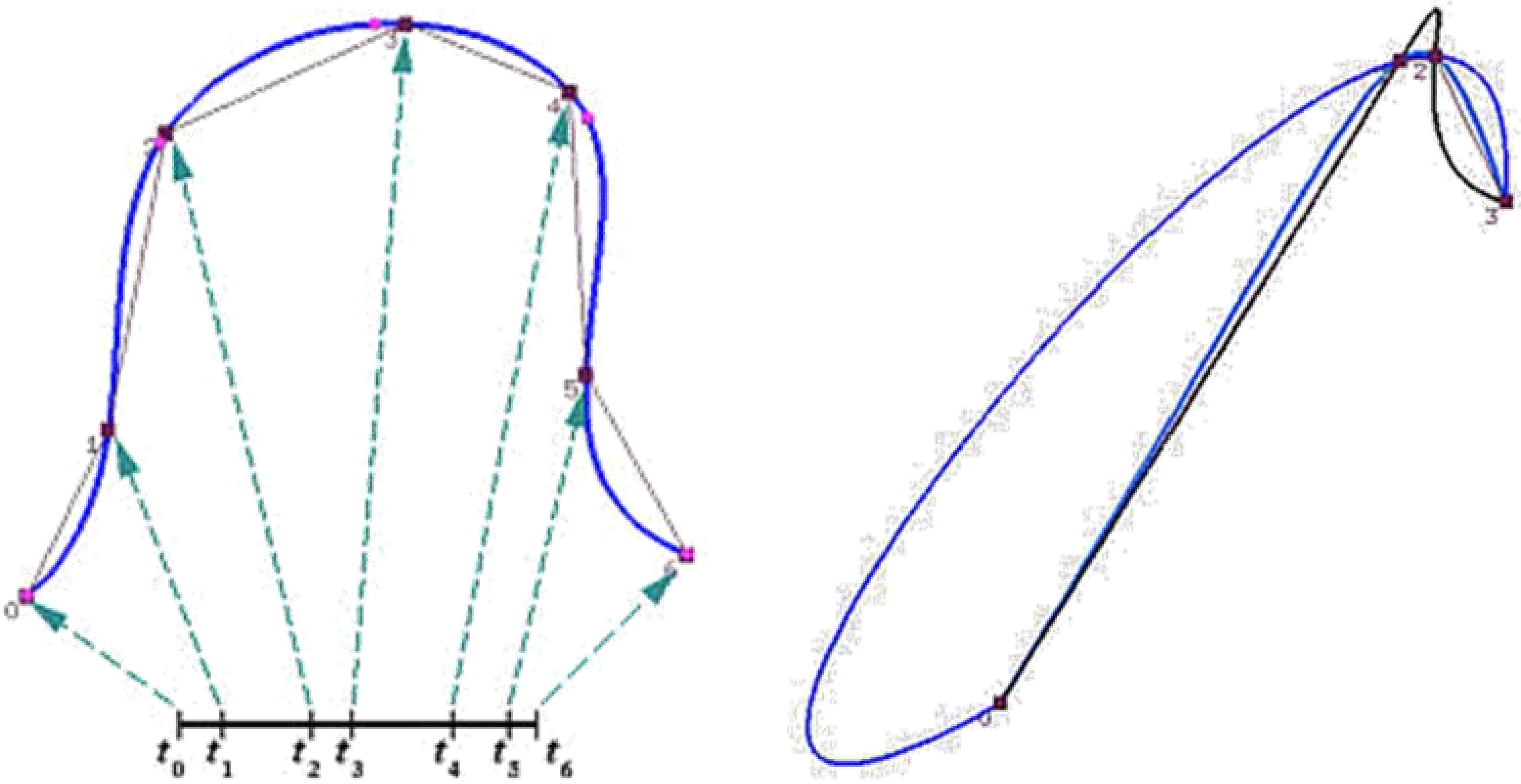
Overview Parameter Selection and Knot Vector Generation

- B-spline interpolation
  - Input a set of data points $\mathbf{D}_0, \ldots, \mathbf{D}_n$
  - Find
    - A B-Spline Curve: $\mathbf{C}=\mathbf{C}(t)$
    - $n+1$ parameters $t_0, \ldots, t_n$
  - Such that

$$\mathbf{D}_k = \mathbf{C}(t_k) \text{ for all } 0 \le k \le n.$$

Overview Parameter Selection and Knot Vector Generation



Examples of B-Spline Interpolation

Overview Parameter Selection and Knot Vector Generation

- How to choose these parameters ?
  - Infinite number of possibilities !
  - Parameters' selection will greatly influence
    - shape of the curve
    - parameterization of the curve

- There are other methods for selecting parameters besides our introduction

The Uniformly Spaced Method

- There is $n+1$ data points $\{\mathbf{D}_k\}$ $k=0,1,\ldots,n$
- The parametric domain is $[0,1]$
  - The end parameters: $t_0 = 0$ and $t_n=1$
  - The other parameters:

$$1/n,\ 2/n,\ 3/n,\ \ldots,\ (n\text{-}1)/n$$

- Example: $n = 7$, uniformly spaced parameters

$$0,\ 1/7,\ 2/7,\ 3/7,\ 4/7,\ 5/7,\ 6/7,\ 1$$

The Uniformly Spaced Method

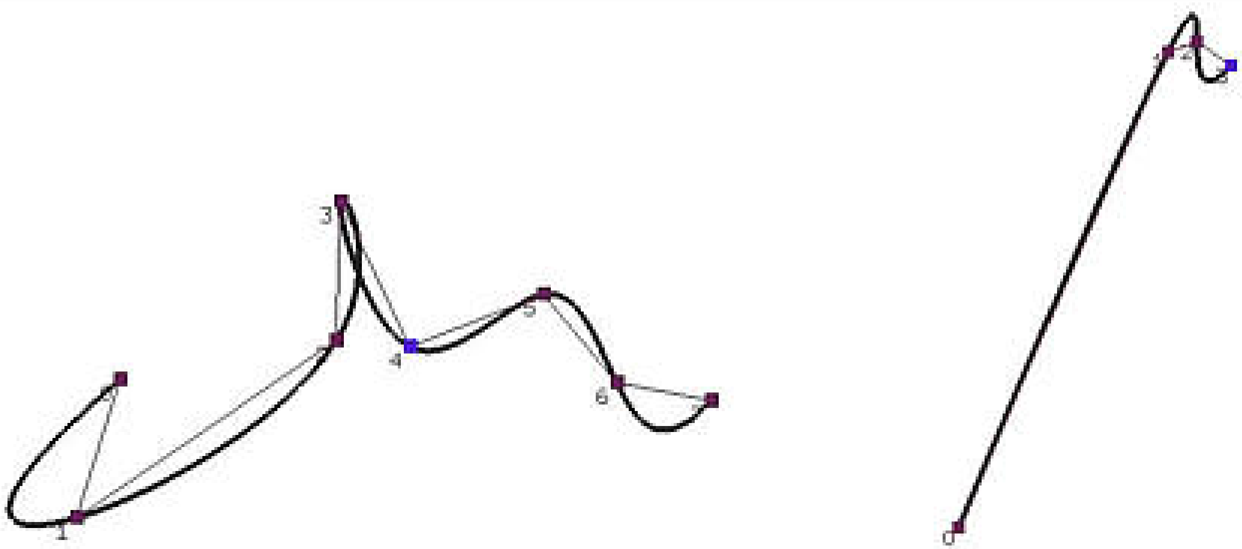
- The parametric domain is $[a,b]$
  - The end parameters: $t_0 = a$ and $t_n=b$
  - The other parameters:

$$\begin{cases} t_0 = a \\ t_i = a + i\dfrac{b-a}{n} & \text{for } 1 \le i \le n-1 \\ t_n = b \end{cases}$$

The Uniformly Spaced Method

- About the uniformly spaced method
  - Simple
  - May cause unpleasant results
    - big bulges, sharp peaks and loops
  - These problems are unique to the uniformly spaced method, it does occur more frequently than with other methods

Example: The Uniformly Spaced Method



Examples: B-Spline curve interpolation with the uniformly spaced method

The Chord Length Method

- Arc-length parameterization
  - If an interpolating curve follows very closely to the data polygon
    - Between two adjacent data points

      arc-length ≅ *chord*
    - For the curve

      arc-length ≅ polygon *chord*
  - If the domain is subdivided according to the distribution of the chord lengths, we can get an approximation of the arc-length parameterization

approximation of arc-length parameterization

The Chord Length Method

- The data points are $\mathbf{D}_0, \mathbf{D}_1, \ldots, \mathbf{D}_n$
- The total chord length L

$$L = \sum_{i=1}^{n} \left| \mathbf{D}_i - \mathbf{D}_{i-1} \right|$$

- The chord length from $\mathbf{D}_0$ to $\mathbf{D}_k$: L_k

$$L_k = \sum_{i=1}^{k} \left| \mathbf{D}_i - \mathbf{D}_{i-1} \right|$$

The Chord Length Method

- The domain is $[0,1]$, then parameter t_k should be located at the value of L_k

$$t_0 = 0$$

$$t_k = \frac{\sum_{i=1}^{k} \left| \mathbf{D}_i - \mathbf{D}_{i-1} \right|}{L} \qquad \text{for } k = 1, \cdots, n-1$$

$$t_n = 1$$

Example: the Chord Length Method

- Four data points: $\mathbf{D}_0$=(0,0), $\mathbf{D}_1$=(1,2), $\mathbf{D}_2$=(3,4) , $\mathbf{D}_3$=(4,0)

$$|\mathbf{D}_1 - \mathbf{D}_0| = \sqrt{5} = 2.236$$

$$|\mathbf{D}_2 - \mathbf{D}_1| = 2\sqrt{2} = 2.828$$

$$|\mathbf{D}_3 - \mathbf{D}_2| = \sqrt{17} = 4.123$$

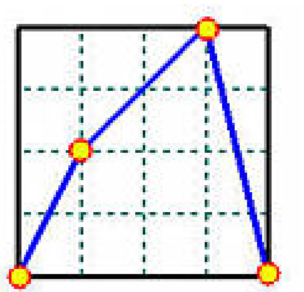
$$L = \sqrt{5} + 2\sqrt{2} + \sqrt{17} = 9.8176$$

$$t_0 = 0$$

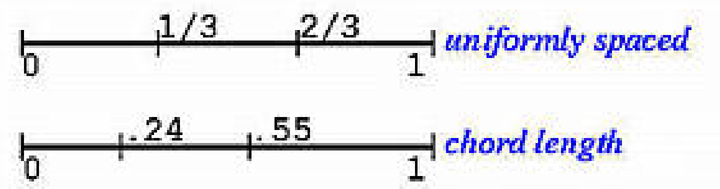
$$t_1 = \frac{|\mathbf{D}_1 - \mathbf{D}_0|}{L} = 0.2434$$

$$t_2 = \frac{|\mathbf{D}_1 - \mathbf{D}_0| + |\mathbf{D}_2 - \mathbf{D}_1|}{L} = 0.5512$$

$$t_3 = 1$$



Four data points



Parameters

The Chord Length Method

- The parametric domain is $[a,b]$
  - The end parameters: $t_0 = a$ and $t_n=b$
  - The other parameters

$$t_0 = a$$

$$t_i = a + \frac{L_k}{L}(b-a) \qquad \text{for } k = 1, \cdots, n-1$$

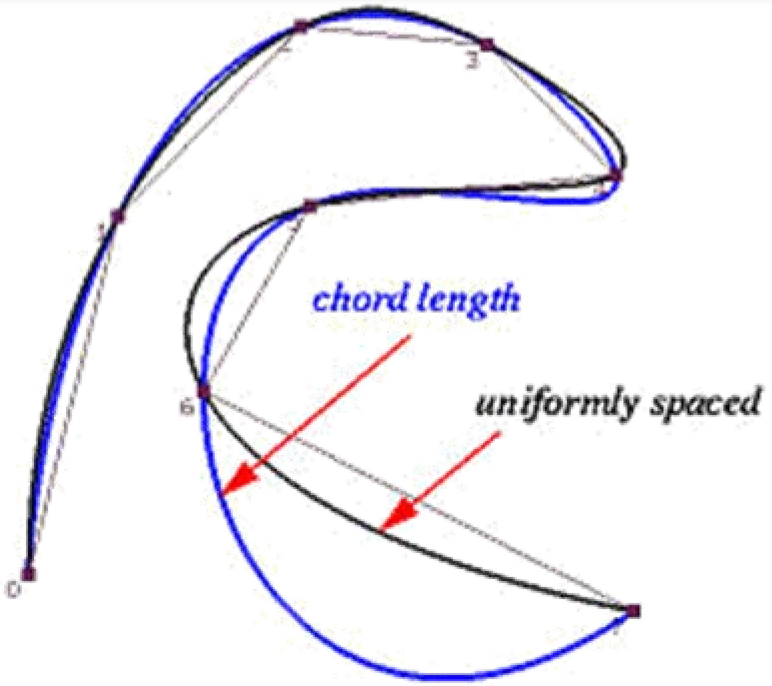
$$t_n = b$$

  - The L_k is defined as the chord length from $\mathbf{D}_0$ to $\mathbf{D}_k$

The Chord Length Method

- About the chord length method
  - The chord length method is widely used and usually performs well
  - The polynomial curves cannot be arc-length parameterized, the chord length can only be an approximation
  - A longer chord may cause its curve segment to have a bulge bigger than necessary

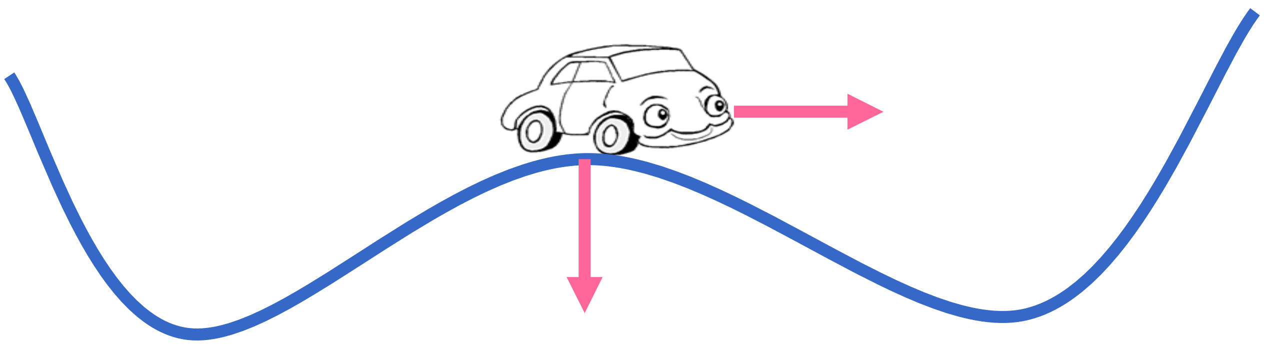
The Chord Length Method



For chord length method, most of the curve segments wiggling a little. The last curve segments has a large bulge and twists away from the black curve produced by the uniformly spaced method

The Centripetal Method

- Physical interpretation:
  - Suppose we are driving a car through a slalom course. We have to be very careful at sharp turns so that the normal acceleration (*i.e.*, centripetal force) should not be too large.
  - The normal force along the path should be proportional to the change in angle. The centripetal method is an approximation to this model
  - The centripetal method is an extension to the chord length method



The Centripetal Method

- The data points are $\mathbf{D}_0, \mathbf{D}_1, \ldots, \mathbf{D}_n$
- A positive "power" value α
  - α=1/2: square root of chord length

- The total “length” L:

$$L = \sum_{i=1}^{n} \left| \mathbf{D}_i - \mathbf{D}_{i-1} \right|^{\alpha}$$

The Centripetal Method

- The parameters on $[0,1]$:

$$t_0 = 0$$

$$t_k = \frac{\sum_{i=1}^{k} \left| \mathbf{D}_i - \mathbf{D}_{i-1} \right|^{\alpha}}{L} \qquad \text{for } k = 1, 2, \cdots, n-1$$

$$t_n = 1$$

Discussion of α

- If $\alpha = 1$, the centripetal method = the chord length method
- If $\alpha < 1$, say $\alpha = 1/2$ (*i.e.*, square root)
  - $|\mathbf{D}_k - \mathbf{D}_{k-1}|^\alpha < |\mathbf{D}_k - \mathbf{D}_{k-1}|$ if length > 1

    the impact of a longer chord on the length of the data polygon is reduced
  - $|\mathbf{D}_k - \mathbf{D}_{k-1}|^\alpha > |\mathbf{D}_k - \mathbf{D}_{k-1}|$ if length < 1

    the impact of a shorter chord on the length of the data polygon is increased

Example: The Centripetal Method

- Four data points: $\mathbf{D}_0$=(0,0), $\mathbf{D}_1$=(1,2), $\mathbf{D}_2$=(3,4) , $\mathbf{D}_3$=(4,0)

$$\left|\mathbf{D}_1 - \mathbf{D}_0\right|^{1/2} = \sqrt{\sqrt{5}} = 1.495$$

$$\left|\mathbf{D}_2 - \mathbf{D}\right|^{1/2} = \sqrt{2\sqrt{2}} = 1.682$$

$$\left|\mathbf{D}_3 - \mathbf{D}_2\right|^{1/2} = \sqrt{\sqrt{17}} = 2.031$$

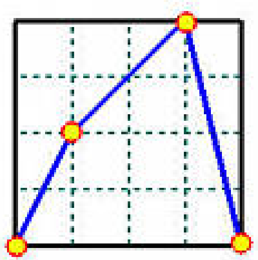
$$L = \sqrt{\sqrt{5}} + \sqrt{2\sqrt{2}} + \sqrt{\sqrt{17}} = 5.208$$

$$t_0 = 0$$

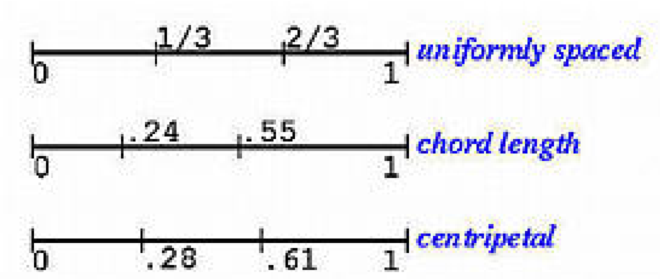
$$t_1 = \frac{\left|\mathbf{D}_1 - \mathbf{D}_0\right|^{1/2}}{L} = 0.2871$$

$$t_2 = \frac{\left|\mathbf{D}_1 - \mathbf{D}_0\right|^{1/2} + \left|\mathbf{D}_2 - \mathbf{D}_1\right|^{1/2}}{L} = 0.6101$$

$$t_3 = 1$$

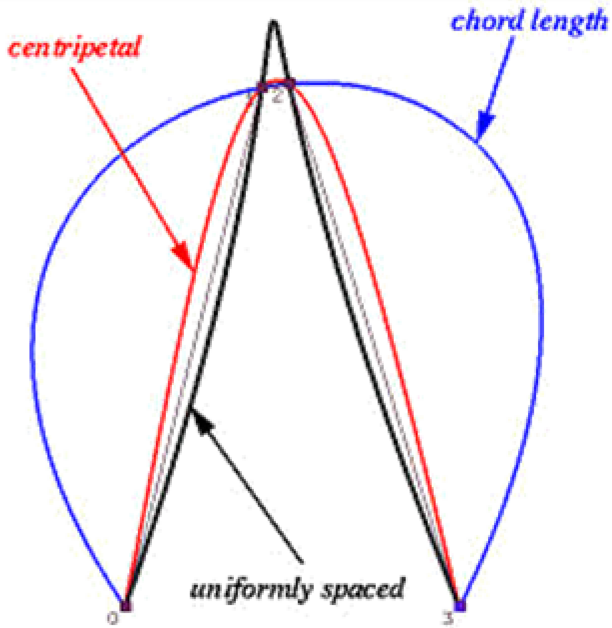


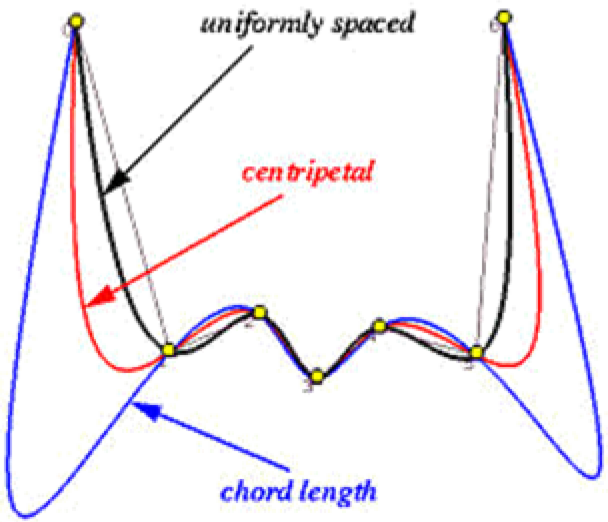
Four data points



Parameters comparison among three methods

Example: The Centripetal Method





- The uniformly spaced method has a peak
- The chord length method have two big bulges
- The centripetal method interpolates the two very close adjacent points nicely

- The uniformly spaced method provides a very tight interpolation.
- The centripetal method is slightly off the tight result using the uniformly spaced method.
- The chord length method wiggles through the two longest chords too much

Knot Vector Generation

- How to generate a knot vector for B-spline after a set of parameters is obtained ?
  - Available:

    $n+1$ parameters $t_0, t_1, ..., t_n$

    degree p for a B-spline curve
  - Compute: $m+1$ knots, where $m=n+p+1$

    The curve is clamped

    $$u_0 = u_1 = ... = u_p = 0$$

    $$u_{m-p} = u_{m-p+1} = ... = u_m = 1.$$

    For the remaining n-p knots ($u_{p+1}, ..., u_{m-p-1}$)

    uniformly spaced

    satisfy some desired conditions.

Knot Vector Generation

- Uniformly spaced knot vector

$$u_0 = u_1 = \cdots = u_p = 0$$

$$u_{j+p} = \frac{j}{n-p+1} \qquad \text{for } j = 1, 2, \cdots, n-p$$

$$u_{m-p} = u_{m-p+1} = \cdots = u_m = 1$$

  - Simple
  - Irrelevant with the parameter $\{t_i\}$

Example: uniformly spaced knot vector

- Have 6 (n = 5) parameters
- The degree $p = 3$
- Find $(n+p+1)+1 = (5+3+1)+1 = 10$ knots (*i.e.*, m=9) as

$$0, 0, 0, 0, u_4, u_5, 1, 1, 1, 1$$

Finally the knot vector is

$$\{0, 0, 0, 0, 1/3, 2/3, 1, 1, 1, 1\}.$$

Knot Vector Generation

- Problem with the uniformly spaced knot vector

    if it is used with the **chord length** method for **global interpolation**, the system of linear equations would be **singular** !

Knot Vector Generation

- "Average" method: relevant with parameters $\{t_i\}$

$$u_0 = u_1 = \cdots = u_p = 0$$

$$u_{j+p} = \frac{1}{p}\sum_{i=j}^{j+p-1} t_i \qquad \text{for } j = 1, 2, \cdots, n-p$$

$$u_{m-p} = u_{m-p+1} = \cdots = u_m = 1$$

  - First internal knot is the average of p parameters t_1, t_2, ..., t_p;
  - Second internal knot is the average of the next p parameters, t_2, t_3, ..., t_{p+1}

Example: "Average" method

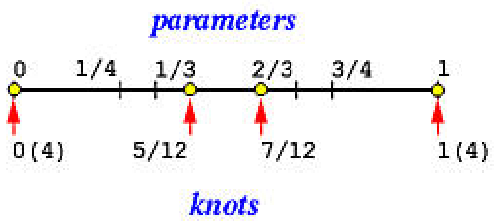
For cubic B-spine curve

The parameters

t_0	t_1	t_2	t_3	t_4	t_5
0	1/4	1/3	2/3	3/4	1

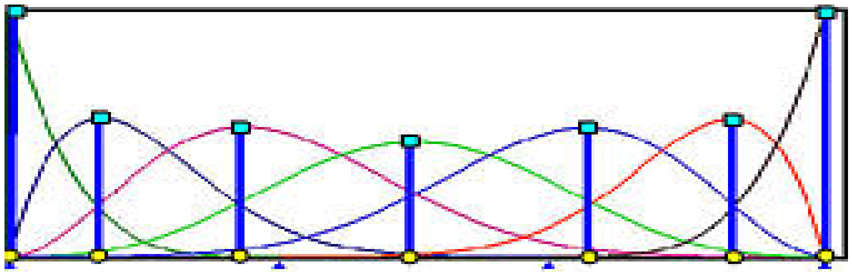
The average knot vector

$u_0=u_1=u_2=u_3$	u_4	u_5	$u_6=u_7=u_8=u_9$
0	(1/4+ 1/3 + 2/3)/3 = 5/12	(1/3 + 2/3 + 3/4)/3 = 7/12	1



The Universal Method

- In previously discussed methods
  - the parameters $\rightarrow$ a knot vector
- The Universal Method
  - uniformly spaced knots $\rightarrow$ computing the parameters
  - the parameters at which their corresponding basis functions reach a maximum



Example: the Universal Method

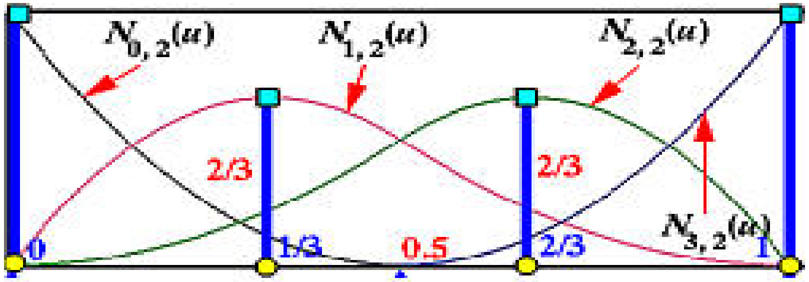
- Example
  - 4 data points (*i.e.*, n=3)
  - degree p=2
  - number of knots is 7 (*i.e.*, $m = n + p + 1 = 6$)
  - uniformly spaced knot vector

$u_0 = u_1 = u_2$	u_3	$u_4 = u_5 = u_6$
0	0.5	1

Example: the Universal Method

B-Spline base functions

Basis Function	***Equation***	***Non-zero Range***
$N_{0,2}(u)$	$(1\text{-}2u)^2$	[0,0.5)
$N_{1,2}(u)$	$2u(2\text{-}3u)$	[0,0.5)
	$2(1\text{-}u)^2$	[0.5,1)
$N_{2,2}(u)$	$2u^2$	[0,0.5)
	$-2(1\text{-}4u+3u^2)$	[0.5,1)
$N_{3,2}(u)$	$(2u\text{-}1)^2$	[0.5,1)



The parameter vector is { 0, 1/3, 2/3, 1}

Discussion: the universal method

- The maximum of a B-spline basis function does not have to be computed precisely
  - Sampling some values in the non-zero domain and choosing the one with maximum function value usually provides approximate result
  - One-dimensional search techniques such as the Golden Section Search can be used for accurate result

Discussion: the universal method

- The universal method has *affine invariant* property

  The transformed interpolating B-spline curve can be obtained by transforming the data points

- The uniformly spaced method is *affine invariant*

  The knot vector is computed from a set of uniformly spaced parameters which are not changed before and after a transformation

- The chord length method and centripetal method are ***not*** affine invariant

Parameters and Knot Vectors for Surfaces

- A B-spline surface of degree (p,q) defined by $e+1$ rows and $f+1$ columns of control points has the following equation

$$\mathbf{S}(u,v) = \sum_{i=0}^{e} \sum_{j=0}^{f} \mathbf{P}_{ij} N_{i,p}(u) N_{j,q}(v)$$

it requires *two* sets of parameters for surface interpolation and approximation

Parameters and Knot Vectors for Surfaces

- Suppose: data points $\mathbf{D}_{ij}$ $(0 \le i \le m$ and $0 \le j \le n)$
- Computation:
  - $m+1$ parameters $s_0, \ldots, s_m$ in the u-direction (*i.e.*, one for each row of data points)
  - $n+1$ parameters $t_0, \ldots, t_n$ in the v-direction (*i.e.*, one for each column of data points)
  - Satisfy:

$$\mathbf{D}_{cd} = \mathbf{S}(s_c, t_d) = \sum_{i=0}^{e} \sum_{j=0}^{f} \mathbf{P}_{ij} N_{i,p}(s_c) N_{j,q}(t_d)$$

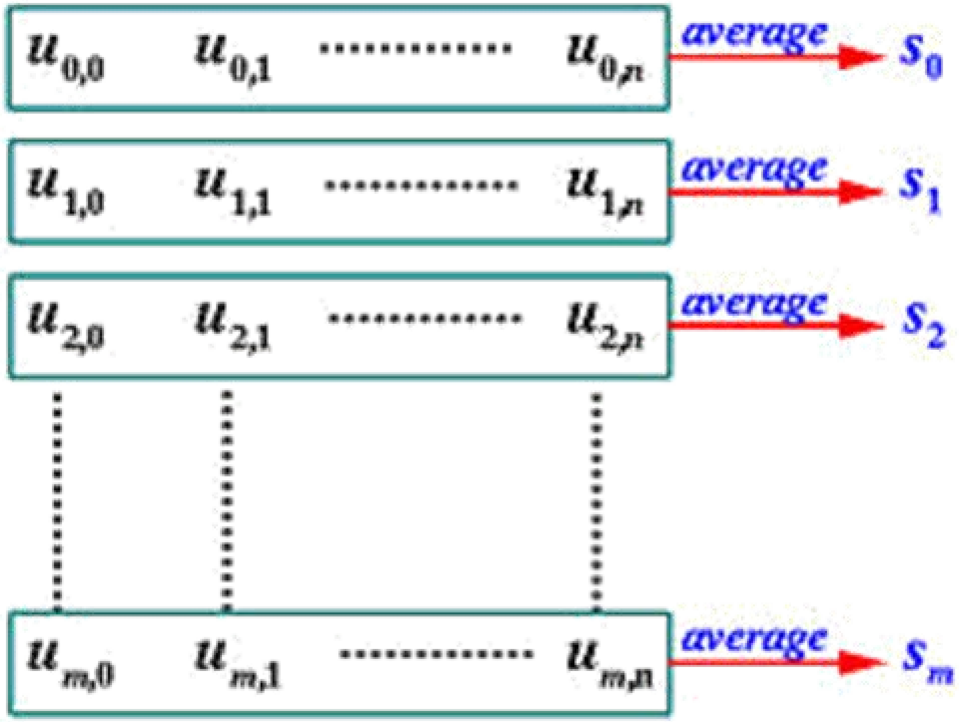
Parameters and Knot Vectors for Surfaces

- For data points on column j (j=0,1,..,m), compute m+1 parameters $u_{0,j}$, $u_{1,j}$, …, $u_{m,j}$
- The desired parameters s_0, s_1, …, s_m are simply the average of each row

$$s_i = (u_{i,0} + u_{i,1} + \ldots + u_{i,n})/(n+1)$$

- For the t-direction can be evaluated similarly

Parameters and Knot Vectors for Surfaces



Parameters and Knot Vectors for Surfaces

Discussion: Parameters and Knot Vectors for Surfaces

- The above algorithm works for the uniformly spaced, chord length and centripetal methods
- For the universal method, because there is no data points involved, we can apply uniform knots to one row and one column of data points for computing the parameters

Global Curve Interpolation

- Problem Statement
- Solution for Global Curve Interpolation
- The Impact of Parameters and Knots
- The Impact of Degree
- Why “Global”

Problem Statement

Global Curve Interpolation

Given a set of $n+1$ data points, $D_0, D_1, \ldots, D_n$ and a degree p, find a B-spline curve of degree p defined by $n+1$ control points that passes all data points in the given order.

Solution for Global Curve Interpolation

- The interpolating B-spline curve of degree p :

$$\mathbf{C}(u) = \sum_{i=0}^{n} N_{i,p}(u)\mathbf{P}_i$$

- The parameter values $t_0, t_1, \ldots, t_n$
- The knot vectors can be determined by above methods

Solution for Global Curve Interpolation

- The parameter t_k corresponds to data point $\mathbf{D}_k$, substituting t_k into equation

$$\mathbf{D}_k = \mathbf{C}(u_k) = \sum_{i=0}^{n} N_{i,p}(u_k)\mathbf{P}_i \qquad \text{for } 0 \le k \le n$$

- Rewrite it as the matrix form:

$$\mathbf{D} = \mathbf{N} \cdot \mathbf{P}$$

Solution for Global Curve Interpolation

where

$$\mathbf{N}=\begin{bmatrix} N_{0,p}(t_0) & N_{1,p}(t_0) & N_{2,p}(t_0) & \cdots & N_{n,p}(t_0) \\ N_{0,p}(t_1) & N_{1,p}(t_1) & N_{2,p}(t_1) & \cdots & N_{n,p}(t_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ N_{0,p}(t_n) & N_{1,p}(t_n) & N_{2,p}(t_n) & \cdots & N_{n,p}(t_n) \end{bmatrix}$$

$$\mathbf{D}=\begin{bmatrix} d_{01} & d_{02} & d_{03} & \cdots & d_{0s} \\ d_{11} & d_{12} & d_{13} & \cdots & d_{1s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & d_{n3} & \cdots & d_{ns} \end{bmatrix} \qquad \mathbf{P}=\begin{bmatrix} p_{01} & p_{02} & p_{03} & \cdots & p_{0s} \\ p_{11} & p_{12} & p_{13} & \cdots & p_{1s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & p_{n3} & \cdots & p_{ns} \end{bmatrix}$$

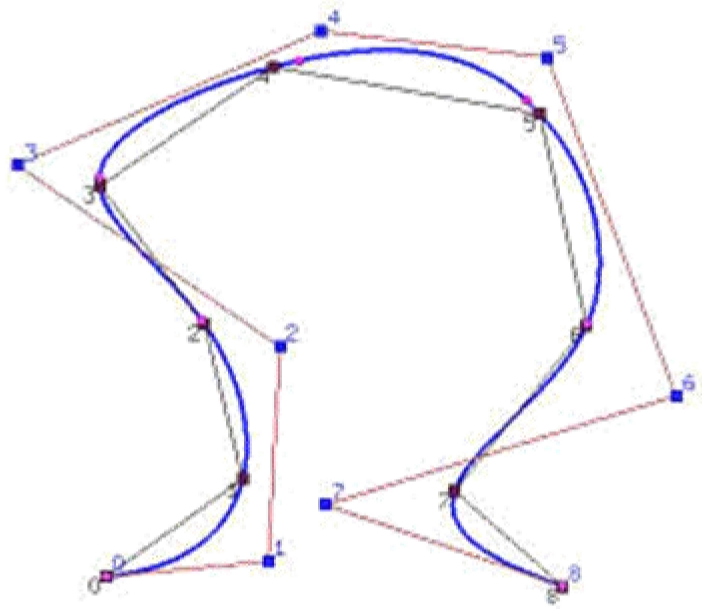
The $\mathbf{D}_k$ is a vector in s-dimensional space (*i.e.*, $\mathbf{D}_k = [\, d_{k1}, \ldots\ldots, d_{ks}]$)

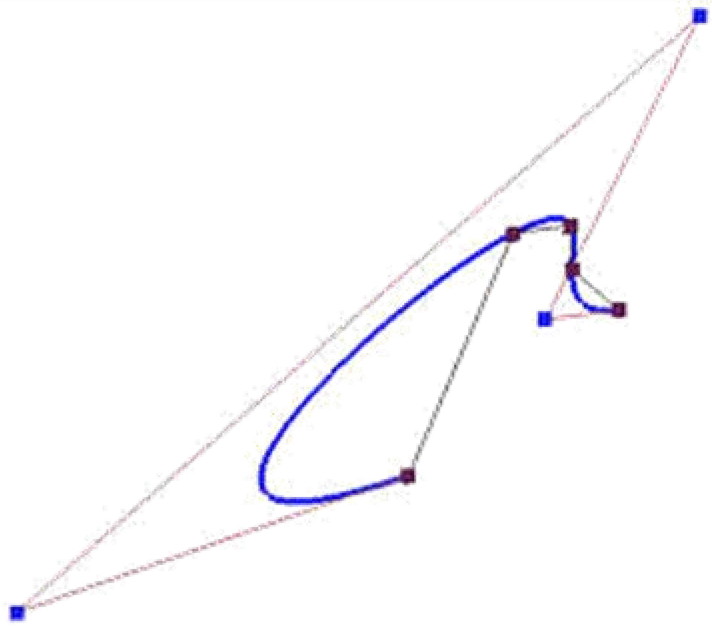
The $\mathbf{P}_i$ is also a vector in s-dimensional space (*i.e.*, $\mathbf{P}_i = [\, p_{i1}, \ldots\ldots, p_{is}]$)

Solution for Global Curve Interpolation

- It is important to point out that matrix $\mathbf{N}$ is totally positive and banded with a semi-bandwidth less than p (*i.e.*, $N_{i,p}(t_k) = 0$ if $|i - k| \geq p$) if the knots are computed by averaging consecutive p parameters.

Example: Global Curve Interpolation





The small red dots on the interpolating curve are points corresponding to the knots computed using the chord length method

The data polygon and the control polygon are very different

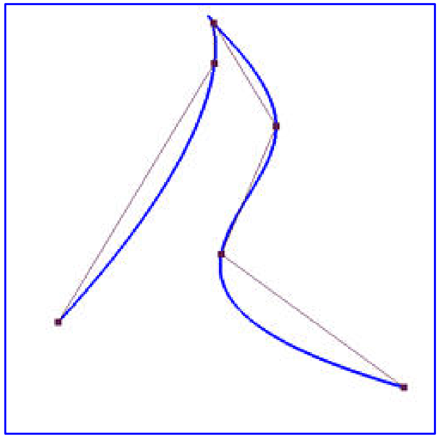
The Impact of Parameters and Knots

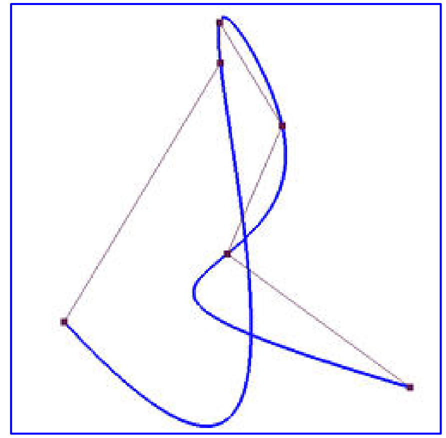
- In general: the impact of the selected parameters and knots cannot be predicted easily
- If the chord length distribution is about the same, four parameter selection methods should perform similarly
- The universal method should perform similar to the uniform method because the maximums of B-spline basis functions with uniform knots are distributed quite uniformly
- The centripetal method should work similar to the chord length method because the former is an extension to the latter for distribution of chord lengths change not wildly

The Impact of Parameters and Knots

generates a cusp

uniform



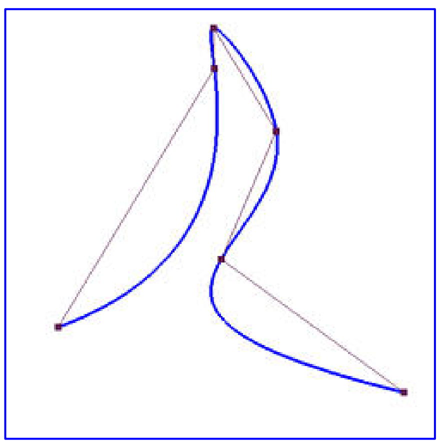


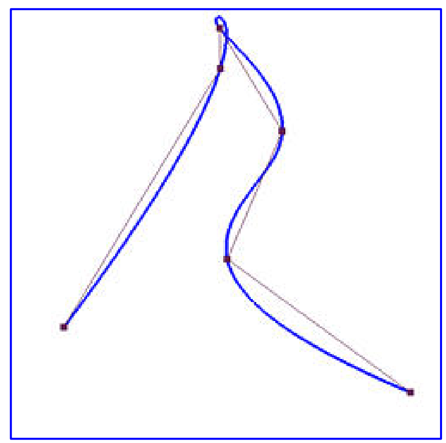
curve wiggle passes through data points wildly

chord length

performs better than chord length

centripetal





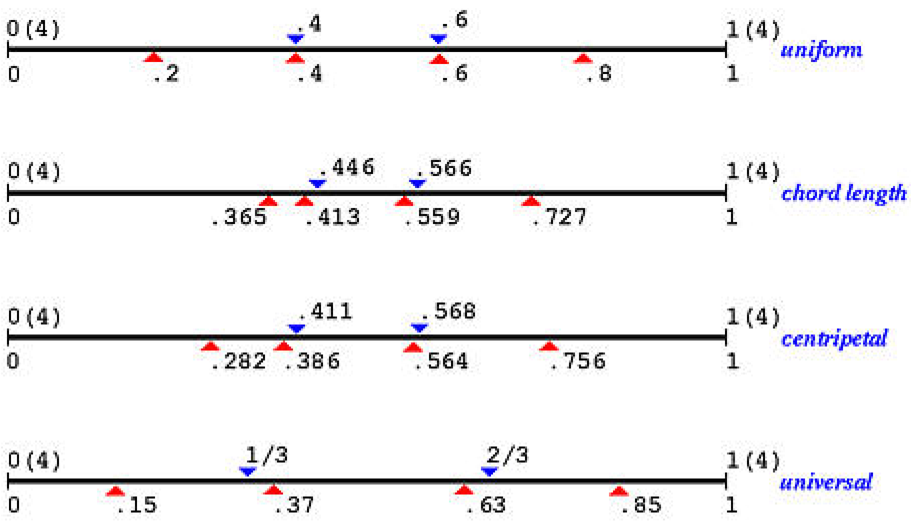
follows the data polygon closely but produces a small loop

universal

The Impact of Parameters and Knots

- The parameters and knots obtained from the universal method is more evenly distributed than those of the chord length method and the centripetal method
- The parameters and knots obtained from the centripetal method stretch the shorter (*resp*, longer) chords longer (*resp.*, shorter) and hence are more evenly distributed

  The longer curve segments obtained by the chord length method become shorter in the centripetal method and the curve does not wiggle wildly through data points.



The Impact of Degree

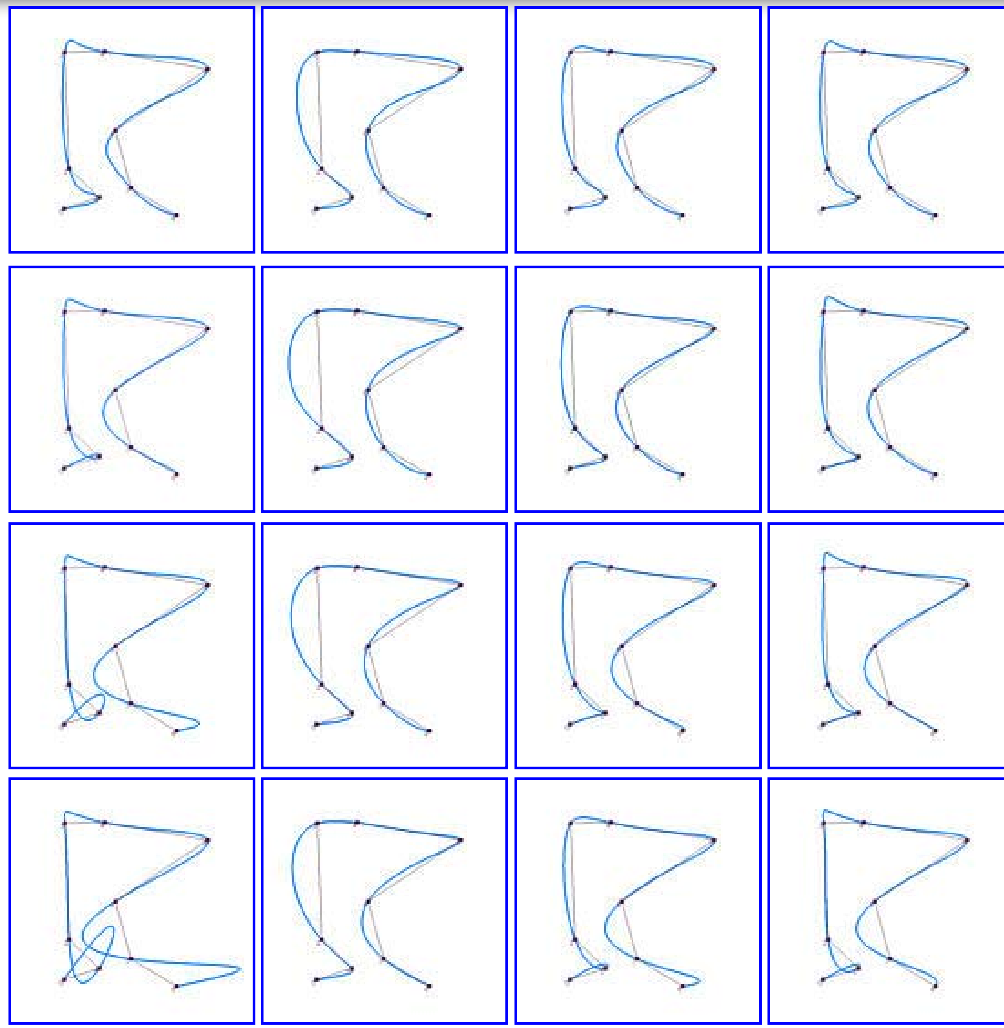
Uniform Chord Length Centripetal Universal

Degree=2

Degree=3

Degree=4

Degree=5



The Impact of Degree

- The impact of degree to the shape of the interpolating B-spline curve is also difficult to predict
- The uniformly spaced method and universal method usually follow long chords very well
  - On the other hand, these two methods have problems with short chords: peaks and loops.
  - This situation gets worse with higher degree curves because higher degree curves provide more freedom to wiggle.

The Impact of Degree

- The chord length method does not work very well for longer chords
  - Big bulges may occur
  - There is no significant impact of degree on curves
- The centripetal method and the universal method perform similarly
  - The generated interpolating curves follow longer chords closely
  - Loops may occur for shorter chords when degree increases

The Impact of Degree

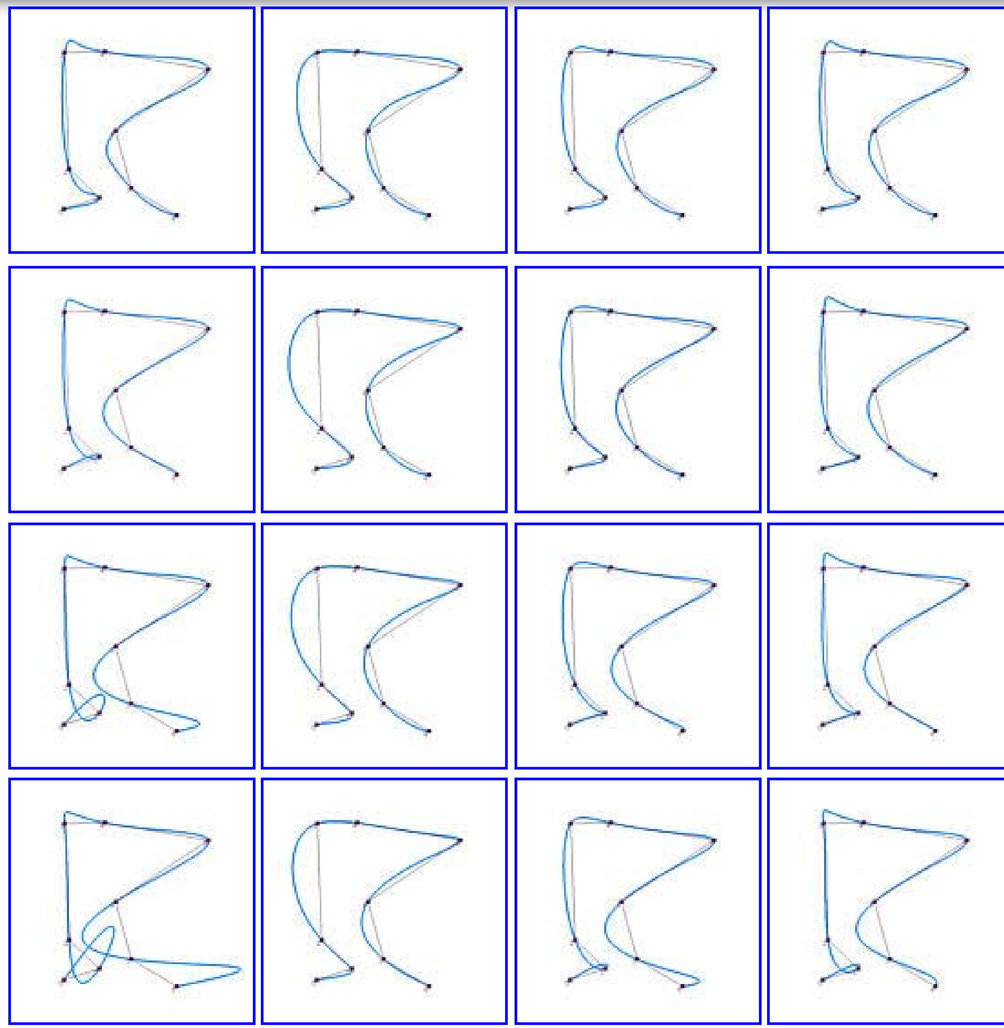
Uniform | Chord Length | Centripetal | Universal

Degree=2

Degree=3

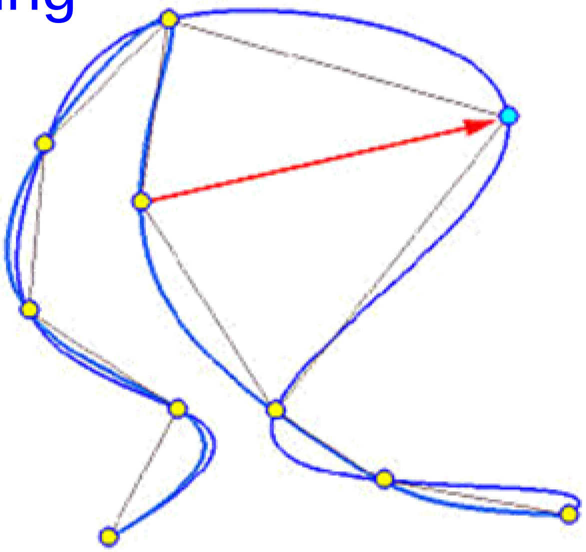
Degree=4

Degree=5



Why “Global”

- The B-spline curves satisfy the local modification property
- Changing the position of single data point changes the shape of the interpolating curve globally !



Global Curve Approximation

- Problem statement
- The Meaning of Least Square
- Solution for Global Curve Approximation
- The Impact of Degree and Number of Control Points
- Why “Global”

Problem statement

- The interpolating curve may wiggle through all data points
- The approximating curve overcomes this problem
  - Pass the first and last data points, does not have to contain any other point
- Approximation is measured by “*error distance* ”
  - The error distance is the distance between a data point and its "corresponding" point on the curve
  - For the interpolating curve: *error distance* = 0

Problem statement

Global Curve Approximation

Given a set of n+1 data points, D_0, D_1, ..., D_n, a degree p, and a number h, where $n > h \geq p \geq 1$, find a B-spline curve of degree p defined by h+1 control points that satisfies the following conditions:

1. **This curve contains the first and last data points (*i.e.*, D_0 and D_n)**
2. **This curve *approximates* the data polygon in the sense of *least square*.**

Problem statement

- The approximation B-spline of degree p is

$$\mathbf{C}(u) = \sum_{i=0}^{h} N_{i,p}(u)\mathbf{P}_i$$

  - where $\mathbf{P}_0$, $\mathbf{P}_1$, ..., $\mathbf{P}_h$ are the $h+1$ *unknown* control points
  - The curve passes the first and last data points

$$\mathbf{D}_0 = \mathbf{C}(0) = \mathbf{P}_0 \quad \mathbf{D}_n = \mathbf{C}(1) = \mathbf{P}_h$$

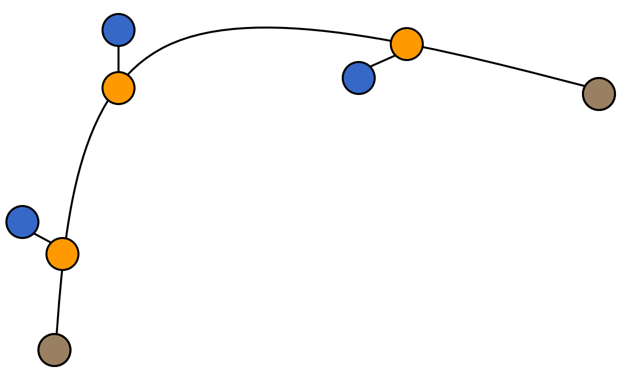
  - $h - 1$ unknown control points $\mathbf{P}_0$, $\mathbf{P}_1$, ..., $\mathbf{P}_h$

$$\mathbf{C}(u) = N_{0,p}(u)\mathbf{P}_0 + \sum_{i=1}^{h-1} N_{i,p}(u)\mathbf{P}_i + N_{h,p}(u)\mathbf{P}_h$$

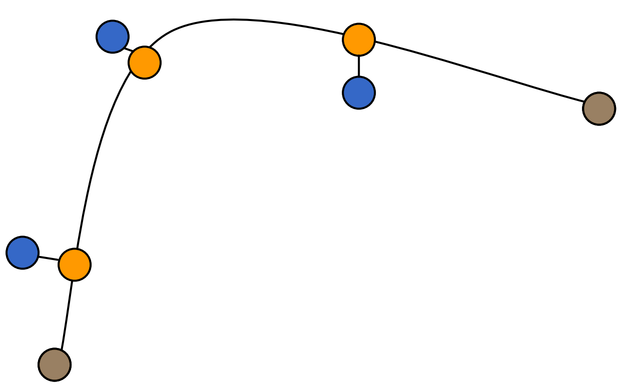
The Meaning of Least Square

- The sum of all *squared* error distances

$$f(\mathbf{P}_1,\ldots,\mathbf{P}_{h-1}) = \sum_{k=1}^{h-1} \left| D_k - C(t_k) \right|^2$$



Algebraic distance



Geometric distance

Solution for Global Curve Approximation

- Optimal problem: least square
- Rewrite $\mathbf{D}_k$ - $\mathbf{C}(t_k)$ into a different form

$$\mathbf{D}_k - C(t_k) = \mathbf{D}_k - \left[N_{0,p}(u)\mathbf{P}_0 + \sum_{i=1}^{h-1} N_{i,p}(u)\mathbf{P}_i + N_{h,p}(u)\mathbf{P}_h \right]$$

$$= \left(\mathbf{D}_k - N_{0,p}(u)\mathbf{P}_0 - N_{h,p}(u)\mathbf{P}_h \right) - \sum_{i=1}^{h-1} N_{i,p}(u)\mathbf{P}_i$$

Note: $\mathbf{Q}_k = \left(\mathbf{D}_k - N_{0,p}(u)\mathbf{P}_0 - N_{h,p}(u)\mathbf{P}_h \right)$

- The sum-of-square function $f()$ can be written as

$$f(\mathbf{P}_1, \ldots, \mathbf{P}_{h-1}) = \sum_{k=1}^{h-1} \left| \mathbf{Q}_k - \sum_{i=1}^{h-1} N_{i,p}(t_k)\mathbf{P}_i \right|^2$$

Solution for Global Curve Approximation

- Minimize $f()$
  - Function $f()$ is an elliptic paraboloid in variables $\mathbf{P}_1, \ldots, \mathbf{P}_{h\text{-}1}$
  - We can differentiate $f()$ with respect to each $\mathbf{p}_g$ and find the common zeros of these partial derivatives
  - These zeros are the values at which function $f()$ attends its minimum

Solution for Global Curve Approximation

- By computing partial differentiation

$$\frac{\partial f}{\partial \mathbf{P}_g} = -2N_{g,p}(t_k)\mathbf{Q}_k + 2N_{g,p}(t_k)\left(\sum_{i=1}^{h-1} N_{i,p}(t_k)\mathbf{P}_i\right) = 0$$

for $g=1,2,..,h\text{-}1$

$$\sum_{k=1}^{n-1} N_{g,p}(t_k)\sum_{i=1}^{h-1} N_{i,p}(t_k)\mathbf{P}_i = \sum_{k=1}^{n-1} N_{g,p}(t_k)\mathbf{Q}_k$$

Rewrite the equation as matrix form

Please refer to the course for detail

Solution for Global Curve Approximation

Finally we get $(\mathbf{N}^T\mathbf{N})\mathbf{P}=\mathbf{Q}$

where

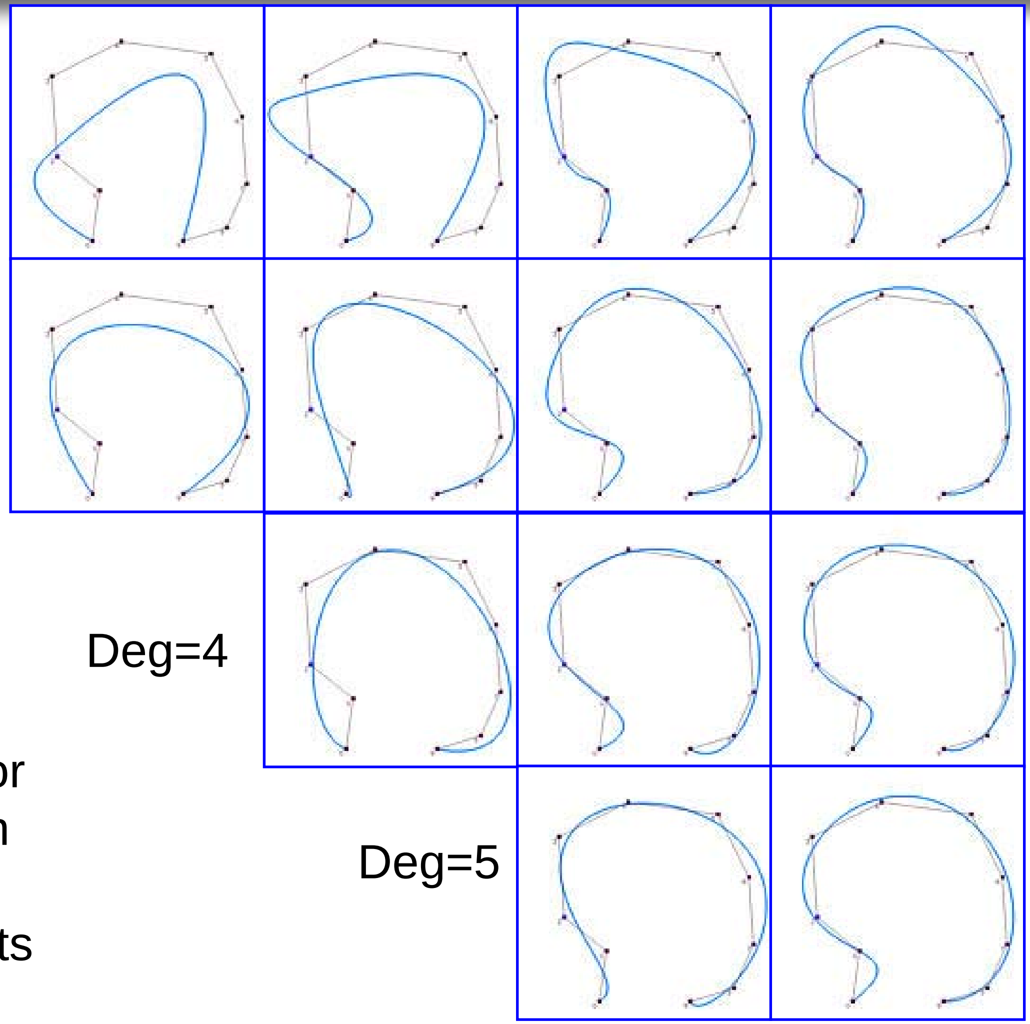
$$\mathbf{N} = \begin{bmatrix} N_{1,p}(t_1) & N_{2,p}(t_1) & \cdots & N_{h-1,p}(t_1) \\ N_{1,p}(t_2) & N_{2,p}(t_2) & \cdots & N_{h-1,p}(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ N_{1,p}(t_{n-1}) & N_{2,p}(t_{n-1}) & \cdots & N_{h-1,p}(t_{n-1}) \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \vdots \\ \mathbf{P}_{h-1} \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \sum_{k=1}^{n-1} N_{1,p}(t_k)\mathbf{Q}_k \\ \sum_{k=1}^{n-1} N_{2,p}(t_k)\mathbf{Q}_k \\ \vdots \\ \sum_{k=1}^{n-1} N_{h-1,p}(t_k)\mathbf{Q}_k \end{bmatrix}$$

Solving this system of linear equations for **P** gives us the desired control points!

The Impact of Degree and Number of Control Points



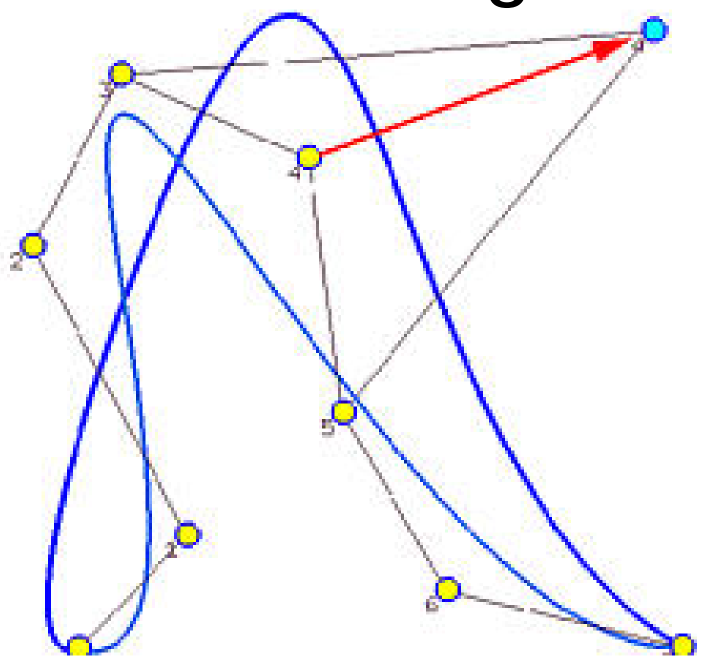
Approximation curves for 10 data points (n=9) with various degrees and numbers of control points

The Impact of Degree and Number of Control Points

- On each column, a higher the degree yields a better result (*i.e.*, closer to the data polygon) since high degree B-Spline is more flexible
- On each row, as the number of control points increases, the curve becomes closer to the data polygon since more control points offer higher flexibility of the approximation curve
- Higher degree and many control points? No!
  - The purpose of using approximation: fewer number of control points than global interpolation
  - As for degree: computational efficiency!

Why "Global"

- Changing the position of data point causes the entire curve to change !



Global Surface Interpolation

- Problem statement
- Solution for global surface interpolation
- Why “Global”

Problem statement

Global Surface Interpolation

Given a grid of $(m+1)\times(n+1)$ data points D_{ij} $(0 \le i \le m$ and $0 \le j \le n)$ and a degree (p, q), find a B-spline surface of degree (p,q) defined by $(m+1)\times(n+1)$ control points that passes all data points in the given order.

Solution for global surface interpolation

- The solution is two-step global curve interpolations
  - First for u-direction, then v-direction or
  - First for v-direction, then u-direction

- Suppose the B-Spline surface is given as

$$\mathbf{S}(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}\mathbf{P}_{ij}N_{i,p}(u)N_{j,q}(v)$$

Solution for global surface interpolation

Since the surface passes all data points

$$\mathbf{D}_{cd} = \mathbf{S}(s_c, t_d) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{P}_{ij} N_{i,p}(s_c) N_{j,q}(t_d)$$

Rewrite above formula

$$\mathbf{D}_{cd} = \mathbf{S}(s_c, t_d) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{P}_{ij} N_{i,p}(s_c) N_{j,q}(t_d)$$

$$= \sum_{i=0}^{m} N_{i,p}(s_c) \left(\sum_{j=0}^{n} \mathbf{P}_{ij} N_{j,q}(t_d) \right)$$

Solution for global surface interpolation

- Define the inner term as

$$\boxed{\mathbf{Q}_{id} = \sum_{j=0}^{n} \mathbf{P}_{ij} N_{j,q}(t_d)}$$

2nd

Global curve interpolation!

- Together with the equation

$$\mathbf{D}_{cd} = \sum_{i=0}^{m} N_{i,p}(s_c) \left(\sum_{j=0}^{n} \mathbf{P}_{ij} N_{j,q}(t_d) \right)$$

$$\boxed{\mathbf{D}_{cd} = \sum_{i=0}^{m} N_{i,p}(s_c) \mathbf{Q}_{id}}$$

1st

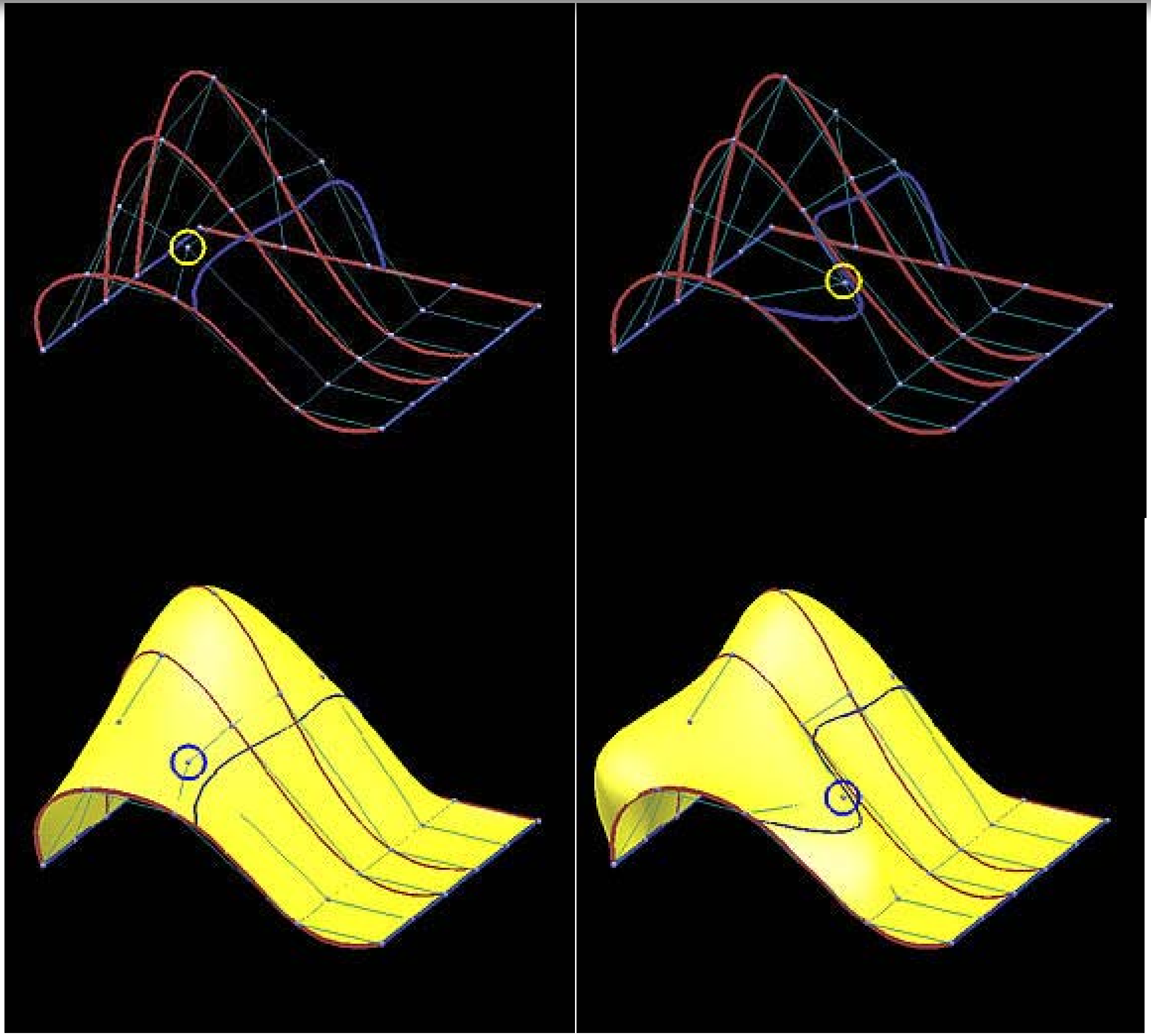
Global curve interpolation!

Why "Global"

Before Move | After Move

Knot curve

Surface



Global Surface Approximation

- Problem statement
- Solution for Global Surface Approximation
- A Simple Comparison

Problem statement

Global Surface Approximation

Given a grid of $(m+1)\times(n+1)$ data points D_{ij} $(0 \le i \le m$ and $0 \le j \le n)$ a degree (p, q), and e and f satisfying $m > e \ge p \ge 1$ and $n > f \ge q \ge 1$, find a B-spline surface of degree (p,q) defined by $(e+1)\times(f+1)$ control points P_{ij} $(0 \le i \le e$ and $0 \le j \le f)$ that approximates the data point grid in the given order.

Solution for Global Surface Approximation

- The desired B-spline surface is

$$\mathbf{S}(u,v) = \sum_{i=0}^{e}\sum_{j=0}^{f}\mathbf{P}_{ij}N_{i,p}(u)N_{j,q}(v)$$

- The point on surface that corresponds to data point $\mathbf{D}_{cd}$ is computed as

$$\mathbf{S}(s_c,t_d) = \sum_{i=0}^{e}\sum_{j=0}^{f}\mathbf{P}_{ij}N_{i,p}(s_c)N_{j,q}(t_d)$$

Solution for Global Surface Approximation

- The sum of all squared error distances is

$$f(\mathbf{P}_{00}, \mathbf{P}_{01}, \ldots, \mathbf{P}_{ef}) = \sum_{c=0}^{m} \sum_{d=0}^{n} \left| \mathbf{D}_{cd} - \mathbf{S}(s_c, t_d) \right|^2$$

- To minimize ***f***(), we can compute the partial derivatives and set them to zero

$$\frac{\partial f}{\partial \mathbf{P}_{ij}} = 0$$

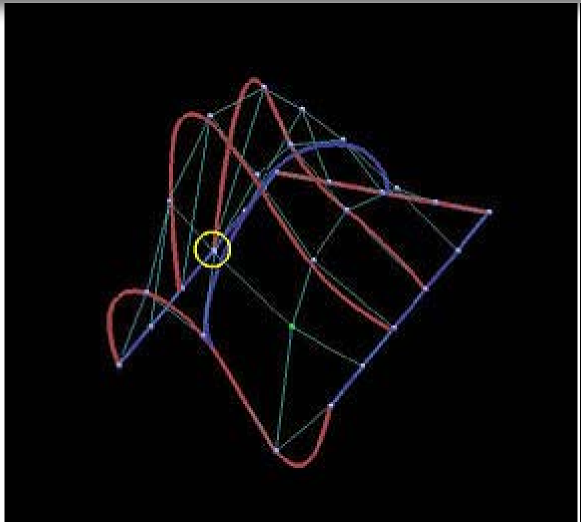
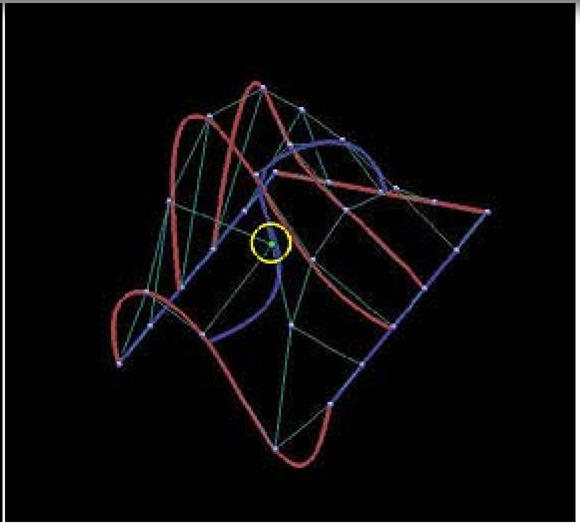
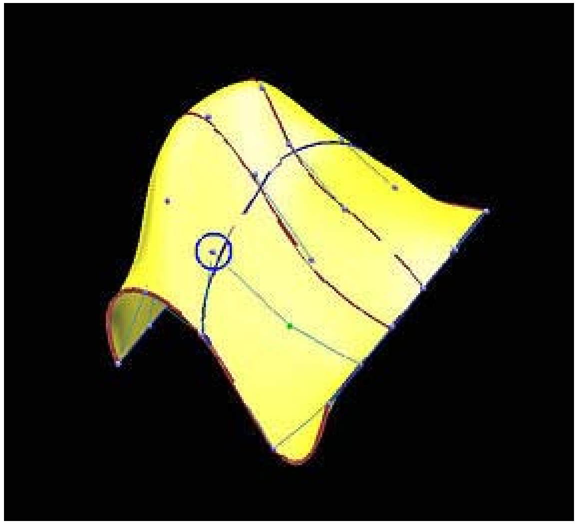
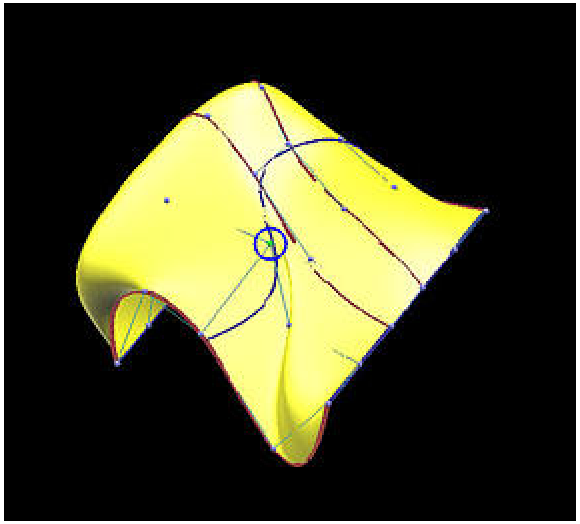
$$\boxed{\text{The result is non-linear equations!}}$$

Solution for Global Surface Approximation

- To find a non-optimal solution, we use two-step **Global Curve Interpolations**
  - First for u-direction, then v-direction or
  - First for v-direction, then u-direction

- This algorithm **does not** minimize function $f()$, it is **not** an optimal one,
  - it is adequate for many applications !

A Simple Comparison

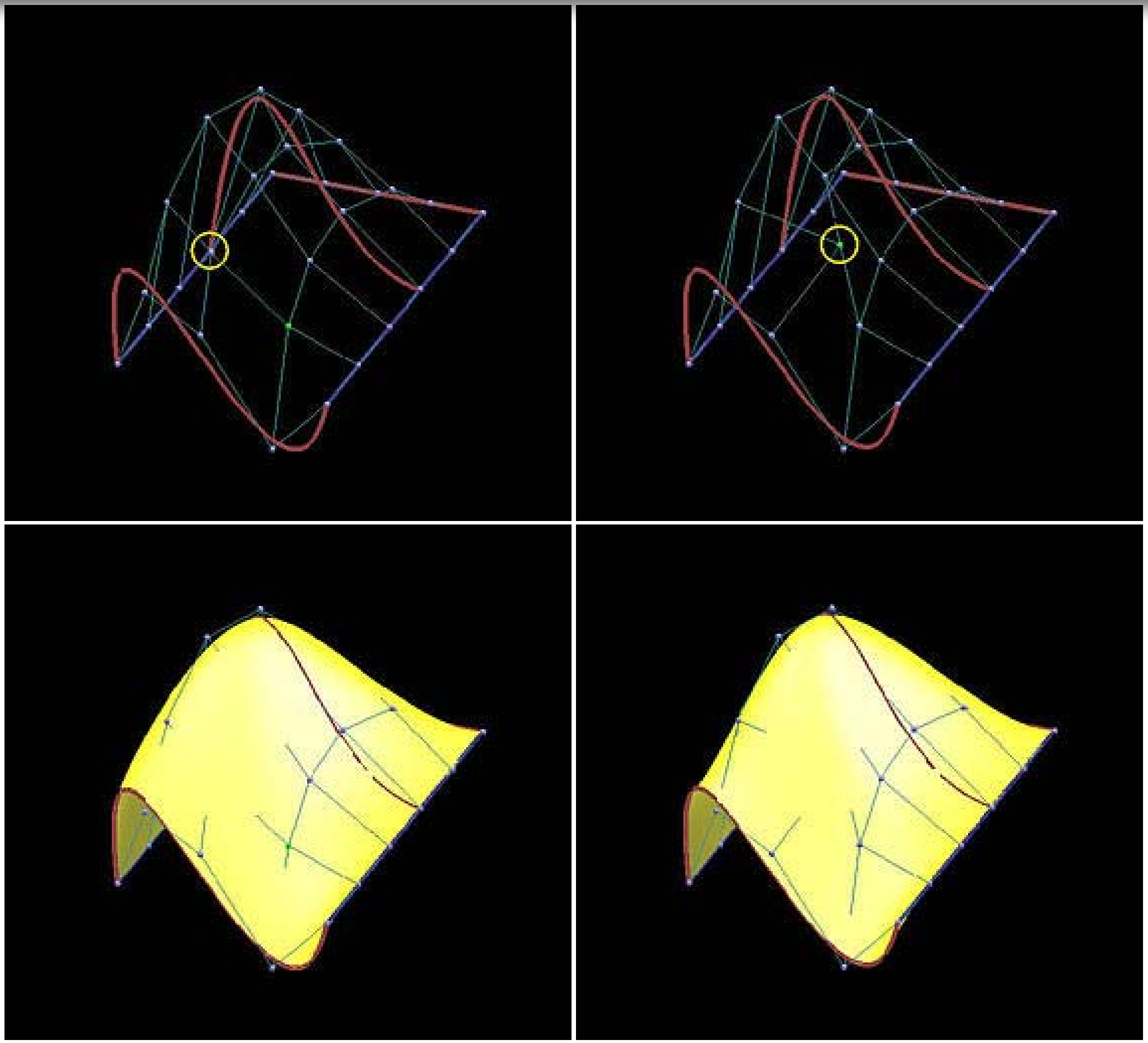
	Before Move	After Move
Interpolation Knot curve		
Interpolation Surface		

A Simple Comparison

Before Move | After Move

Approximation Knot curve

Approximation Surface



A Simple Comparison

- Global interpolation surface is sensitive for the movement of the data point
- Global Approximation surface is not sensitive for the movement of the data point

Assignments

- Project (choose one among four topics)
  - Implementing Doo-Sabin, Catmull-Clark and Loop subdivision surfaces with texture mapping
  - Implementing B-Spline surface interpolation and approximation with texture mapping
- Literature reading
  - Choosing one topic from the following Siggraph Course Notes
  - Writing a report about your reading topic
  - No restriction on the length of the report

Assignments

Siggraph2004 Course 15: Shape-Based Retrieval and Analysis of 3D Models

Overview : This course covers concepts, methods, and applications for retrieving and analyzing 3D models in large databases. Emphasis is placed on geometric representations and algorithms for indexing and matching 3D objects based on their shapes. A survey of current shape descriptors, query interfaces, and shape-based retrieval applications will be included.

Topics (choose one topic)

1. Shape Retrieval and Analysis Motivation
2. Shape Retrieval Challenges
3. Survey of Statistical Shape Descriptors
4. Survey of Structural Shape Descriptors
5. Case study: Search Engine for 3D Models

Assignments

Siggraph2004 Course 30: Visualizing Geospatial Data (www.siggraph.org/s2004/conference/courses/handson.php)

Overview : This course reviews concepts and highlights new directions in GeoVisualization. We review four levels of integrating geospatial data and geographic information systems (GIS) with scientific and information visualization (VIS) methods.

Topics (choose one topic)

1. Overview of integrating geospatial data with visualization methods
2. GeoVRML Applications for Landscape Planning & Visibility Studies
3. New Directions in Distributed GeoVisualization
4. The GeoVISTA Studio Project

Assignments

Siggraph2003 Course 4: L-Systems and Beyond

Overview : L-systems are a biologically motivated formalism for modeling and visualizing complex structures with a dynamically changing topology. Applications of L-systems and their extensions include modeling of plants and geometric modeling of curves and surfaces (for example, subdivision algorithms). This course presents recent theoretical results, implementations, applications, and research directions.

Topics (choose one topic)

1. Fundamentals of L-systems;
2. Solving systems of equations with L-systems;
3. Modeling of plant genetics, physiology, biomechanics, and ecology;
4. Geometric modeling of curves and surfaces;
5. Modeling programs and languages;
6. Implementation of L-systems on graphics hardware

Assignments

Siggraph2003 Course 6: Creating Advanced X3D Graphics

Overview: A review of open-standards X3D content creation tools with an emphasis on open-source code and how to use some of those to tools create advanced 3D graphics. Creation of multi-texturing, animation, interactivity, and scripting effects is demonstrated with an opportunity for participants to create their own effects.

Topics (choose one topic)

1. Advanced web-based 3D graphics with X3D,
2. open-source and proprietary tools for creation of open-standards X3D content
3. hands-on development and use of X3D content-creation tools to produce high-quality advanced 3D interactive graphics for multitexturing, animation, interactivity, and scripting (using Script Authoring Interface)

Assignments

Siggraph2003 Course 13: Beyond Blobs: Recent Advances in Implicit Surfaces

Overview

This course covers exciting advances in implicit surfaces that are useful but seldom covered by standard graphics courses. It reviews recently developed implicit modeling tools such as radial-basis functions, level sets, skeletal extraction, and topology, and demonstrates their utility for real-world applications from character animation to medical modeling.

Topics(choose one topic)

Implicit surfaces that interpolate point data, implicit surfaces for shape transformation, surface reconstruction from computer-vision data, medical applications, modern level sets, implicit methods to compute medial structures, digital Morse theory, and a library of software tools for interactive modeling with implicit surfaces.

Assignments

Siggraph2003 Course15: 3D Models From Photos and Videos

Overview: How 3D models can be obtained from photos or video acquired with a hand-held camera. The approach is based on advanced automatic techniques that avoid camera calibration and a priori scene knowledge, and that gradually retrieve more and more information about the images, the cameras, and the scene.

Topics (choose one topic)

Feature extraction, feature tracking, (wide-baseline) feature matching, multi-view relations, projective structure and motion recovery, self-calibration, bundle adjustment, image-pair rectification, dense stereo matching, multi-view matching, 3D surface modeling and texturing, image-based rendering, and applications.

Assignments

Siggraph2003 Course 16: Geometric Data Structures for Computer Graphics

Overview

This course provides working knowledge of essential geometric data structures and their elegant use in several representative and current areas of research in computer graphics: terrain visualization, texture synthesis, modeling, and others. Attendees learn to recognize geometric problems and acquire thorough understanding of suitable algorithms.

Topics (choose one topic)

Geometric data structures (quadtrees, Voronoi/Delaunay diagrams, distance fields, and bounding volume hierarchies). Algorithms and applications (terrain visualization, iso-surfaces, point location, texture synthesis, NURBS tesselation, motion planning, modeling, collision detection, occlusion culling, and a generic dynamization technique).

Assignments

Siggraph2002 Course 41: Non-Traditional Modeling

Overview

The state of the art in three areas of modeling that might be considered non-traditional; procedural-volume modeling, implicit surface modeling, and point-based modeling. The course presents methods for designing, storing, manipulating, and rendering these models, and summarizes their advantages, their practical applications, and future directions for research.

Topics (choose one topic)

General procedural modeling techniques; algorithmic representations of geometry; L-systems, fractals, and procedural cloud modeling; data structures and algorithms for implicit modeling, including controlled blending techniques, precise contact modeling, constructive solid geometry, space warping; point-based rendering methods; spectral processing of point sampled geometry.

Assignments

Siggraph2001 Course 17: Geometric Signal Processing on Large Polygonal Meshes

ABSTRACT (choose one topic)

Very large polyhedral models, which are used in more and more graphics applications today, are routinely generated by a variety of methods such as surface reconstruction algorithms from 3D scanned data, isosurface construction algorithms from volumetric data, and photogrametric methods from aerial photography.

The course will provide an overview of several closely related methods designed to smooth, denoise, edit, compress, transmit, and animate very large polygonal models, based on signal processing techniques, constrained energy minimization, and the solution of diffusion differential equations.

Assignments

Siggraph2001 Course 33: New Directions in Shape Representations

Course Summary

Several recently developed shape representations go beyond conventional surface and volume techniques and offer advantages for compression, transmission, high resolution, editing, and rendering of complex shapes.

In this course, some of the world's leading computer graphics researchers and practitioners summarize the state of the art in shape representations and provide detailed information on how to implement the various methods. The course includes a discussion of various applications, including sculpting and 3D scanning of real-life objects.

Topics(choose one topic)

Introduction and overview; Displaced subdivision surfaces; Normal meshes; Point-based graphics and visualization; Surface representations and signal processing; Adaptively sampled distance fields; Image-based representations.

Assignments

Siggraph2001 Course 50: Digital Geometry Processing

Course Summary Topics(choose one topic)

The arrival of 3D scanning has created a new wave of digital media after sound, images, and video, raising the need for digital processing algorithms. Traditionally fine detail geometry is represented through unstructured polygonal meshes. Such meshes are awkward for editing, filtering, and compression applications. In this course we propose a new paradigm based on semi-regular meshes, constructed through a process of recursive quadrisection. Several research results have shown their many advantages. We will show how to build semi-regular meshes from unstructured polygonal meshes and raw range data, and how to build applications such as filtering, editing, simulation, and compression using semi-regular meshes.

Assignments

Siggraph2000 Course 23: Subdivision for Modeling and Animation

Course Summary Topics (choose one topic)

An introduction to subdivision, a technique for generating smooth curves and surfaces that extends classical spline modeling approaches. In presentations that are accessible to a wide audience, the course covers the basic ideas of subdivision, the particulars of a number of different subdivision algorithms, and the most recent contributions in the area.

Course download

http://www.cad.zju.edu.cn/home/jqfeng/GM/GM05.zip

Geometric Modeling in Siggraph courses

ftp://210.32.131.214/GeoMod (pending)