B-Spline Curves and Surfaces (2)

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Contents

Uniform B-Spline Definition: Convolution Form

- Overview
- Definition of the Blending Functions Utilizing Convolution
- The First Order Blending Function
- The Second Order Blending Function
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Overview

- The uniform B-splines are based upon a knot sequence that has uniform spacing
  - Translation property

$$N_{i,k}(t) = N_{0,k}(t-i)$$

- The single blending function can be defined by convolution of blending functions of lower degree

Definition of the Blending Functions Utilizing Convolution

The uniform kth order B-spline blending function N_k is defined as:

$$N_1(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$N_k(t)=(N_{k\text{-}1} * N_1)(t)$$

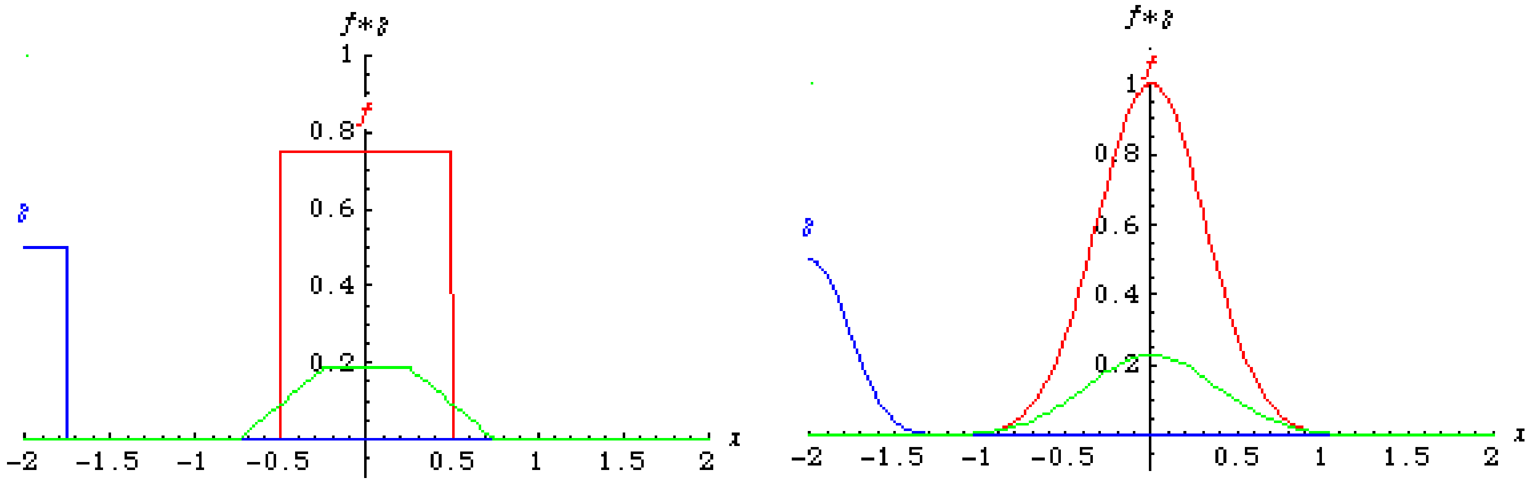
The kth order blending function is defined by convolving the k-1st order blending function with the *first* order blending function.

Definition of the Blending Functions Utilizing Convolution

By expanding above equation, the convolution can be formulated as:

$$\begin{aligned} N_k(t) &= (N_{k-1} * N_1)(t) \\ &= \int_{-\infty}^{\infty} N_{k-1}(x) N_1(t-x)\,dx \\ &= \int_{t-1}^{t} N_{k-1}(x)\,dx \end{aligned}$$

Definition of the Blending Functions Utilizing Convolution



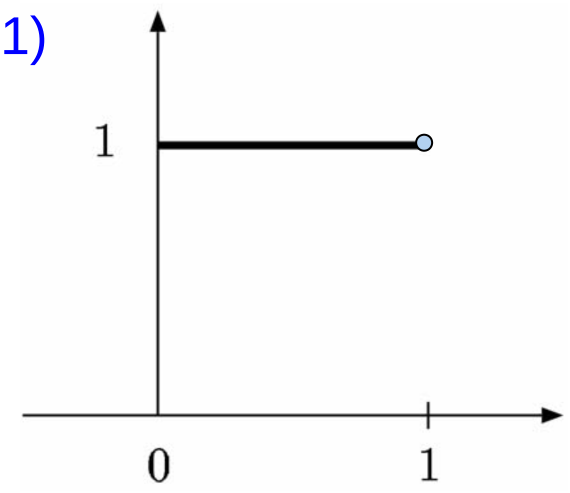
The animations above graphically illustrate the convolution of two rectangle functions (left) and two Gaussians (right). In the plots, the green curve shows the convolution of the blue and red curves as a function of t, the position indicated by the vertical green line. The gray region indicates the product $g(s) * f(t-s)$ as a function of t, so its area as a function of t is precisely the convolution.

The First Order Blending Function

The first order blending function is the Haar scaling function

$$N_1(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{else} \end{cases}$$

The support of this function is the interval [0,1)



The Second Order Blending Function

The second order blending function

$$N_2(t) = \int_{t-1}^{t} N_1(x)dx$$

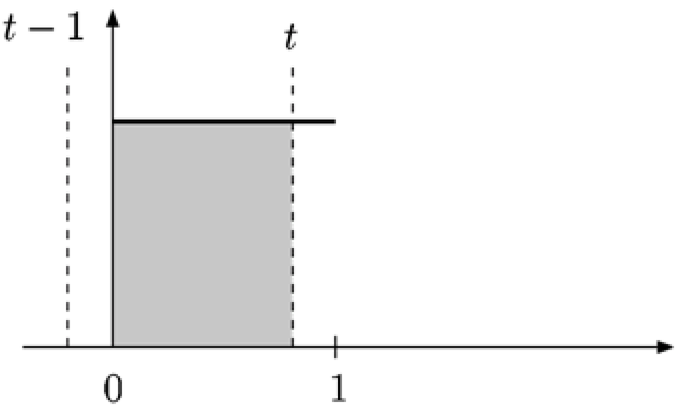
- The function $N_1(x)$ is nonzero when $0 \leqslant x < 1$
- The integral interval is $[t-1, t]$, where $0 \leqslant t < 2$
- The integral splits naturally into the two cases
  - $0 \leqslant t < 1$
  - $1 \leqslant t < 2$

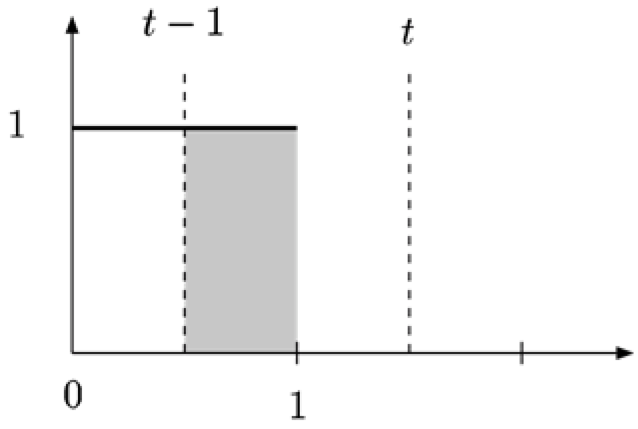
The Second Order Blending Function

$$N_2(t) = \int_{t-1}^{t} N_1(x)dx$$

$$= \begin{cases} \int_0^t dx & \text{if } 0 \le t < 1 \\ \int_{t-1}^{1} dx & \text{if } 1 \le t < 2 \end{cases}$$

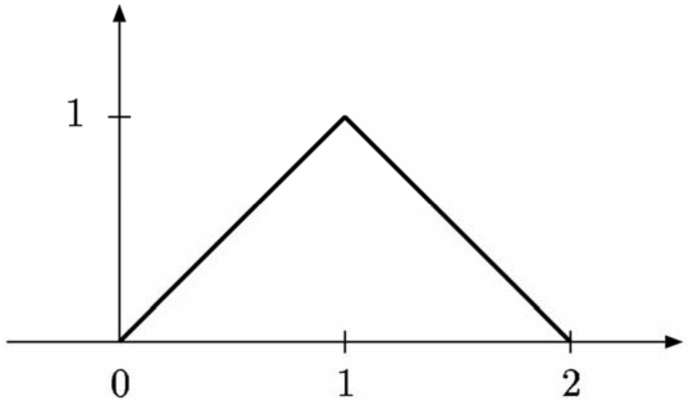
$$= \begin{cases} t & \text{if } 0 \le t < 1 \\ 2-t & \text{if } 1 \le t < 2 \end{cases}$$





The Second Order Blending Function

The second order B-spline blending function can be illustrated as



The support of $N_2(t)$ is [0,2)

The Third Order Blending Function

The third order blending function is

$$N_3(t) = \int_{t-1}^{t} N_2(x)dx$$

- The function $N_2(x)$ is nonzero only when $0 \leqslant x < 2$
- The nonzero values in the integrand for any t are in $0 \leqslant t < 3$.
- The integral can be decomposed as three parts illustrated below.

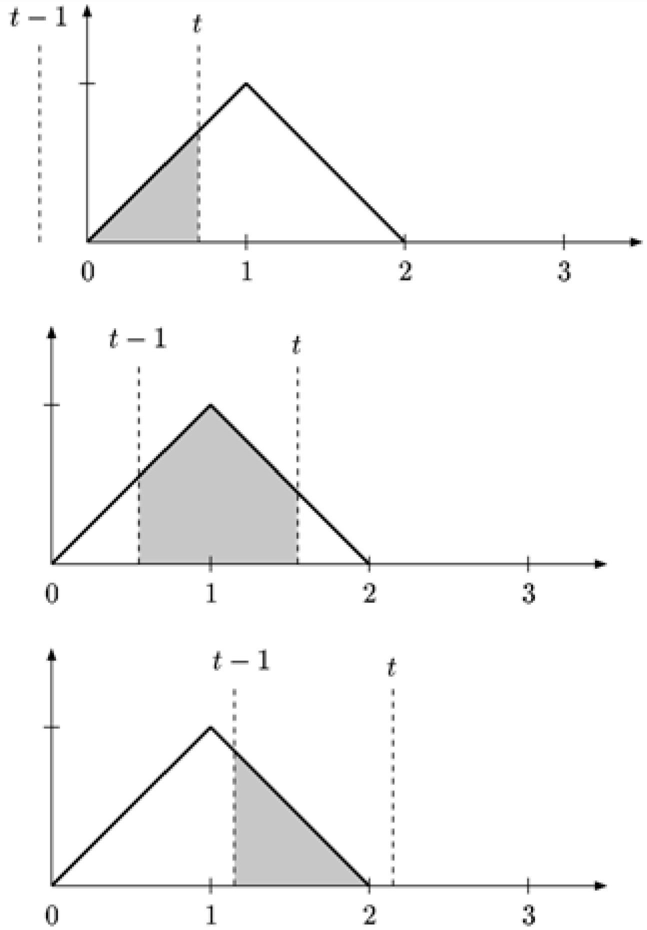
The Third Order Blending Function

$$N_3(t) = \int_{t-1}^{t} N_2(x)dx$$

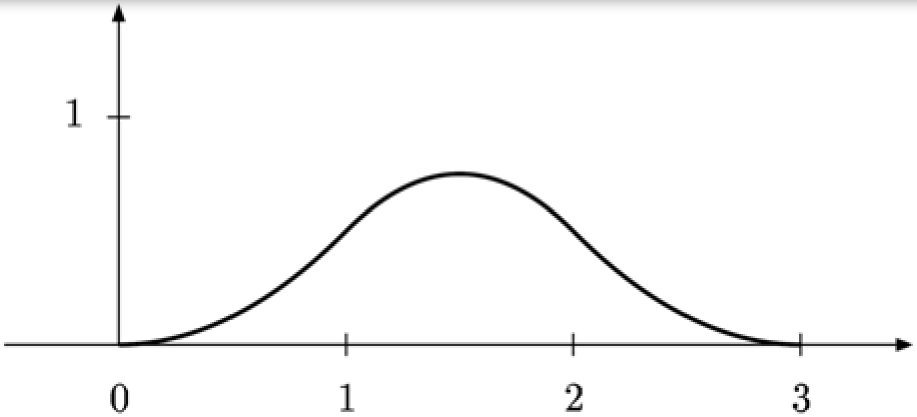
$$= \begin{cases} \int_0^t N_2(x)dx & if\ \ 0 \le t < 1 \\ \int_{t-1}^1 N_2(x)dx + \int_1^t N_2(x)dx & if\ \ 1 \le t < 2 \\ \int_{t-2}^1 N_2(x)dx & if\ \ 2 \le t < 3 \end{cases}$$

$$= \begin{cases} \int_0^t xdx & if\ \ 0 \le t < 1 \\ \int_{t-1}^1 (2-x)dx + \int_1^t xdx & if\ \ 1 \le t < 2 \\ \int_{t-2}^1 (2-x)dx & if\ \ 2 \le t < 3 \end{cases}$$

$$= \begin{cases} \frac{1}{2}t^2 & if\ \ 0 \le t < 1 \\ \frac{1}{2}\left(-2t^2 + 6t - 3\right) & if\ \ 1 \le t < 2 \\ \frac{1}{2}\left(t^2 - 6t + 9\right) & if\ \ 2 \le t < 3 \end{cases}$$



The Third Order Blending Function



- The curve (of the third order B-spline blending function) is a *piecewise quadratic* - i.e. it has quadratic pieces that are smoothly joined together
- The function $N_3(x)$ is nonzero only when $0 \leqslant x < 3$

The Two-Scale Relation for Uniform B-Splines

- Overview
- Translating and Scaling the Blending Function
- The Two-Scale Relation for Uniform B-Splines
- The Two-Scale Relation for Uniform Linear B-Splines
- The Two-Scale Relation for Uniform Quadratic B-Splines

Overview

- Translation property

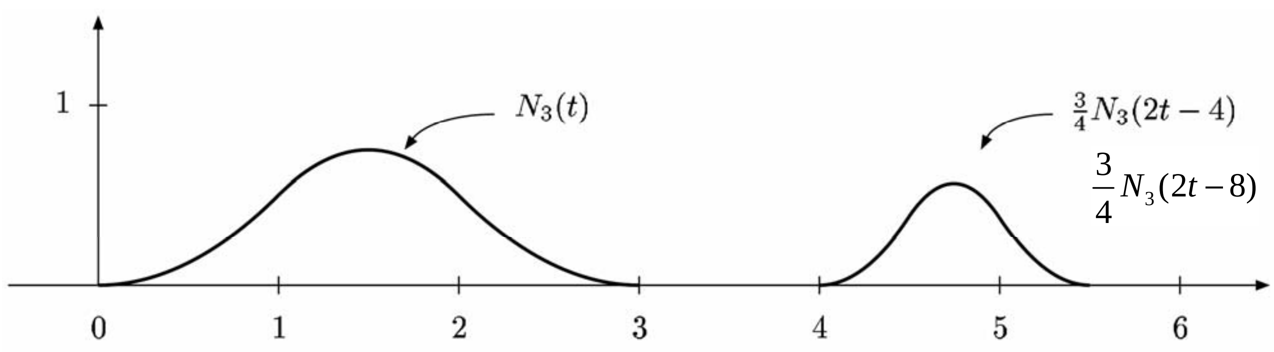
$$N_{i,k}(t) = N_{0,k}(t-i)$$

- Two-scale relation
  - The single blending function can be written as a sum of scaled and translated copies of itself.
  - It is essential to defining wavelets on spaces of functions and subdivision surfaces

Translating and Scaling the Blending Function

- The uniform B-spline blending function $N_k(t)$ can be scaled and translated simply by redefining the parameterization of the function

- Example $\frac{3}{4}N_3(2t-8)$
  - The support of above function is $[4,5.5]$
  - The height of function is 0.75



The Two-Scale Relation for Uniform B-Splines

For the given order k, the two-scale relation is written as

$$N_k(t) = \sum_{i=0}^{k} p_i N_k(2t - i)$$

where

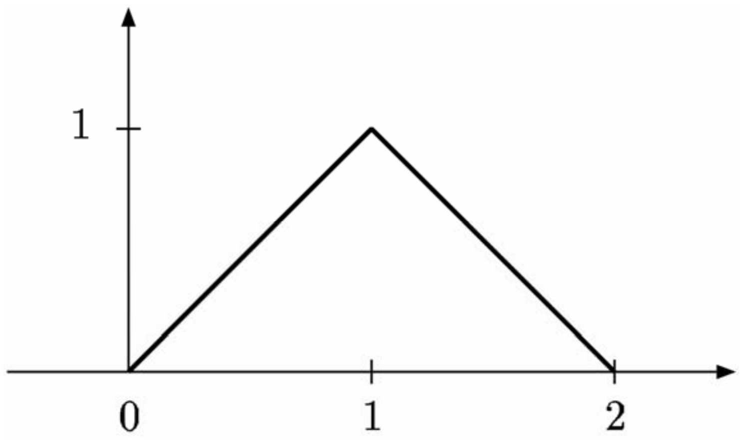
$$p_i = \frac{1}{2^{k-1}} \binom{k}{i}$$

- The basic function can be written as linear combination of the translated and scaled copies of the basic function
- The coefficients can be developed by the fact that the uniform B-spline blending function can be defined by convolution

The Two-Scale Relation for Uniform Linear B-Splines

- The uniform 2nd order B-spline blending function $N_2(t)$ is defined by

$$N_2(t) = \begin{cases} t & \text{if } 0 \leq t < 1 \\ 2 - t & \text{if } 1 \leq t < 2 \end{cases}$$

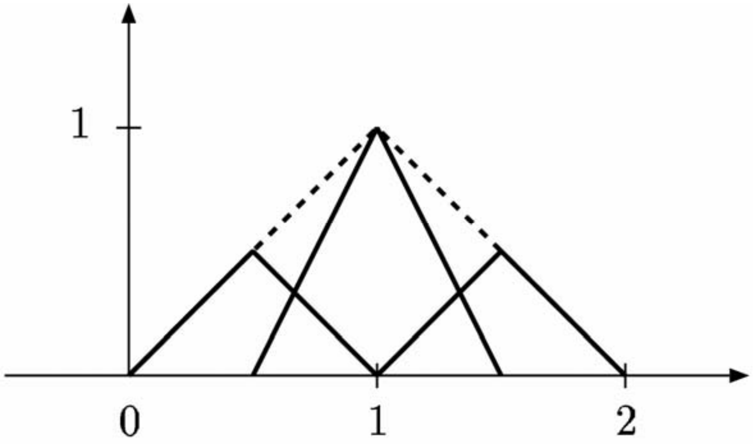


The Two-Scale Relation for Uniform Linear B-Splines

- The two-scale relation for $N_2(t)$ is given by

$$N_2(t) = \frac{1}{2}N_2(2t) + N_2(2t-1) + \frac{1}{2}N_2(2t-2)$$

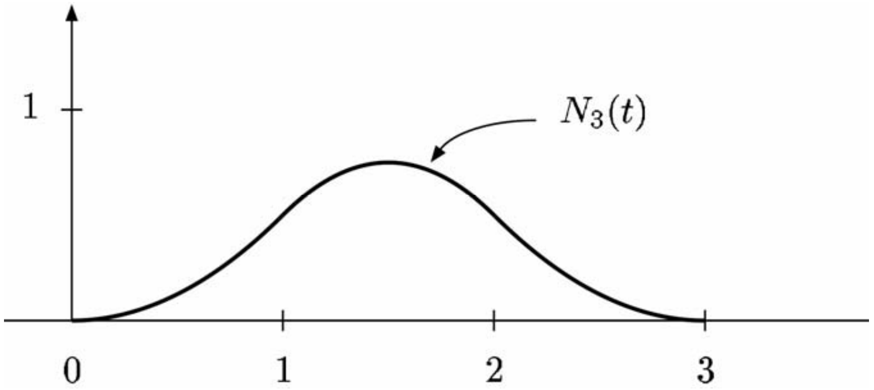
- The original blending function is shown with dashed lines
- The three scaled and translated functions are shown using solid lines.



The Two-Scale Relation for Uniform Quadratic B-Splines

- The uniform 3rd order B-spline blending function $N_3(t)$ is defined by

$$N_3(t) = \begin{cases} \frac{1}{2}t^2 & \text{if } 0 \le t < 1 \\ \frac{1}{2}\left(-2t^2 + 6t - 3\right) & \text{if } 1 \le t < 2 \\ \frac{1}{2}\left(t^2 - 6t + 9\right) & \text{if } 2 \le t < 3 \end{cases}$$

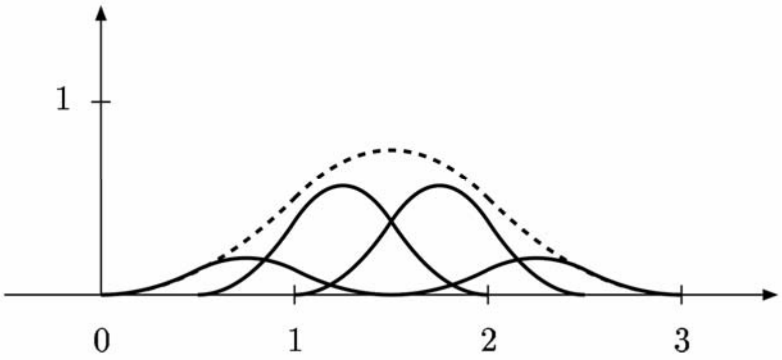


The Two-Scale Relation for Uniform Quadratic B-Splines

- The two-scale relation for $N_3(t)$ is given by

$$N_3(t) = \frac{1}{4}N_3(2t) + \frac{3}{4}N_3(2t-1) + \frac{3}{4}N_3(2t-2) + \frac{1}{4}N_3(2t-3)$$

- The original blending function is shown with dashed lines
- The three scaled and translated functions are shown using solid lines.



A Proof of the Two-Scale Relation for Uniform Splines

- The Two-Scale Relation for Uniform B-Splines
- The Fourier Transform
- Proof of the Two-Scale Relation

The Two-Scale Relation for Uniform B-Splines

For the given order k, the two-scale relation is written as

$$N_k(t) = \sum_{i=0}^{k} p_i N_k(2t - i)$$

where

$$p_i = \frac{1}{2^{k-1}} \binom{k}{i}$$

- The basic function can be written as linear combination of the translated and scaled copies of the basic function
- The coefficients can be developed by the fact that the uniform B-spline blending function can be defined by convolution

The Fourier Transform

- Let $\hat{N}_k(\omega)$ be the Fourier Transformation of $N_k(t)$

$$\hat{N}_k(\omega) = \int_{-\infty}^{\infty} e^{-\imath\omega x} N_k(x)dx$$

- Properties
  - For any k, $N_k(t)=(N_{k-1}*N_1)(t)$, then

$$\hat{N}_k(\omega) = (\widehat{N_{k-1} * N_1})(\omega) = \hat{N}_{k-1}(\omega)\hat{N}_1(\omega)$$

$$\hat{N}_1(\omega) = \frac{1 - e^{-\imath\omega}}{\imath\omega}$$

$$\hat{N}_k(\omega) = \left(\frac{1 - e^{-\imath\omega}}{\imath\omega}\right)^k$$

Proof of the Two-Scale Relation

The two-scale relation is $N_k(t) = \sum_{j=-\infty}^{\infty} p_j N_k(2t - j)$

Taking the Fourier Transform of both sides of the equation

$$\hat{N}_k(\omega) = \frac{1}{2}\left(\sum_{j=-\infty}^{\infty} p_j e^{-\frac{\imath j\omega}{2}}\right)\hat{N}_k\left(\frac{\omega}{2}\right)$$

Thus

$$\left(\frac{1-e^{-\imath\omega}}{\imath\omega}\right)^k = \frac{1}{2}\left(\sum_{j=-\infty}^{\infty} p_j e^{-\frac{\imath j\omega}{2}}\right)\left(\frac{1-e^{-\imath\frac{\omega}{2}}}{\imath\frac{\omega}{2}}\right)^k$$

Proof of the Two-Scale Relation

By using binomial theorem

$$\frac{1}{2}\left(\sum_{j=-\infty}^{\infty} p_j e^{-\frac{\imath j\omega}{2}}\right) = \left(\frac{1-e^{-\imath\omega}}{\imath\omega}\right)^k \left(\frac{\imath\frac{\omega}{2}}{1-e^{-\imath\frac{\omega}{2}}}\right)^k$$

$$= \left(\frac{1-e^{-\imath\frac{\omega}{2}}}{2}\right)^k$$

$$= 2^{-k}\sum_{j=0}^{k}\binom{k}{j} e^{-\frac{\imath j\omega}{2}}$$

So

$$p_j = \begin{cases} \frac{1}{2^{k-1}}\binom{k}{j} & \text{for } 0 \le j \le k \\ 0 & \text{otherwise} \end{cases}$$

B-Spline Curves

- Definition
- Properties
- Knot vectors
- Controls
- Other Topics

B-Spline Curves – Definition

$$P(t) = \sum_{i=1}^{n+1} B_i N_{i,k}(t) \quad t_{\mathbf{min}} \le t < t_{\mathbf{max}}, \quad 2 \le k \le n+1$$

B_i **s** — **are the polygon control vertices**

$N_{i,k}(t)$ — **are the normalized B-spline basis functions of order** k

$n+1$ — **is the number of control vertices**

B-Spline Curves – Basis functions

$$N_{i,1}(t) = \begin{cases} 1 & \textbf{if } x_i \le t < x_{i+1} \\ 0 & \textbf{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{(t - x_i)N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t)N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}$$

x_i **s** **are the elements of a knot vector**

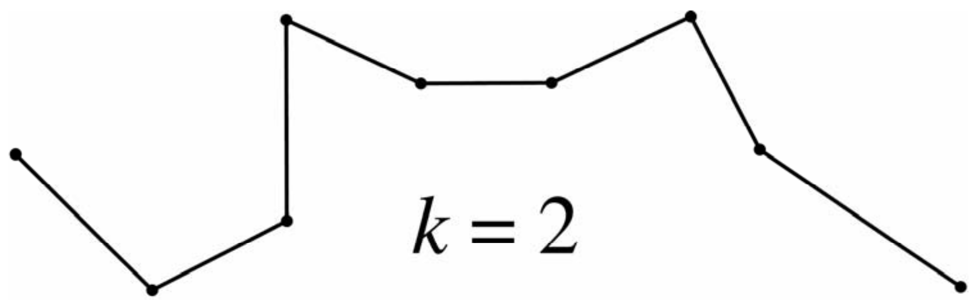
Note $0/0 \equiv 0$

B-Spline Curves – Properties

- $\sum_i N_{i,k}(t) \equiv 1$ for all t
- $N_{i,k}(t) \geqslant 0$ for all t
- Maximum order $k_{max}=n+1$
- Maximum degree, n, is one less than the order
- Exhibits the variation diminishing property
- Follows shape of the control polygon
- Transform curve ⟺ transform control polygon
- Everywhere C^{k-2} continuous
- Convex hull

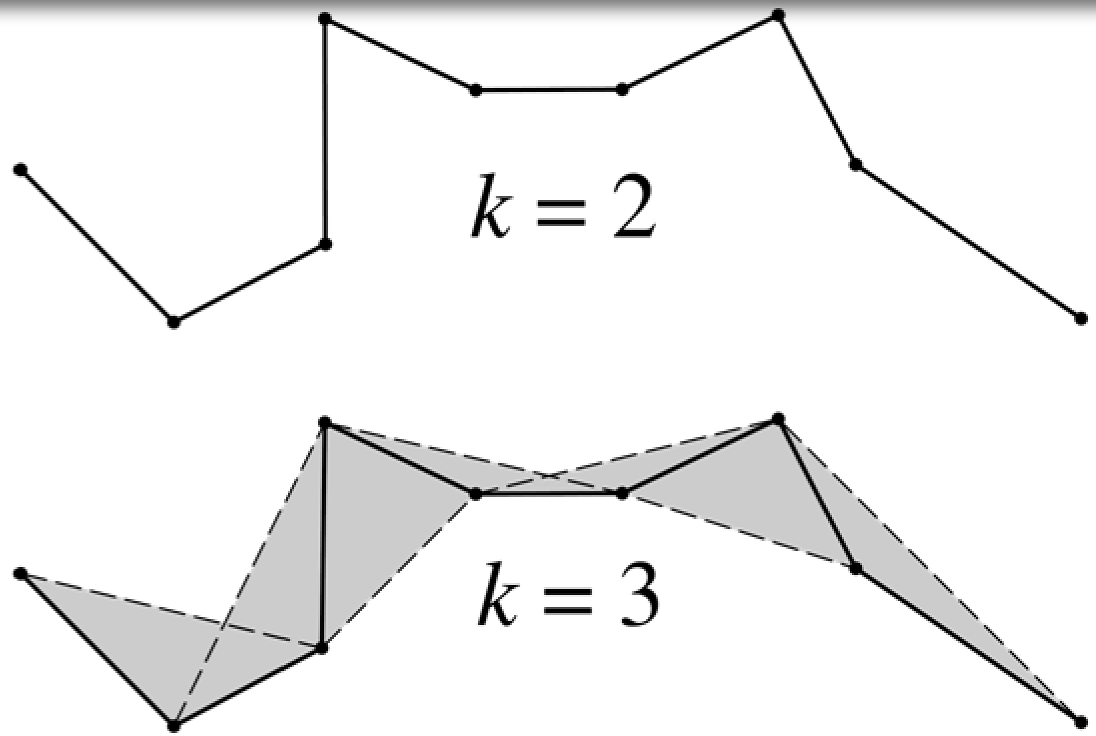
B-Spline Curves – Convex Hulls

- Stronger than for Bézier curves
- A point on the curve $\mathbf{P}(t)$ lies within the convex hull of k neighboring control vertices



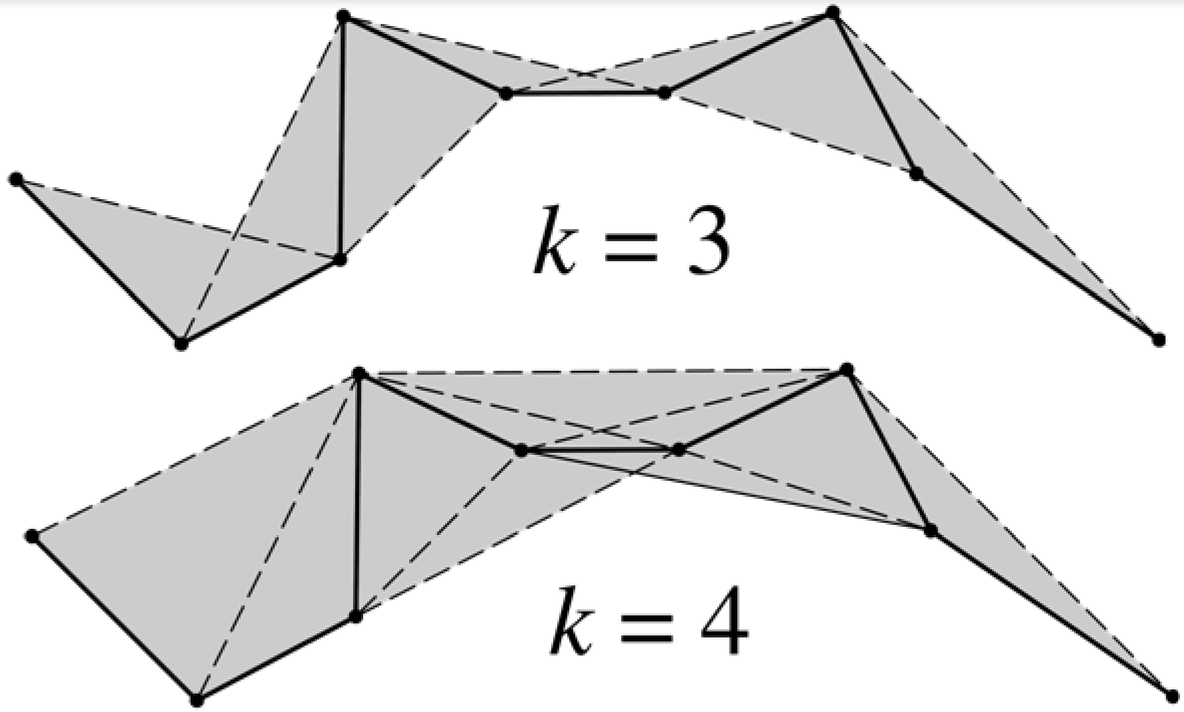
- Notice for order, $k=2$ the degree is one – a straight line
  - The B-Spline curve is the control polygon

B-Spline Curves – Convex Hulls



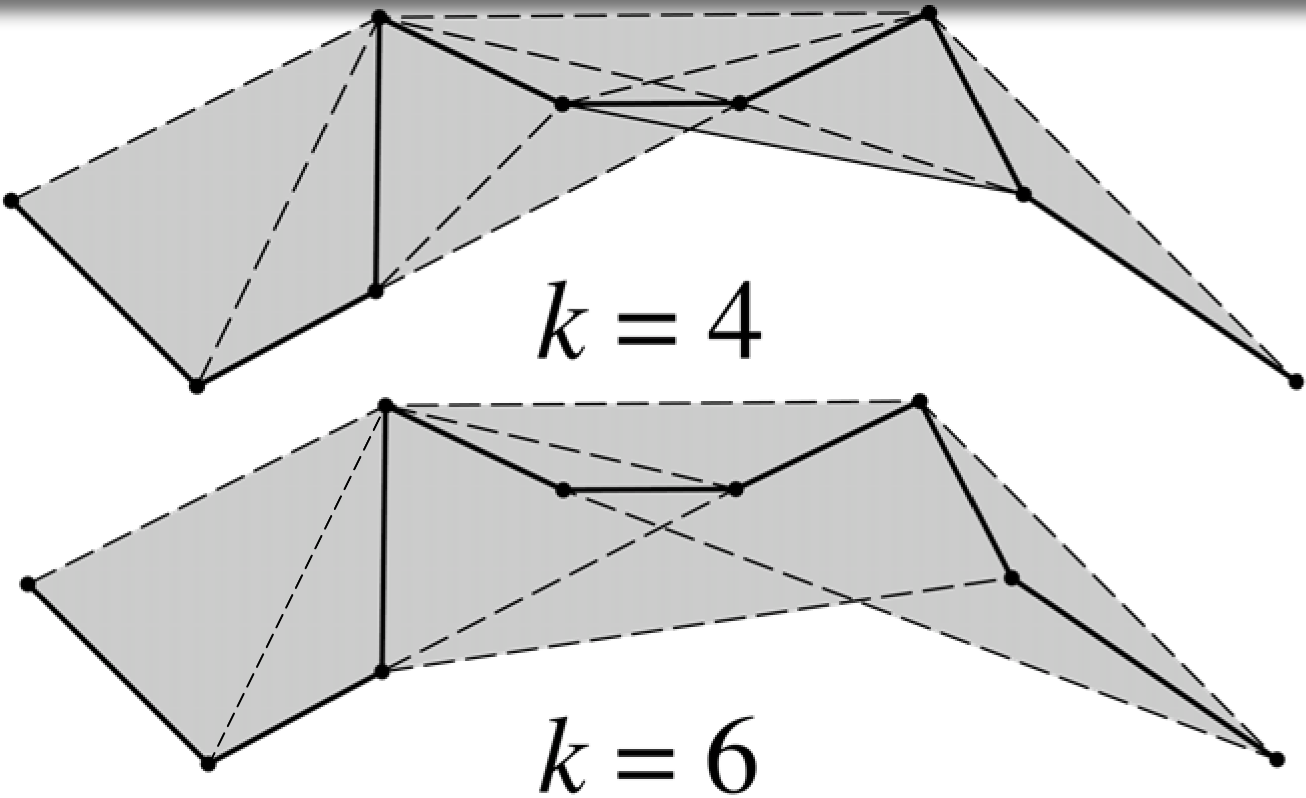
- For $k = 3$ a larger region may contain the curve
- The B-spline curve will not exactly follow polygon

B-Spline Curves – Convex Hulls



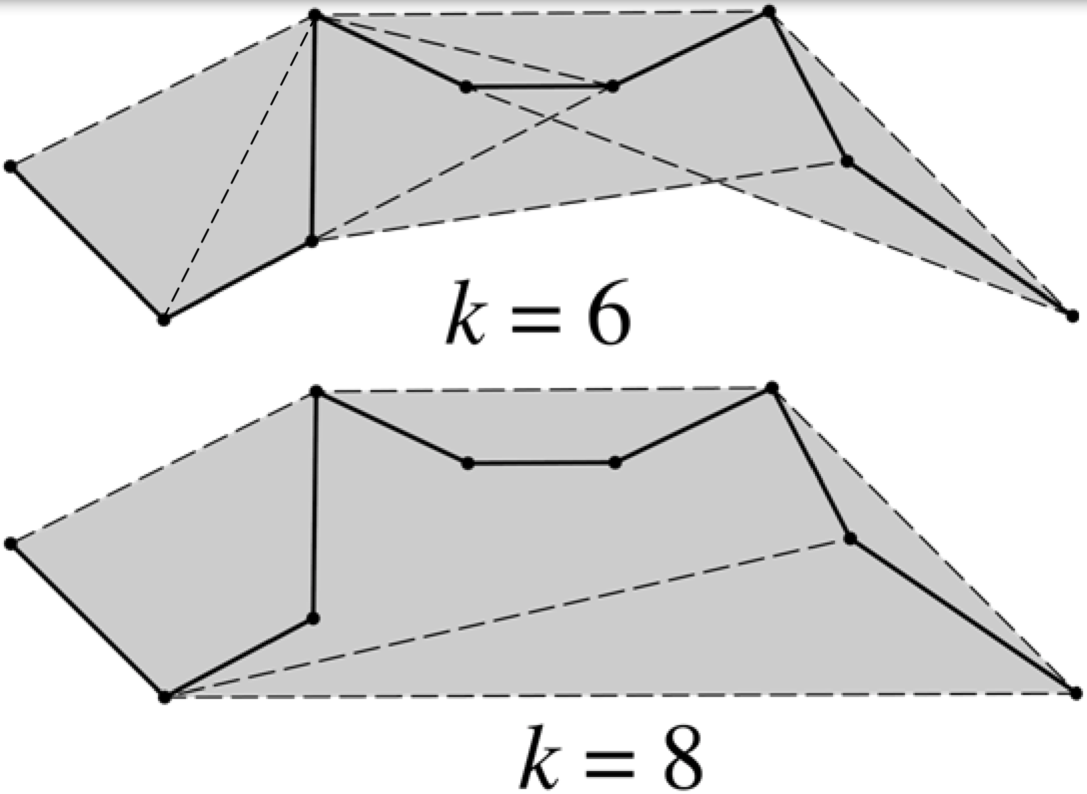
The higher the order the less closely the B-spline curve follows the control polygon

B-Spline Curves – Convex Hulls



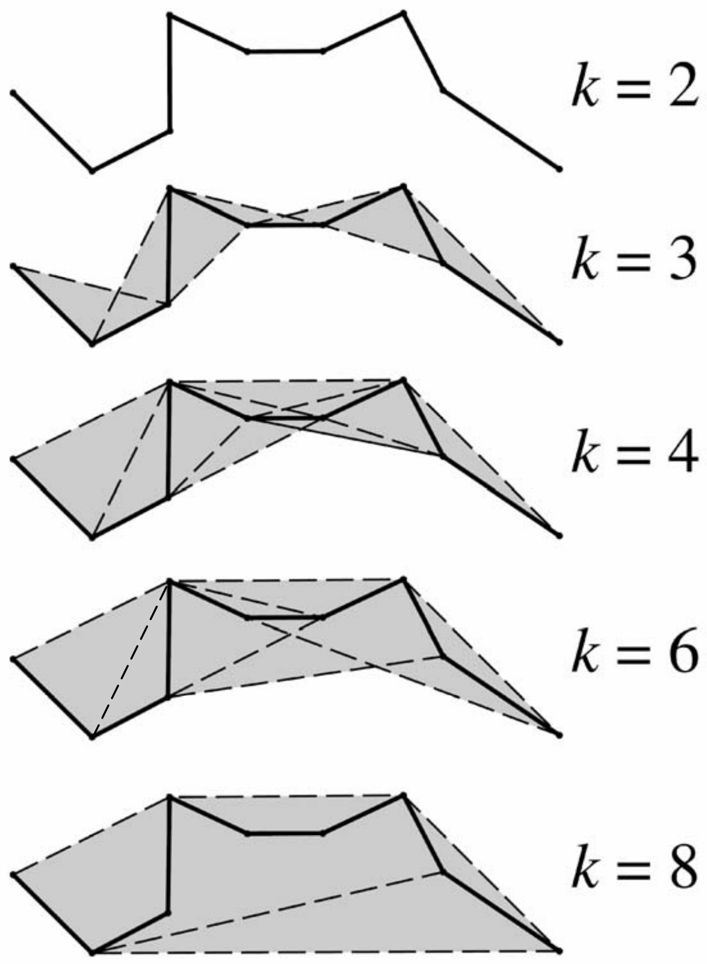
The higher the order the less closely the B-spline curve follows the control polygon

B-Spline Curves – Convex Hulls



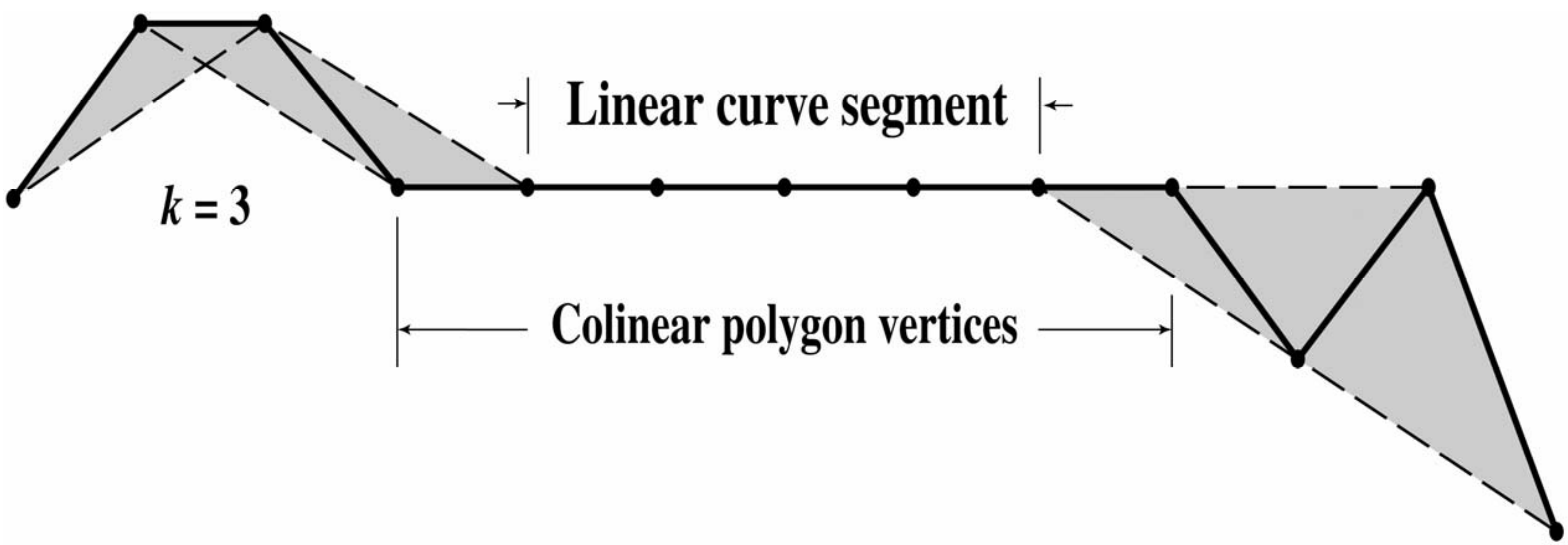
The higher the order the less closely the B-spline curve follows the control polygon

B-Spline Curves – Convex Hulls



B-Spline Curves – Convex Hulls

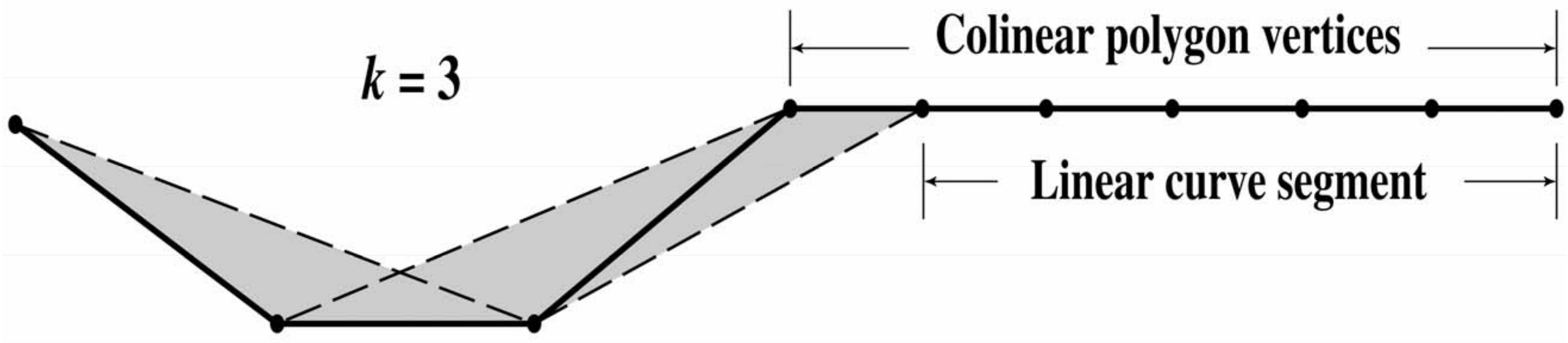
Straight segments



Straight line results start and stop k-2 spans from the ends of the co-linear segments

B-Spline Curves – Convex Hulls

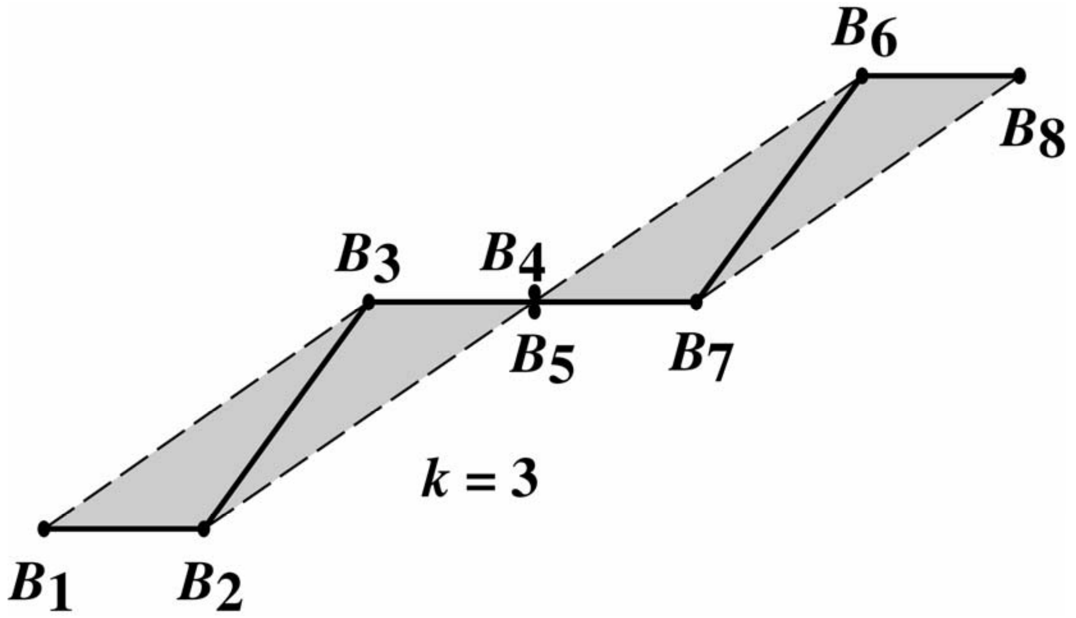
Straight segments at ends



For l colinear vertices then the number of linear segments at the end is at least $l-k+1$

B-Spline Curves – Convex Hulls

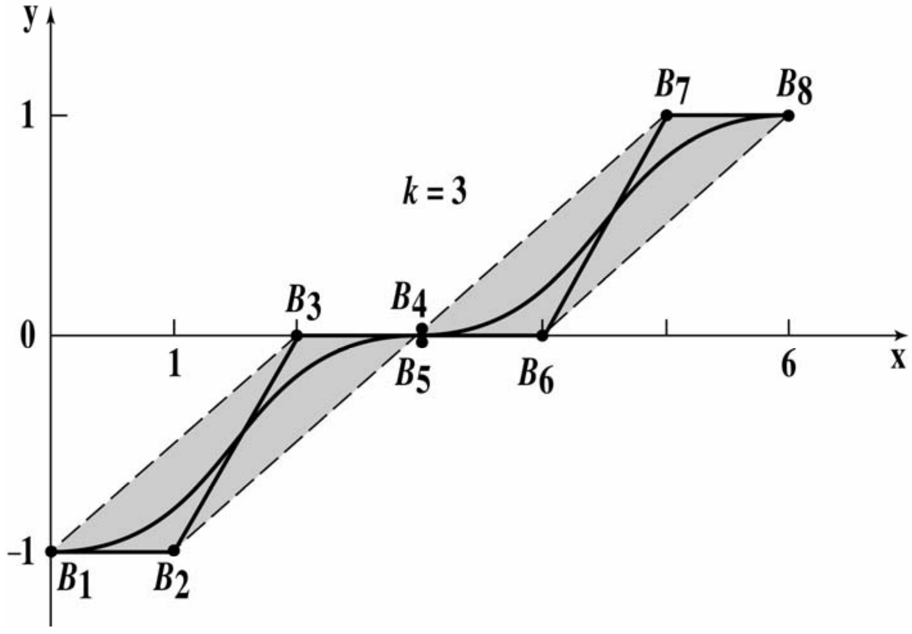
Coincident vertices



k-1 coincident vertices are required for the curve to pass through the vertices

B-Spline Curves – Convex Hulls

Coincident vertices



The curve smoothly transitions through the coincident vertices with C^{k-2} continuity

B-Spline Curves – Knot Vectors

- Only requirement

$$x_i \leqslant x_{i+1}$$

- Uniform – evenly spaced

$$[0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5]$$

$$[-0.2 \quad -0.1 \quad 0 \quad 0.1 \quad 0.2 \quad 0.3]$$

- Typically begin at zero, may normalize to $0 \leqslant x_i \leqslant 1.0$

$$[0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0]$$

Knot Vectors – Open Uniform

- Multiplicity equal to k at the ends

$$k=2 \quad [0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 3]$$

$$k=3 \quad [0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 3 \quad 3]$$

$$k=4 \quad [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 3 \quad 3 \quad 3 \quad 3]$$

- Normalized

$$k=4 \quad [0 \quad 0 \quad 0 \quad 0 \quad 1/3 \quad 2/3 \quad 1 \quad 1 \quad 1 \quad 1]$$

Knot Vectors – Open Uniform

Formal definition

$$x_i = 0 \qquad 1 \le i \le k$$

$$x_i = i - k \qquad k + 1 \le i \le n + 1$$

$$x_i = \underbrace{n - k + 2}_{\textbf{max knot value}} \qquad n + 2 \le i \le \underbrace{n + k + 1}_{\textbf{max no. of knots}}$$

Curves behave most nearly like Bézier curves

Knot vectors – Open nonuniform

[0 0 0 1 3/2 2 2 2]

[0 0 0 1 1 2 2 2]

Repeating knot value

Basis functions

$$N_{i,1}(t) = \begin{cases} 1 & \textbf{if } x_i \le t < x_{i+1} \\ 0 & \textbf{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{(t - x_i)N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t)N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}$$

- x_is are the elements of a knot vector
- Note: $0/0 \equiv 0$
- Recursion relation: dependent on knot vector

Basis functions – Dependencies

Form triangular pattern

$$
\begin{array}{lllll}
N_{i,k} & & & & \\
N_{i,k-1} & N_{i+1,k-1} & & & \\
N_{i,k-2} & N_{i+1,k-2} & N_{i+2,k-2} & & \\
\vdots & & & \vdots & \ddots \\
N_{i,1} & N_{i+1,1} & N_{i+2,1} & N_{i+3,1} \cdot & N_{i+k-1,1}
\end{array}
$$

The single basis function in the first row depends on all those in the last row

Basis Functions – Inverse Dependencies

Form triangular pattern

$$\begin{array}{ccccccc}
N_{i-k+1,k} & \cdot & N_{i+k-1,k} & N_{i,k} & N_{i+1,k} & \cdot & N_{i+k-1,k} \\
\cdot & \cdot & & & & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
 & & N_{i-1,2} & N_{i,2} & N_{i+1,2} & & \\
 & & & N_{i,1} & & &
\end{array}$$

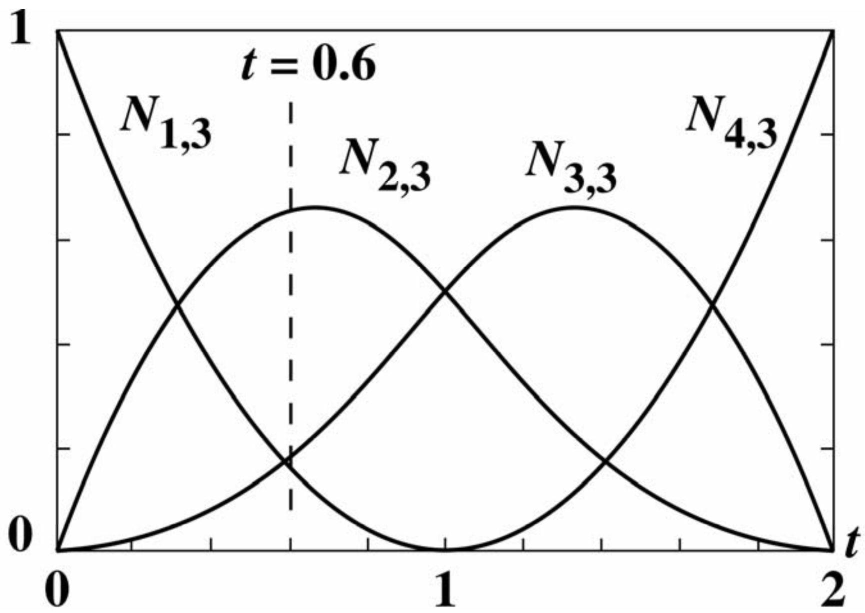
Influence of a single first-order basis function $N_{i,1}$ on higher-order basis functions

Basis Functions – Sum Equals One at Any t

Example: $n+1=4$, $k=3$, $t=0.6$

$[X]=[0 \quad 0 \quad 0 \quad 1 \quad 2 \quad 2 \quad 2]$

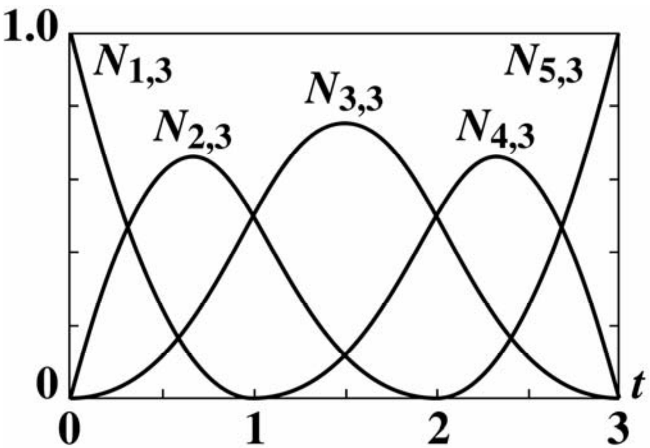
$N_{1,3}+N_{2,3}+N_{3,3}+N_{4,3}=0.16+0.66+0.18+0.0=1.0$

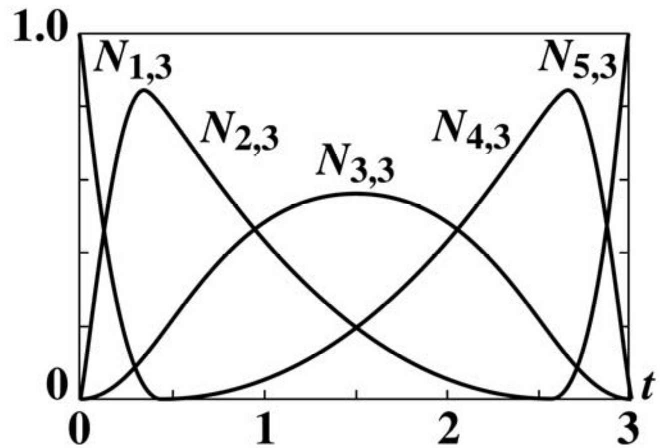


Basis Functions – Comparisons

Uniform and nonuniform knot vectors: $k=3$, $n+1=5$

$[X] = [0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 3\ \ 3\ \ 3]$ $[X] = [0\ \ 0\ \ 0\ \ 0.4\ \ 2.6\ \ 3\ \ 3\ \ 3]$





Notice: $N_{2,3}$ and $N_{4,3}$ pulled left and right.

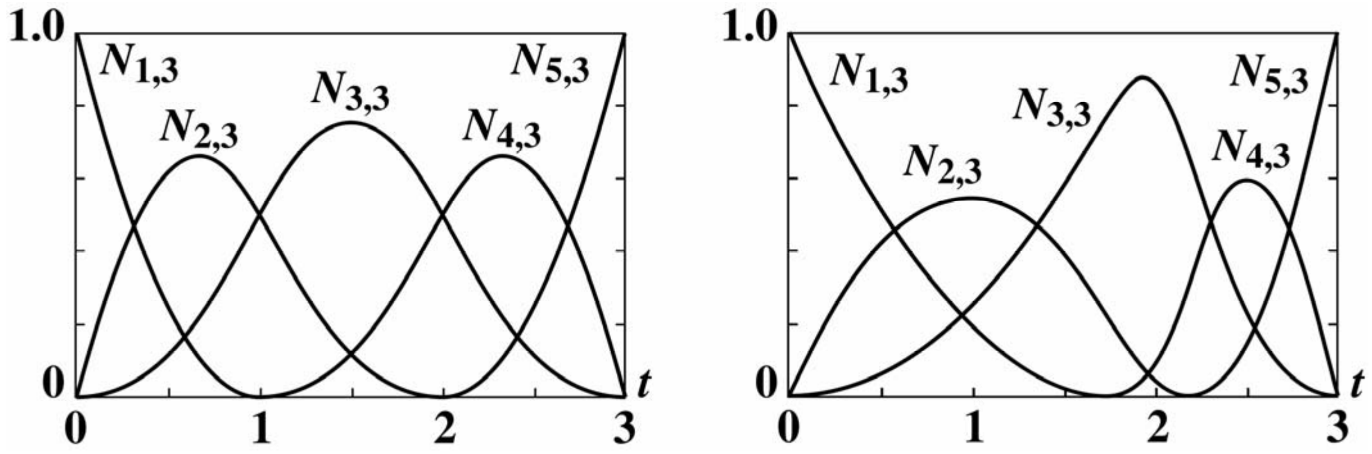
More influence for $B_{2,3}$ and $B_{4,3}$ control vertices

Less for others

Basis Functions – Comparisons

Uniform and nonuniform knot vectors: k=3, n+1=5

$[X] = [0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 3\ \ 3\ \ 3]$ $[X] = [0\ \ 0\ \ 0\ \ 1.8\ \ 2.2\ \ 3\ \ 3\ \ 3]$



Notice: $N_{3,3}$ pulled right and magnitude increased.

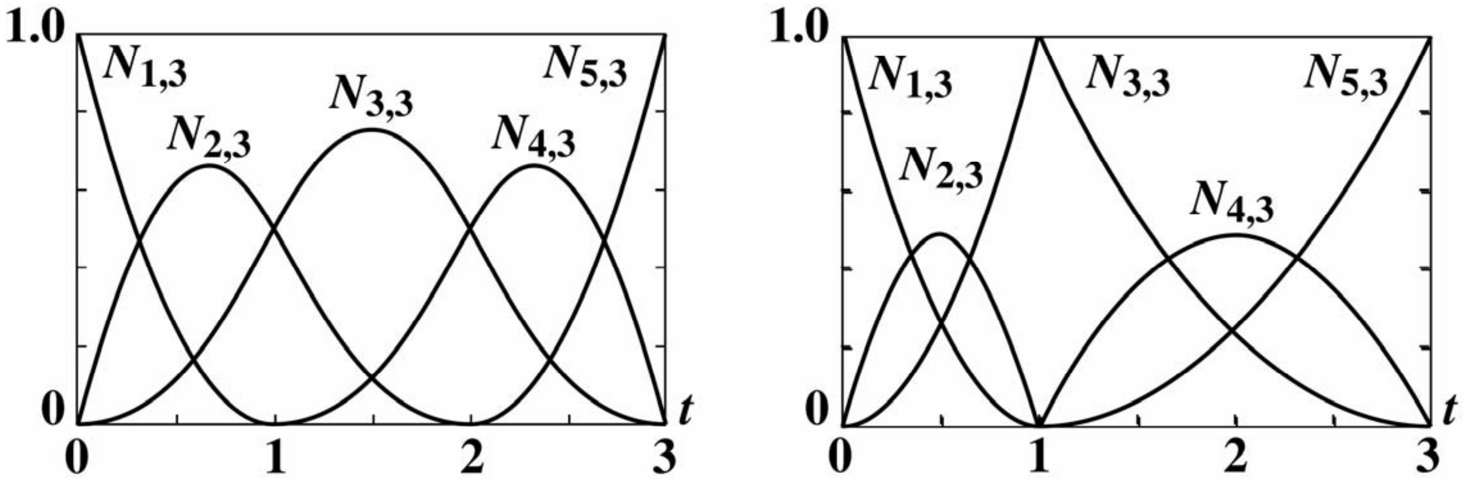
More influence for $B_{3,3}$ control vertices

Less for others

Basis Functions – Comparisons

Multiple duplicate knot values: $k=3$, $n+1=5$

$[X] = [0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 3\ \ 3\ \ 3]$ $[X] = [0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 3\ \ 3\ \ 3]$



Notice: $N_{3,3}=1$ at $t=1$ while all others zero

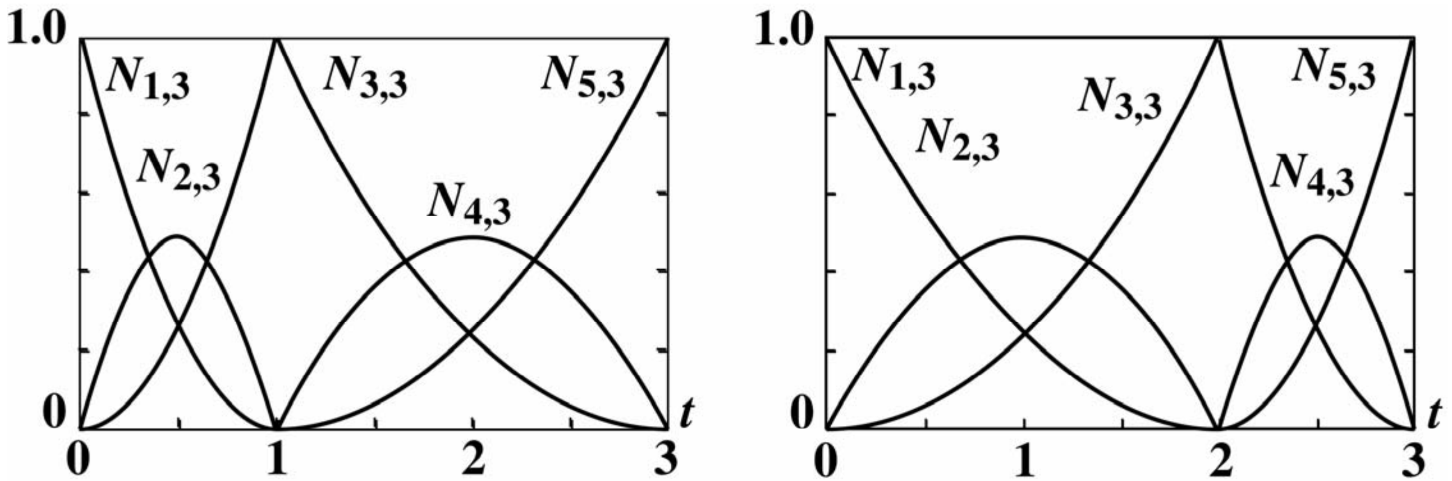
Curve passes through $B_{3,3}$

Continuity reduced

Basis Functions – Comparisons

Multiple duplicate knot values: $k=3$, $n+1=5$

$[X] = [0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 3\ \ 3\ \ 3]$ $[X] = [0\ \ 0\ \ 0\ \ 2\ \ 2\ \ 3\ \ 3\ \ 3]$



Notice: $N_{3,3}=1$ at $t=2$ while all others zero

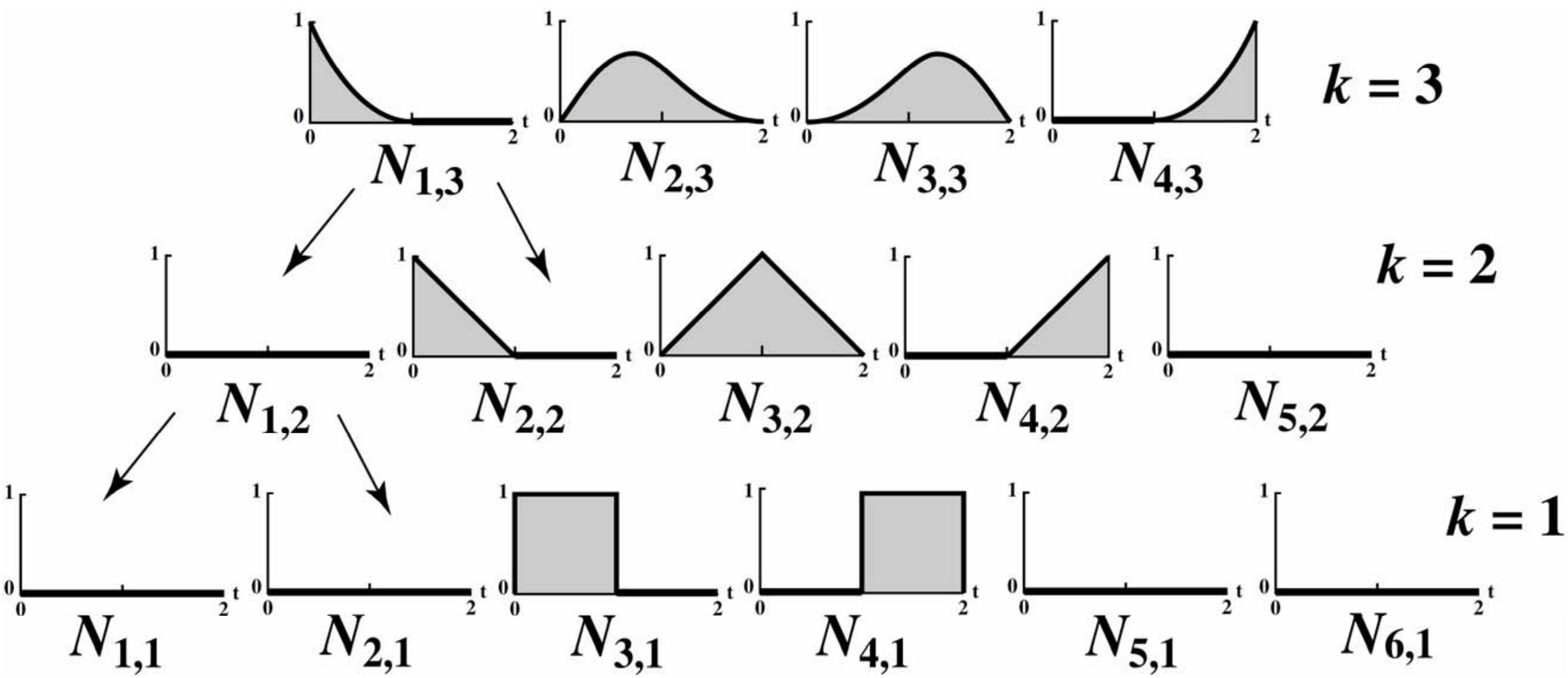
Curve passes through $B_{3,3}$

Continuity reduced

Basis Functions – Buildup

Example: $n+1=4$, $k=3$

$$[X] = [0 \quad 0 \quad \underbrace{0 \quad 1 \quad 2}_{\text{two spans}} \quad 2 \quad 2]$$



Basis Functions – Calculation

Example: $n+1=4$, $k=3$ $\quad [X]=[0\ \ 0\ \ \underbrace{0\ \ 1\ \ 2}_{\text{two spans}}\ \ 2\ \ 2]$

$0 \leqslant t < 1$

$$N_{3,1}(t)=1;\ N_{i,1}(t)=0,\ i\neq 3$$

$$N_{2,2}(t)=1-t;\ N_{3,2}(t)=t;\ N_{i,2}(t)=0,\ i\neq 2,3$$

$$N_{1,3}(t)=(1-t)^2;\ N_{2,3}(t)=t(1-t)+\frac{(2-t)}{2}t$$

$$N_{3,3}(t)=\frac{t^2}{2};\ N_{i,3}(t)=0,\ i\neq 1,2,3$$

Basis Functions – Calculation

Example: $n+1=4$, $k=3$ $\quad [X] = [0 \;\; 0 \;\; \underbrace{0 \;\; 1 \;\; 2}_{\text{two spans}} \;\; 2 \;\; 2]$

$1 \leqslant t < 2$

$$N_{4,1}(t) = 1;\ N_{i,1}(t) = 0,\ i \neq 4$$

$$N_{3,2}(t) = (2-t);\ N_{4,2}(t) = (t-1);\ N_{i,2}(t) = 0,\ i \neq 3, 4$$

$$N_{2,3}(t) = \frac{(2-t)^2}{2};$$

$$N_{3,3}(t) = \frac{t(2-t)}{2} + (2-t)(t-1);$$

$$N_{4,3}(t) = (t-1)^2;\ N_{i,3}(t) = 0,\ i \neq 2, 3, 4$$

B-Spline & Bézier Curves

If $k = n + 1$

and

an open knot vector is used

then

the B-spline curve is a Bézier curve

The knot vector is:

k zeros followed by k ones

k=4, $[X]=[0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1]$

B-Spline Curves – Controls

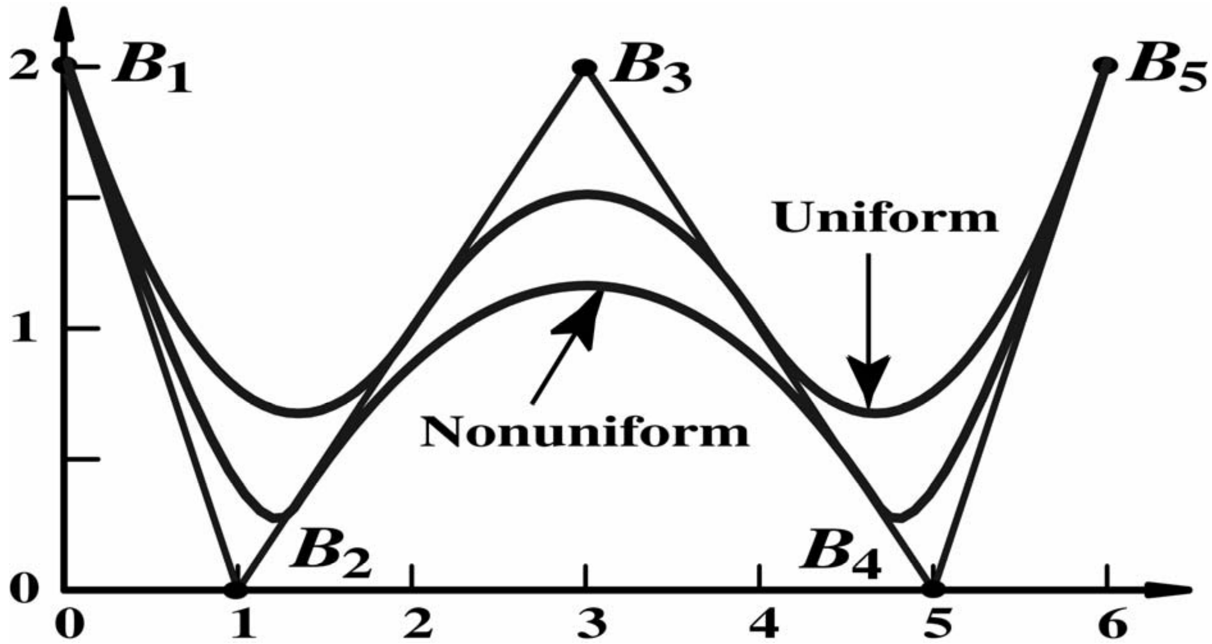
- Change type of knot vector
  - open uniform
  - open nonuniform
- Change order k
- Change number/position of control vertices
- Use multiple coincident control vertices
- Use multiple equal internal knot values

B-spline Curves – Controls

Type of knot vector $n+1=5,\ k=3$

$[X]=[0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 3\ \ 3\ \ 3]$ Open uniform

$[X]=[0\ \ 0\ \ 0\ \ 0.4\ \ 2.6\ \ 3\ \ 3\ \ 3]$ Open nonuniform



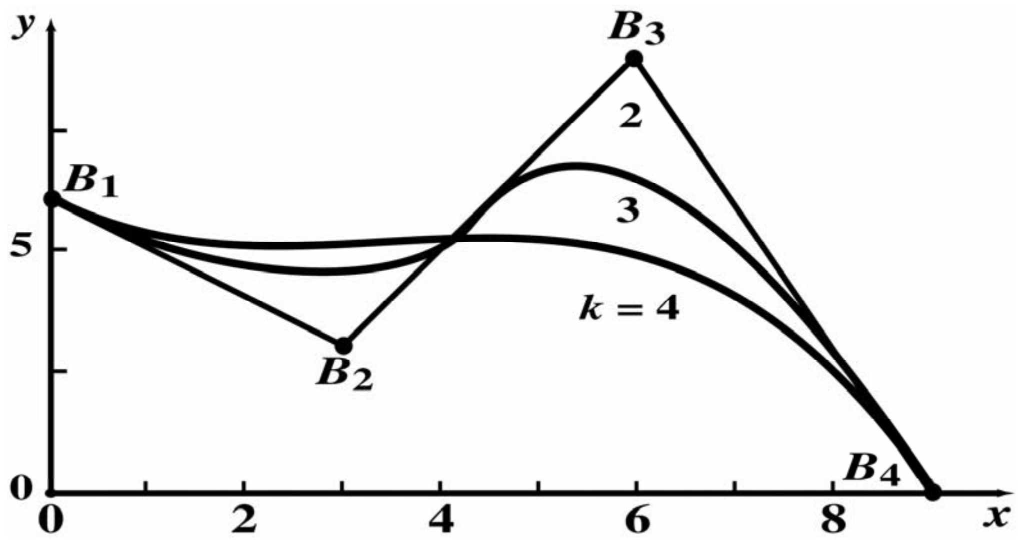
B-spline Curves – Controls

Type of knot vector $n+1=4$, $k=2,3,4$

$k=2$ $[X]=[0\ \ 0\ \ 1\ \ 2\ \ 3\ \ 3]$

$k=3$ $[X]=[0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 2\ \ 2]$

$k=4$ $[X]=[0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1]$



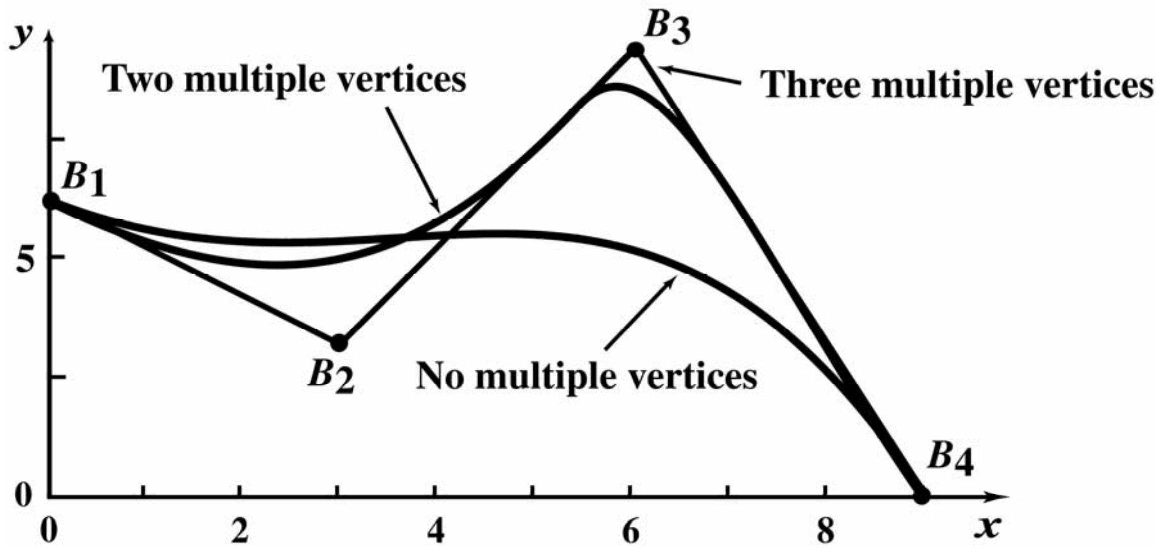
B-spline Curves – Controls

Type of knot vector $k=3,\ n+1=4,5,6$

$[X]=[0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1]$ Single Vertex

$[X]=[0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 2\ \ 2\ \ 2]$ Double Vertex

$[X]=[0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 3\ \ 3\ \ 3\ \ 3]$ Triple Vertex

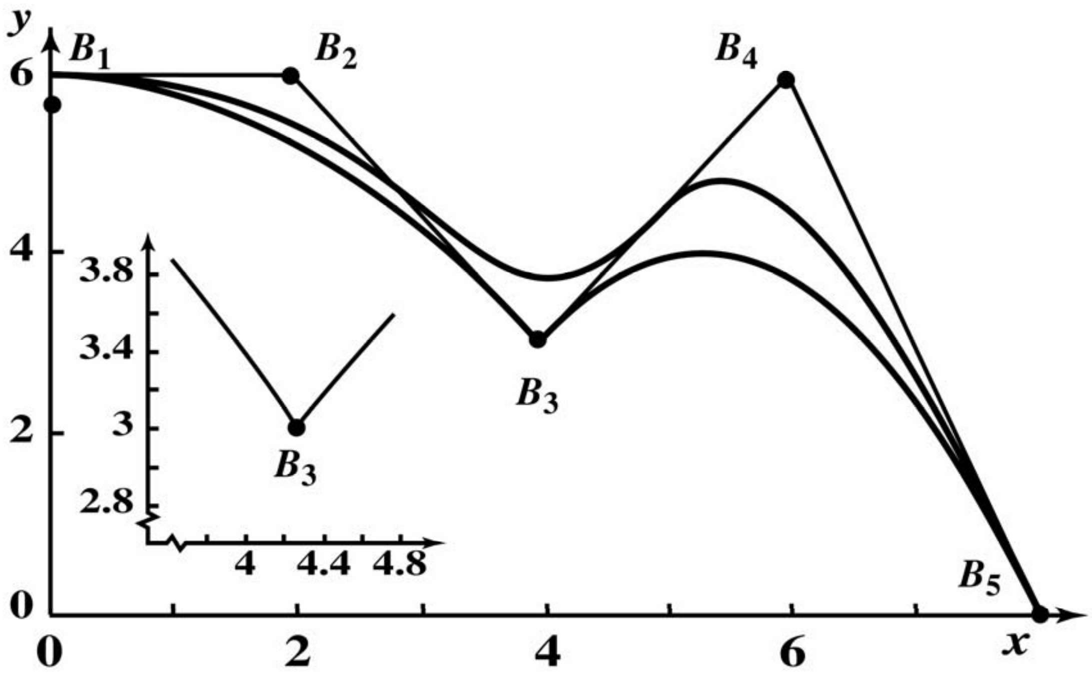


B-spline Curves – Controls

Multiple internal knot values $n+1=5,6 \quad k=3$

$[X]=[0\ \ 0\ \ 0\ \ 1\ \ 2\ \ 3\ \ 3\ \ 3]$

$[X]=[0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 3\ \ 3\ \ 3]$



Other Topics

- Nonrational B-spline curves cannot precisely represent the conic sections
  - Circles
  - Ellipses
  - Parabolas
  - Hyperbolas
- B-spline Curves - Additional Topics
  - Degree elevation
  - Degree reduction
  - Knot insertion
  - Subdivision
  - Reparameterization

Course Download

Course can be downloaded from:

http://www.cad.zju.edu.cn/home/jqfeng/GM/GM04.zip

Interactive Spline Notes:

http://www.people.nnov.ru/fractal/Splines/Intro.htm