B-Spline Curves and Surfaces (1)

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The Analytic and Geometric Definition of a B-Spline Curve

- Why B-Spline Curve?
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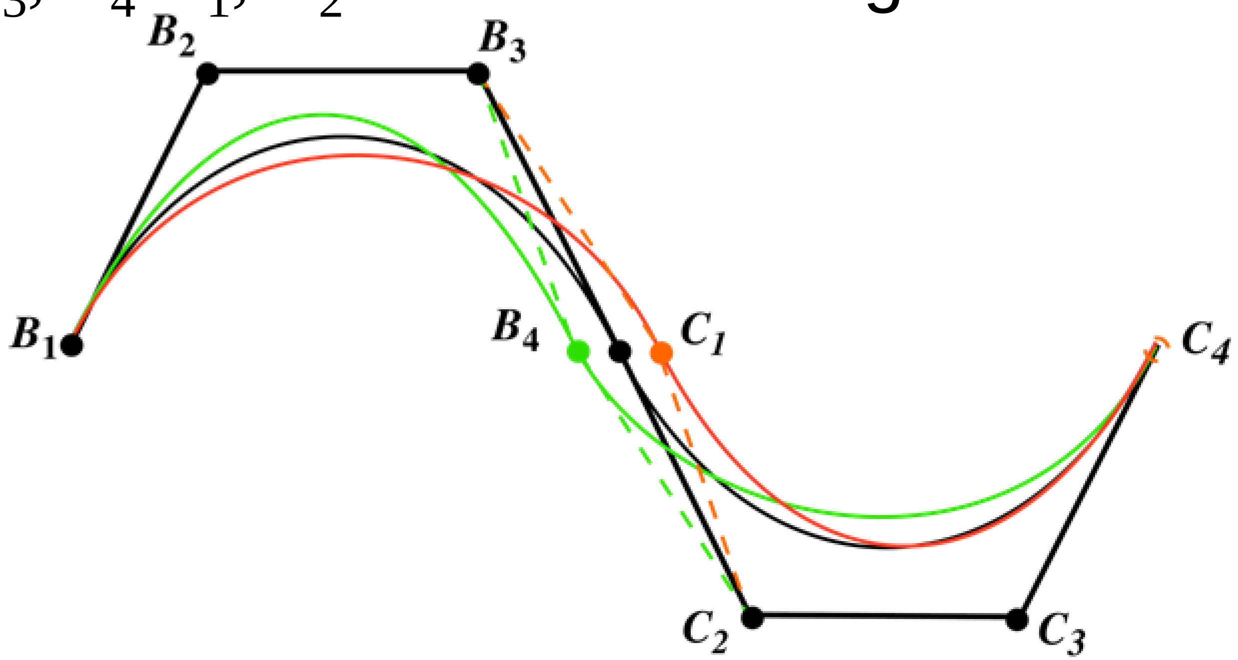
Why B-Spline Curve?

- Composite Bézier Curves – Continuity
  - B_3, B_4/C_1, C_2 must lie in straight line



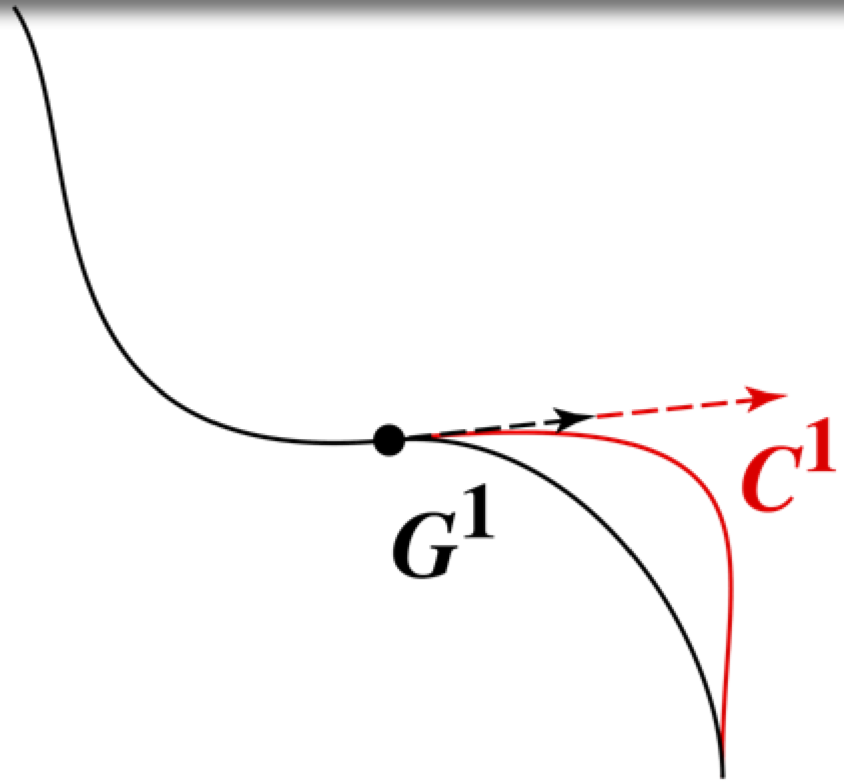
Geometric Continuity

- G^0 continuity: Two curve segments joined at ends
- G^1 continuity: Two curve segments joined at ends *and* tangent vectors point in same direction
  - The tangent vector magnitudes *do not* have to be the same
- Less restrictive than parametric continuity

Parametric Continuity

- C^0 continuity: Two curve segments joined at ends
- C^1 continuity: Two curve segments joined at ends *and* tangent vectors in same direction *and* tangent vectors magnitudes the same
- More restrictive than geometric continuity

Bézier curves – Continuity comparison

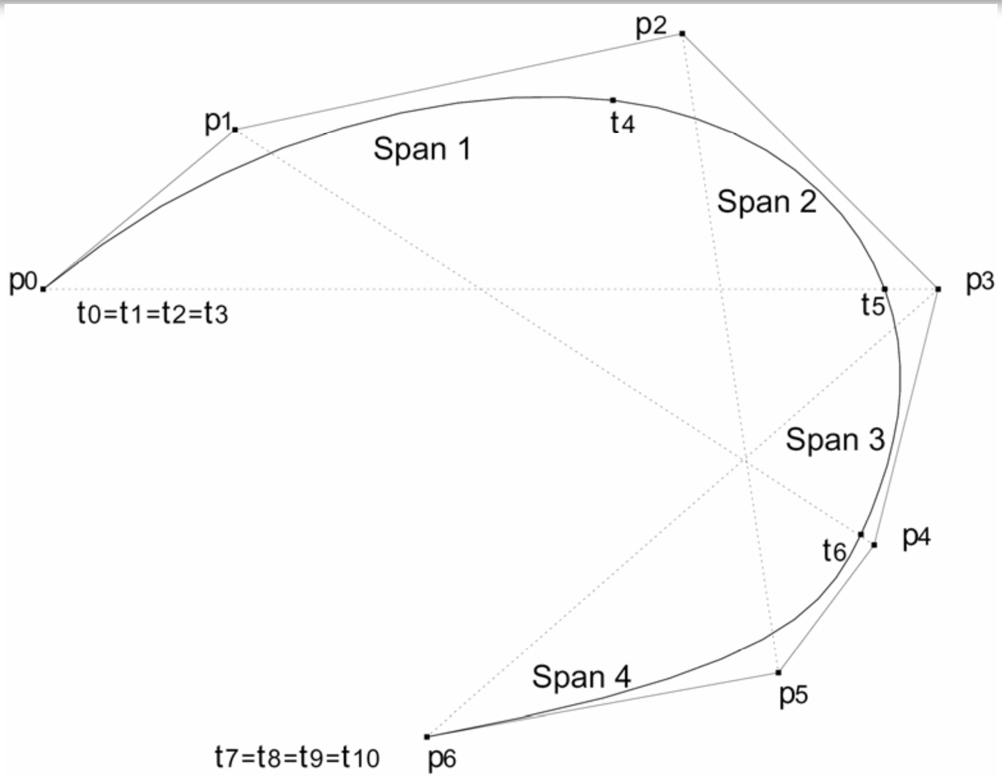


Bézier curves – Parametric (red) and Geometric (black) Continuity comparison

Bézier curve difficulties

- Bernstein basis is global
  - No local control
- Order (degree) fixed
  - Equal to number of control vertices
- High order (degree) required for flexibility
  Wiggles
- Difficult to maintain continuity

The B-Spline Curve - Analytical Definition



A Cubic B-Spline Curve with 7 Control Points (4 Segements)

The B-Spline Curve - Analytical Definition (1)

A B-spline curve $\mathbf{P}(t)$, is defined by

$$\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P}_i N_{i,k}(t)$$

where

- the $\{\mathbf{P}_i : i = 0,1,\ldots,n\}$ are the control points,
- k is the *order* of the polynomial segments of the B-spline curve. *Order* means that the curve is made up of piecewise (k pieces) polynomial segments of degree k-1

The B-Spline Curve - Analytical Definition (2)

- Normalized B-spline blending functions (B-spline base functions) $\{N_{i,k}(t)\}$:
  - The order k
  - A non-decreasing sequence of real numbers

$$\{\ t_i : i=0,\ldots,n+k\}$$

  normally called the "*knot sequence*"

The B-Spline Curve - Analytical Definition (3)

- The $N_{i.k}(t)$ functions are described as follows

$$N_{i,1}(t) = \begin{cases} 1 & \text{if } u \in [t_i, t_{i+1}) \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

and if $k>1$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t) \quad (2)$$

- $t \in [t_{k-1}, t_{n+1})$

The B-Spline Curve - Analytical Definition

- Notes

1. In equation (2), If
   - the *denominator* terms on the right hand side of the equation are *zero*
   - the subscripts are *out of the range* of the summation limits,

   then the associated fraction is not evaluated and the term becomes zero. (Avoiding 0/0).
2. In the equation (1), "*closed-open*" interval.

The B-Spline Curve - Analytical Definition

- Notes

3. The order k is independent of the number of control points $(n+1)$.
    - The Bezier Curve:

      number of control points = degree+1
    - The B-Spline curve,

      number of control points and degree are free

The B-Spline Curve - Geometric Definition

Given a set of Control Points $\{\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n\}$, an order k, and a set of knots $\{t_0, t_1, \ldots, t_{n+k}\}$, the B-Spline curve of order k is defined to be

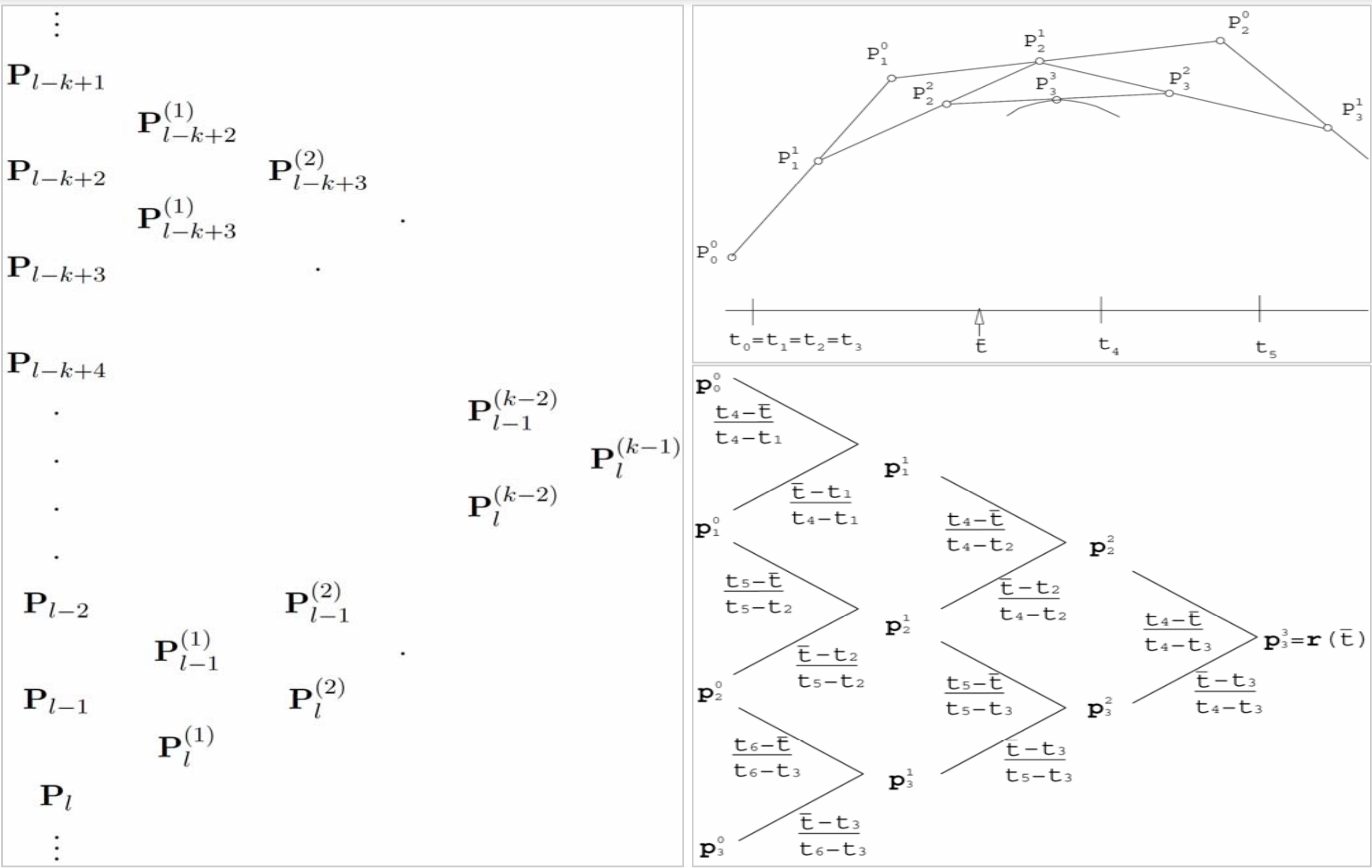
$$\mathbf{P}(t) = \mathbf{P}_l^{(k-1)}(t) \text{ if } u \in [t_l, t_{l+1})$$

where

$$\mathbf{P}_i^{(j)}(t) = \begin{cases} (1-\tau_i^j)\mathbf{P}_{i-1}^{(j-1)}(t) + \tau_i^j \mathbf{P}_i^{(j-1)}(t) & \text{if } j > 0, \\ \mathbf{P}_i & \text{if } j = 0. \end{cases}$$

$$\tau_i^j = \frac{t - t_i}{t_{i+k-j} - t_i}$$

The B-Spline Curve - Geometric Definition: Pyramid Description



The B-Spline Curve - Geometric Definition

- Any **P** in above pyramid is calculated as a convex combination of the two **P** functions immediately to it's left

The Uniform B-Spline Blending Functions

- Calculating the Blending Functions using a Uniform Knot Sequence
- Blending Functions for $k = 1$
- Blending Functions for $k = 2$
- Blending Functions for $k = 3$
- Blending Functions of Higher Orders

Calculating the Blending Functions using a Uniform Knot Sequence

The normalized B-spline blending functions are defined recursively by

$$N_{i,1}(t) = \begin{cases} 1 & \text{if } u \in [t_i, t_{i+1}) \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

and if $k>1$

$$N_{i,k}(t) = \left(\frac{t - t_i}{t_{i+k-1} - t_i}\right) N_{i,k-1}(t) + \left(\frac{t_{i+k} - t}{t_{i+k} - t_{i+1}}\right) N_{i+1,k-1}(t) \qquad (2)$$

where $\{t_0, t_1, ..., t_{n+k}\}$ is a non-decreasing sequence of knots, and k is the order of the curve

Calculating the Blending Functions using a Uniform Knot Sequence

- These blending functions are difficult to calculate directly for a general knot sequence.
- If the knot sequence is uniform, it is quite straightforward to calculate these functions
  - $\{t_0, t_1, \ldots, t_{n+k}\}=\{0,1,\ldots,n+k\}$ *i.e.* $t_i=i$
  - They have some surprising properties

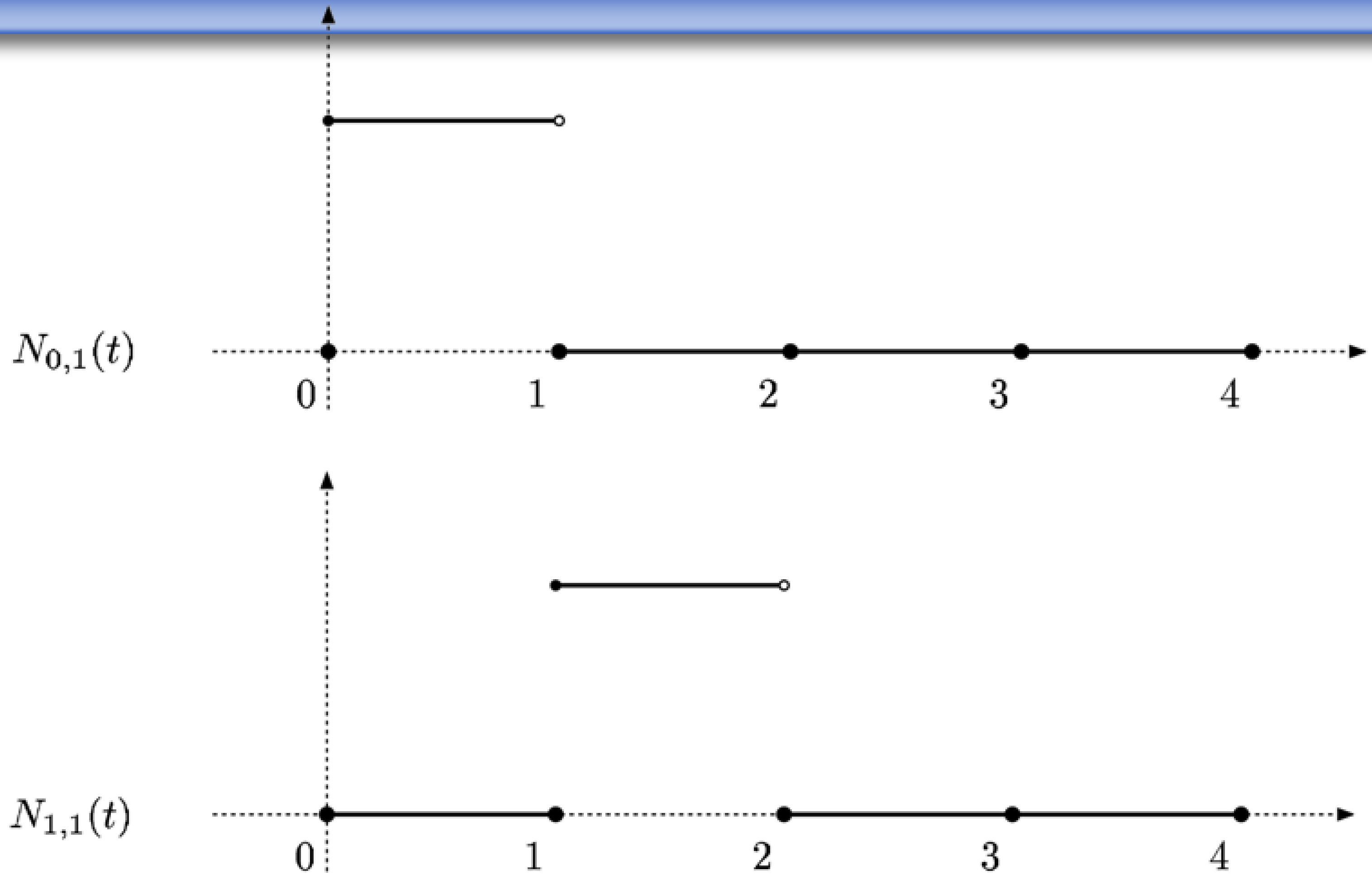
Blending Functions for k = 1

If $k = 1$, from (1), the normalized blending functions are

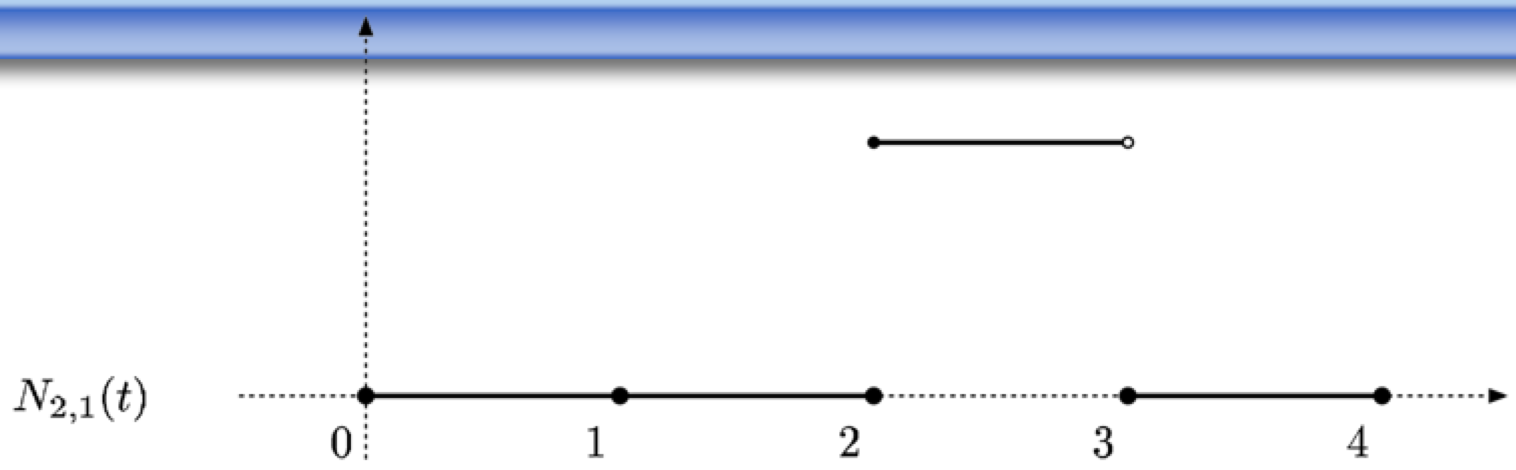
$$N_{i,1}(t) = \begin{cases} 1 & \text{if } u \in [t_i, t_{i+1}) \\ 0 & \text{otherwise} \end{cases} \qquad (3)$$

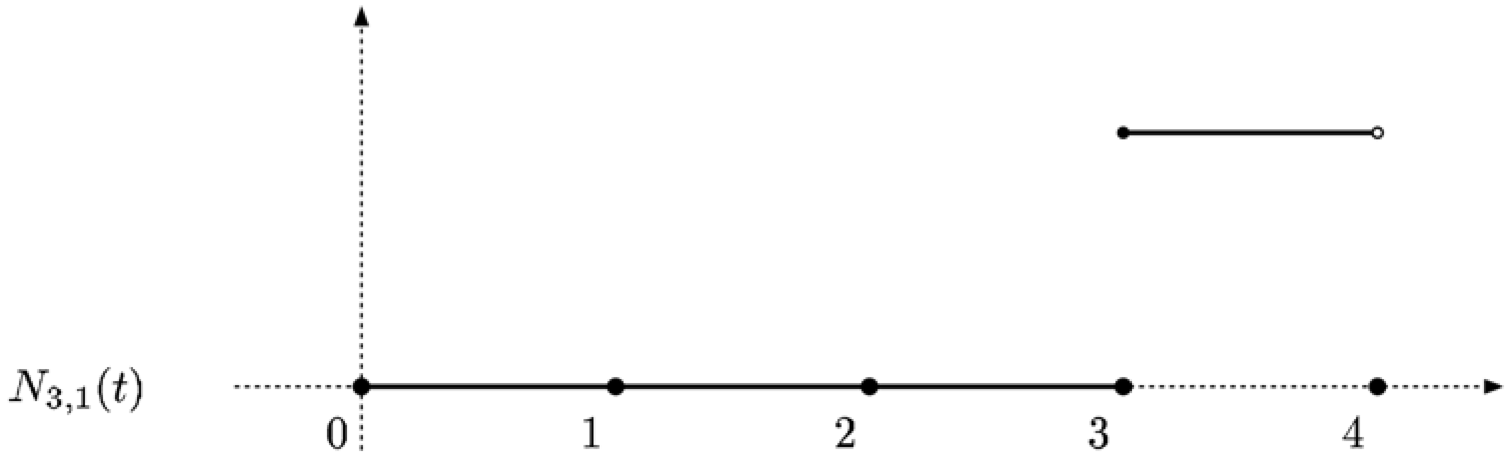
- These are shown together in the following figure, where we have plotted $N_{0,1}$, $N_{1,1}$, $N_{2,1}$ and $N_{3,1}$ respectively, over five of the knots.
- The white circle at the end of the line where the functions value is 0. This represents the affect of the “open-closed” interval found in equation (1).

Blending Functions for *k* = 1



Blending Functions for *k* = 1





Blending Functions for $k = 1$

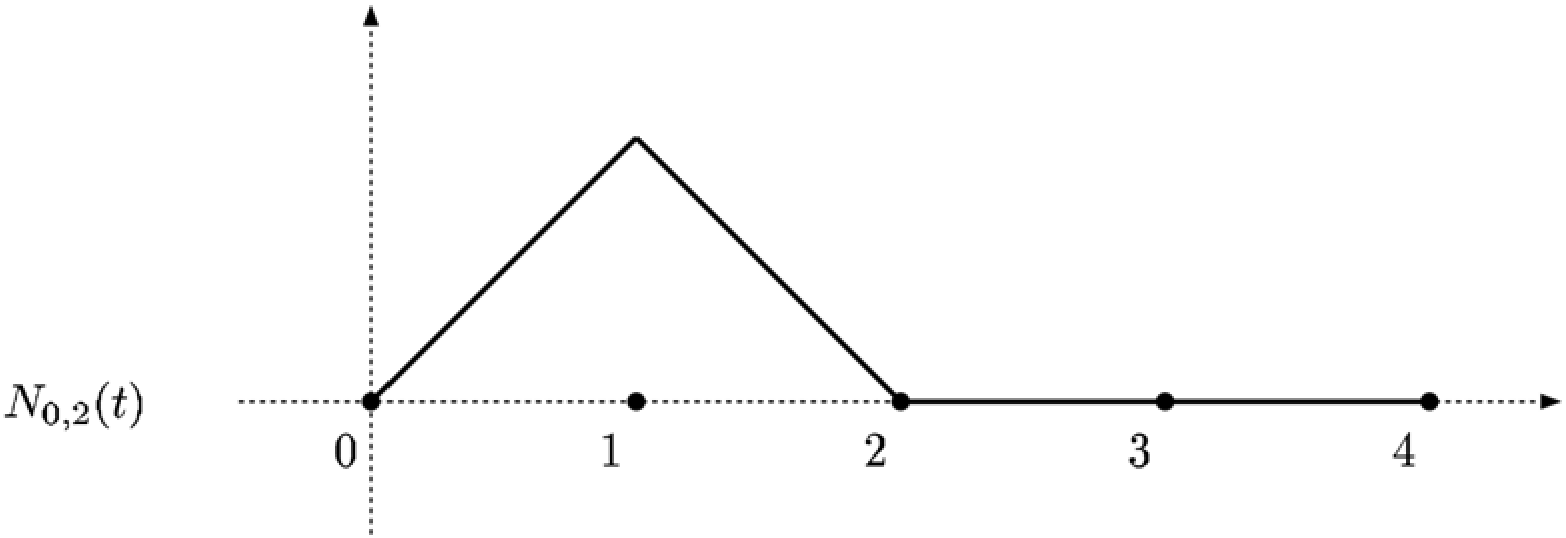
- Notes
  - The white circle at the end of the line where the functions value is 0. This represents the affect of the "open-closed" interval found in equation (1).
  - These functions have support (the region where the curve is nonzero) in an interval, with $N_{i,1}$ having support on $[i,i+1)$.
  - They are also clearly shifted versions of each other – e.g., $N_{i+1,1}$ is just $N_{i,1}$ shifted one unit to the right. In fact, we can write $N_{i,1}(t)=N_{0,1}(t-i)$

Blending Functions for $k = 2$

If k=2 then $N_{0,2}$ can be written as a weighted sum of $N_{0,1}$ and $N_{1,1}$ by equation (2).

$$
\begin{aligned}
N_{0,2}(t) &= \frac{t - t_0}{t_1 - t_0} N_{0,1}(t) + \frac{t_2 - t}{t_2 - t_1} N_{1,1}(t) \\
&= t N_{0,1}(t) + (2 - t) N_{1,1}(t) \\
&= \begin{cases} t & \text{if } 0 \le t < 1 \\ 2 - t & \text{if } 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Blending Functions for $k = 2$



- The curve is piecewise linear, with support in the interval [0,2].
- These functions are commonly referred to as “hat” functions and are used as blending functions in many linear interpolation problems.

Blending Functions for k = 2

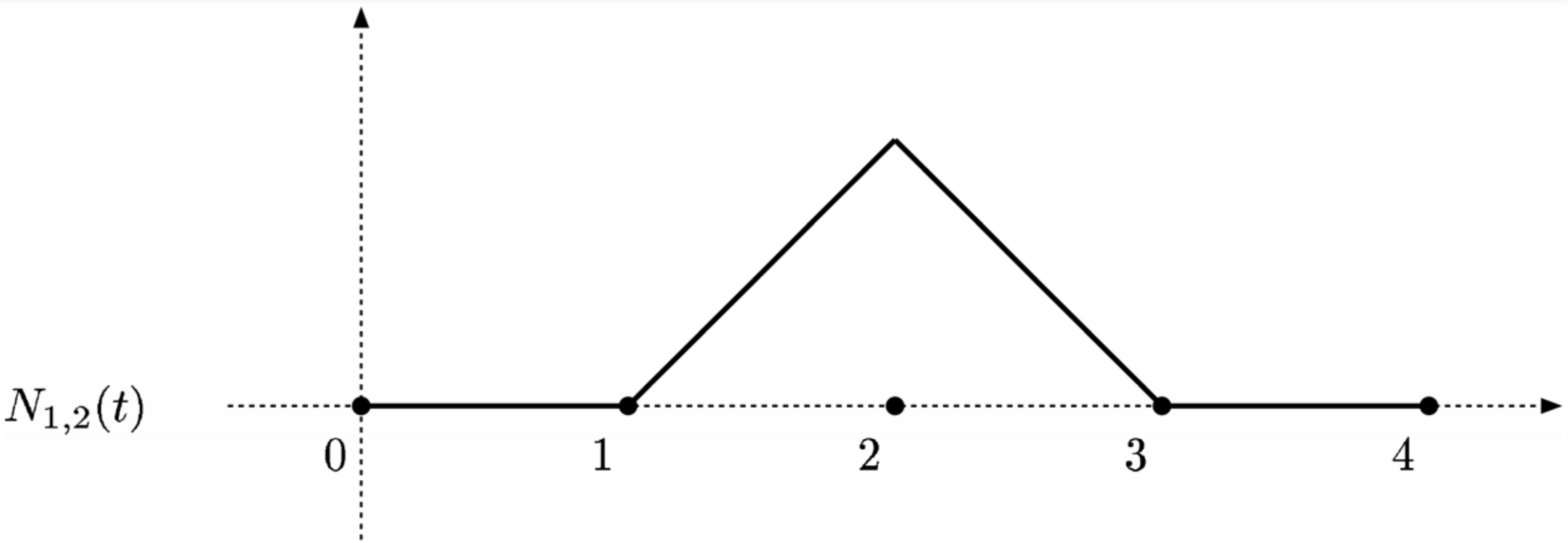
Calculate $N_{1,2}$ to be

$$N_{1,2}(t) = \frac{t - t_1}{t_2 - t_1} N_{1,1}(t) + \frac{t_3 - t}{t_3 - t_2} N_{2,1}(t)$$

$$= (t-1)N_{1,1}(t) + (3-t)N_{2,1}(t)$$

$$= \begin{cases} t - 1 & \text{if } 1 \le t < 2 \\ 3 - t & \text{if } 2 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$$

Blending Functions for $k = 2$



- It is a shifted version of $N_{0,2}$

Blending Functions for $k = 2$

Notes

- Each nonzero portion of these curves covers the intervals spanned by three knots – *e.g.*, $N_{1,2}$ spans the interval $[1,3]$.
- The curves are piecewise linear, made up of two linear segments joined continuously.
- Since the curves are shifted versions of each other, we can write

$$N_{i,2}(t) = N_{0,2}(t-i)$$

Blending Functions for k = 3

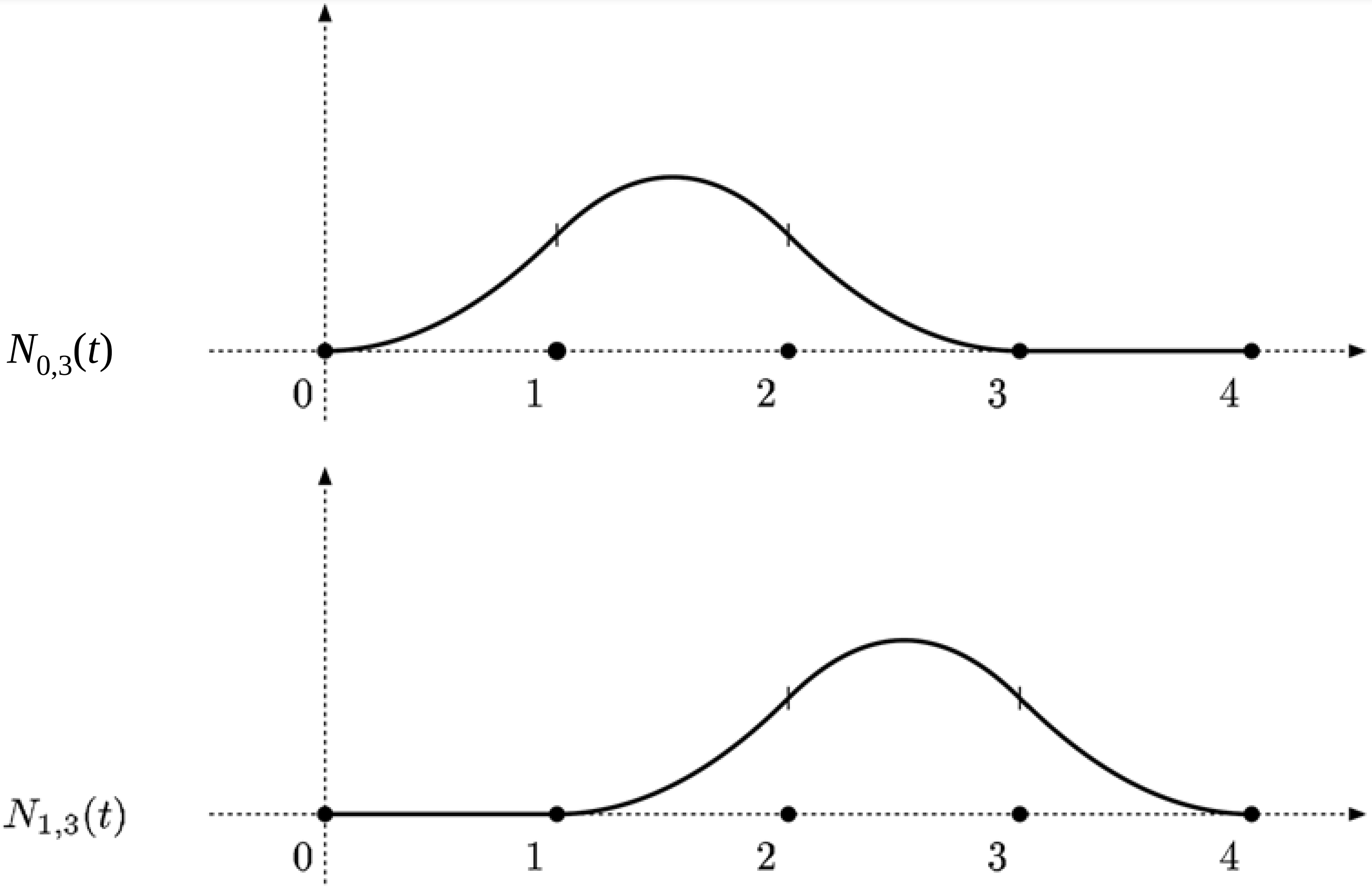
For the case k=3, we again use equation (2) to obtain

$$
\begin{aligned}
N_{0,3}(t) &= \frac{t-t_0}{t_2-t_0}N_{0,2}(t) + \frac{t_3-t}{t_3-t_1}N_{1,2}(t) \\
&= \frac{t}{2}N_{0,2}(t) + \frac{3-t}{2}N_{1,2}(t) \\
&= \begin{cases} \frac{t^2}{2} & \text{if } 0 \le t < 1 \\ \frac{t^2}{2}(2-t) + \frac{3-t}{2}(t-1) & \text{if } 1 \le t < 2 \\ \frac{(3-t)^2}{2} & \text{if } 2 \le t < 3 \\ 0 & \text{otherwise} \end{cases}
= \begin{cases} \frac{t^2}{2} & \text{if } 0 \le t < 1 \\ \frac{-3+6t-2t^2}{2} & \text{if } 1 \le t < 2 \\ \frac{(3-t)^2}{2} & \text{if } 2 \le t < 3 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Blending Functions for $k = 3$

$$
\begin{aligned}
N_{1,3}(t) &= \frac{t - t_1}{t_3 - t_1} N_{1,2}(t) + \frac{t_4 - t}{t_4 - t_2} N_{2,2}(t) \\
&= \frac{t-1}{2} N_{1,2}(t) + \frac{4-t}{2} N_{2,2}(t) \\
&= \begin{cases}
\frac{(t-1)^2}{2} & \text{if } 1 \le t < 2 \\
\frac{-11+10t-2t^2}{2} & \text{if } 2 \le t < 3 \\
\frac{(4-t)^2}{2} & \text{if } 3 \le t < 4 \\
0 & \text{otherwise}
\end{cases}
\end{aligned}
$$

Blending Functions for $k = 3$



Blending Functions for $k = 3$

Notes

- The figures are piecewise quadratic curves, each made up of three parabolic segments that are joined at the knot values (tick marks on the joints).
- The nonzero portion of these two curves each span the interval between four consecutive knots - e.g., the nonzero portion of $N_{1,3}$ spans the interval $[1,4]$
- Again, $N_{1,3}$ can be seen visually to be a shifted version of $N_{0,3}$. Generally,

$$N_{i,3}(t) = N_{0,3}(t-i)$$

Blending Functions of Higher Orders

- When $k = 4$
  - The $N_{i,4}(t)$ blending functions will be piecewise (4 pieces) cubic functions
  - The support of $N_{i,4}(t)$ will be the interval $[i,i+4]$
  - Each of the blending functions will be shifted versions of each other

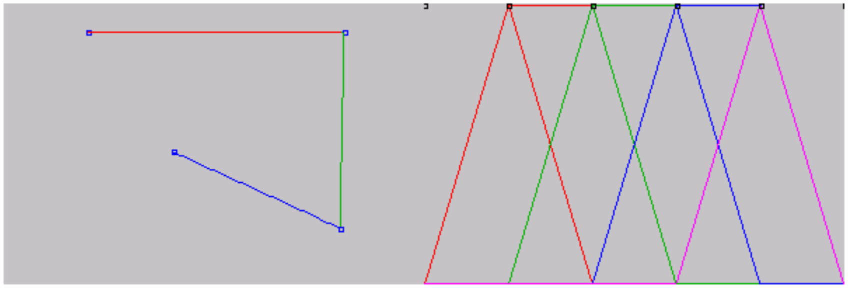
$$N_{i,4}(t) = N_{0,4}(t-i)$$

Blending Functions of Higher Orders

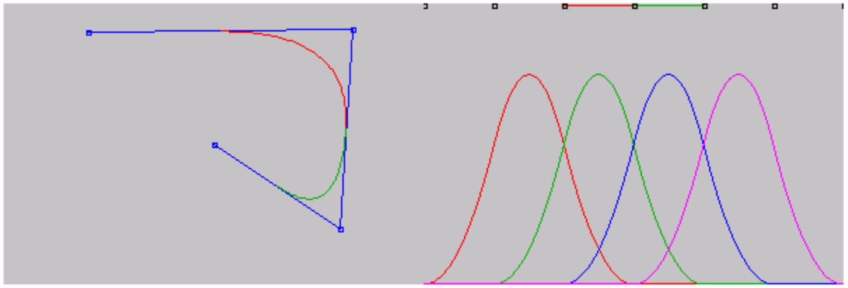
- In general
  - The uniform blending functions $N_{i,k}$ will be piecewise (k pieces) $(k\text{-}1)$*st* degree functions
  - $N_{i,k}$ has support in the interval $[i, i+k)$.
  - They will be shifted versions of each other and each can be written in terms of a "basic" function

$$N_{i,k}(t) = N_{0,k}(t-i)$$

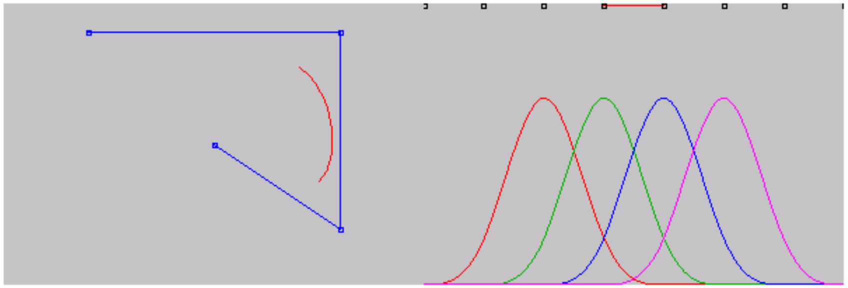
Blending Functions of Higher Orders



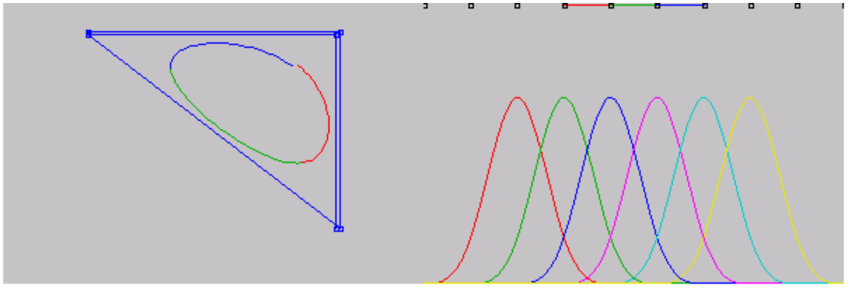
Linear B-Spline n=3, k=2



Quadratic B-Spline n=3, k=3



Cubic B-Spline n=3, k=4



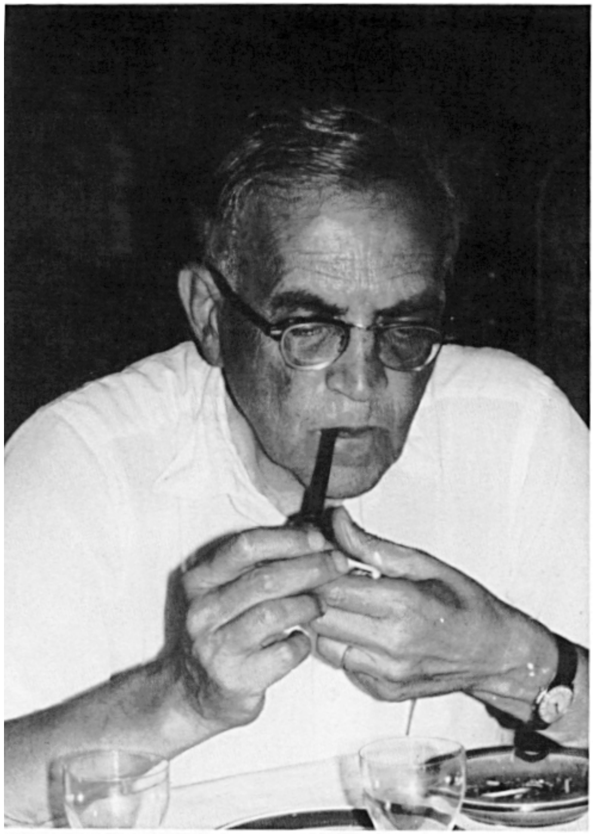
Cubic B-Spline n=5, k=2

[Interactive demo]

The DeBoor-Cox Calculation

- In 1972, Carl DeBoor and M.G. Cox independently discovered the relationship between the analytic and geometric definitions of B-splines.
- Starting with the definition of the normalized B-spline blending functions, these two researchers were able to develop the geometric definition of the B-spline.

VIP for B-Spline



J. Schoenberg

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The DeBoor-Cox Calculation

- Definition of the B-Spline Curve
- The DeBoor-Cox Calculation
- Geometric Definition of the B-Spline Curve

Definition of the B-Spline Curve

A B-spline curve $\mathbf{P}(t)$, is defined by

$$\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P}_i N_{i,k}(t) \qquad (1)$$

where :

- The $\{\mathbf{P}_i: i=0,1,\ldots,n\}$ are the control points,
- k is the order of the polynomial segments of the B-spline curve. Order k means that the curve is made up of piecewise (k pieces) polynomial segments of degree k-1,

Definition of the B-Spline Curve

- $N_{i,k}(t)$ (normalized B-spline blending functions)
  - The order k;
  - Knot sequence: a non-decreasing sequence of real numbers
    $\{t_i: i=0,\ldots,n+k\}$
  - The $N_{i,k}(t)$ functions are described as

$$N_{i,1}(t) = \begin{cases} 1 & \text{if } t \in [t_i, t_{i+1}) \\ 0 & \text{otherwise.} \end{cases} \qquad (2)$$

when $k>1$

$$N_{i,k}(t) = \left(\frac{t - t_i}{t_{i+k-1} - t_i} \right) N_{i,k-1}(t) + \left(\frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} \right) N_{i+1,k-1}(t) \qquad (3)$$

Definition of the B-Spline Curve

- The parameter t ranges throughout interval $[t_{k-1}, t_{n+1})$.

- Notse on definition of the B-Spline curve
  - In equation (3), if either of the *denominator* terms on the right hand side of the equation are zero, or the *subscripts* are out of the range of the summation limits, then the associated fraction is not evaluated and the term becomes *zero*. This is to avoid a zero-over-zero evaluation problem.
  - The order k is independent of the number of control points $(n+1)$. In the B-Spline curve, unlike the Bézier Curve, we have the flexibility of using many control points, and restricting the degree of the polynomial segments

The DeBoor-Cox Calculation

General idea: By substituting the recursively definition of $N_{i,k}(t)$ in equation (3), find the point on the curve in terms of recursively definition of control points $\mathbf{P}_i$

The DeBoor-Cox Calculation

By substituting

$$N_{i,k}(t) = \left(\frac{t - t_i}{t_{i+k-1} - t_i}\right) N_{i,k-1}(t) + \left(\frac{t_{i+k} - t}{t_{i+k} - t_{i+1}}\right) N_{i+1,k-1}(t) \qquad (3)$$

into B-spline curve

$$\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P}_i N_{i,k}(t) \qquad (1)$$

where $t \in [t_{k-1}, t_{n+1}]$

This will give us the definition of $\mathbf{P}(t)$ in terms of $N_{i,k-1}$, which is of lower degree. And continue this process until the sum is written with $N_{i,1}$, functions, which we can evaluate easily.

The DeBoor-Cox Calculation

$$\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P}_i N_{i,k}(t)$$

$$= \sum_{i=0}^{n} \mathbf{P}_i \left[\left(\frac{t - t_i}{t_{i+k-1} - t_i} \right) N_{i,k-1}(t) + \left(\frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} \right) N_{i+1,k-1}(t) \right]$$

$$= \sum_{i=0}^{n} \mathbf{P}_i \left(\frac{t - t_i}{t_{i+k-1} - t_i} \right) N_{i,k-1}(t) + \sum_{i=0}^{n} \mathbf{P}_i \left(\frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} \right) N_{i+1,k-1}(t)$$

Separating out those unique terms of each sum, $N_{0,k-1}$ and $N_{n+1,k-1}$, giving

$$= \mathbf{P}_0 \frac{t - t_0}{t_{k-1} - t_0} N_{0,k-1}(t) + \sum_{i=1}^{n} \mathbf{P}_i \left(\frac{t - t_i}{t_{i+k-1} - t_i} \right) N_{i,k-1}(t)$$

$$+ \sum_{i=0}^{n-1} \mathbf{P}_i \left(\frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} \right) N_{i+1,k-1}(t) + \mathbf{P}_n \left(\frac{t_{n+k} - t}{t_{n+k} - t_{n+1}} \right) N_{n+1,k-1}(t)$$

The DeBoor-Cox Calculation

Because

- The support of a B-spline blending function $N_{i,k}(t)$ is the interval $[t_i, t_{i+k}]$
- The suport of the $N_{0,k\text{-}1}$ is $[t_0, t_{k\text{-}1})$,
- The defined interval of $\mathbf{P}(t)$ is $[t_{k\text{-}1}, t_{n+1})$

Thus

$$N_{0,k\text{-}1}(t) \equiv 0$$

Similarly

- The support of the $N_{n+1,k\text{-}1}$ is $[t_{n+1}, t_{n+k})$, thus

$$N_{n+1,k\text{-}1}(t) \equiv 0$$

The DeBoor-Cox Calculation

$$\mathbf{P}(t) = \sum_{i=1}^{n} \mathbf{P}_i \left[\frac{t - t_i}{t_{i+k-1} - t_i} \right] N_{i,k-1}(t) + \sum_{i=0}^{n-1} \mathbf{P}_i \left[\frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} \right] N_{i+1,k-1}(t)$$

$$= \sum_{i=1}^{n} \mathbf{P}_i \left[\frac{t - t_i}{t_{i+k-1} - t_i} \right] N_{i,k-1}(t) + \sum_{i=1}^{n} \mathbf{P}_{i-1} \left[\frac{t_{i+k-1} - t}{t_{i+k-1} - t_i} \right] N_{i,k-1}(t)$$

$$= \sum_{i=1}^{n} \left[\left(\frac{t_{i+k-1} - t}{t_{i+k-1} - t_i} \right) \mathbf{P}_{i-1} + \left(\frac{t - t_i}{t_{i+k-1} - t_i} \right) \mathbf{P}_i \right] N_{i,k-1}(t)$$

if we denote

$$\mathbf{P}_i^{(1)}(t) = \left(\frac{t_{i+k-1} - t}{t_{i+k-1} - t_i} \right) \mathbf{P}_{i-1} + \left(\frac{t - t_i}{t_{i+k-1} - t_i} \right) \mathbf{P}_i$$

The DeBoor-Cox Calculation

Thus the result is

$$\mathbf{P}(t) = \sum_{i=1}^{n} \mathbf{P}_i^{(1)}(t) N_{i,k-1}(t)$$

where

$$\mathbf{P}_i^{(1)}(t) = \left(\frac{t_{i+k-1} - t}{t_{i+k-1} - t_i}\right)\mathbf{P}_{i-1} + \left(\frac{t - t_i}{t_{i+k-1} - t_i}\right)\mathbf{P}_i$$

- The summation terms of equation (1) is written in terms of blending functions of lower degree.
- We have transferred some of the complexity to the $\mathbf{P}_i^{(1)}$s, but we retain a similar form with control points $\mathbf{P}_i^{(1)}$s weighted by blending functions.
- We can repeat this calculation again.

The DeBoor-Cox Calculation

Repeating the calculation and manipulating the sums

$$\mathbf{P}(t) = \sum_{i=2}^{n} \mathbf{P}_i^{(2)}(t) N_{i,k-2}(t)$$

where

$$\mathbf{P}_i^{(2)}(t) = \left(\frac{t_{i+k-2} - t}{t_{i+k-2} - t_i} \right) \mathbf{P}_{i-1}^{(1)}(t) + \left(\frac{t - t_i}{t_{i+k-2} - t_i} \right) \mathbf{P}_i^{(1)}(t)$$

The DeBoor-Cox Calculation

If we continue with this process repeatedly, eventually we will obtain blending functions of order 1. We are led to the following result: If we define

$$\mathbf{P}_i^{(j)}(t) = \begin{cases} (1-\tau_i^j)\mathbf{P}_{i-1}^{(j-1)}(t) + \tau_i^j \mathbf{P}_i^{(j-1)}(t) & \text{if } j > 0 \\ \mathbf{P}_i & \text{if } j = 0 \end{cases} \tag{4}$$

where

$$\tau_i^j = \frac{t - t_i}{t_{i+k-j} - t_i}$$

if t is in the interval $[t_l, t_{l+1})$, we have

$$\mathbf{P}(t) = \mathbf{P}_l^{(k-1)}(t)$$

The DeBoor-Cox Calculation

By continuing the DeBoor-Cox calculation k-1 times, we arrive at the formula

$$\mathbf{P}(t) = \sum_{i=k-1}^{n} \mathbf{P}_i^{(k-1)}(t) N_{i,1}(t)$$

where

$$\mathbf{P}_i^{(j)}(t) = \begin{cases} (1-\tau_i^j)\mathbf{P}_{i-1}^{(j-1)}(t) + \tau_i^j \mathbf{P}_i^{(j-1)}(t) & \text{if } j > 0 \\ \mathbf{P}_i & \text{if } j = 0 \end{cases} \quad (4)$$

Geometric Definition of the B-Spline Curve

Given

- order of B-spline curve: k
- a set of Control Points: $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$
- a set of knots: $t_0, t_1, \ldots, t_{n+k}$

The corresponding B-spline curve is

$$\mathbf{P}(t) = \mathbf{P}_l^{(k-1)}(t) \text{if } t \in [t_l, t_{l+1})$$

where

$$\mathbf{P}_i^{(j)}(t) = \begin{cases} (1-\tau_i^j)\mathbf{P}_{i-1}^{(j-1)}(t) + \tau_i^j \mathbf{P}_i^{(j-1)}(t) & \text{if } j > 0 \\ \mathbf{P}_i & \text{if } j = 0 \end{cases}$$

$$\tau_i^j = \frac{t - t_i}{t_{i+k-j} - t_i}$$

Geometric Definition of the B-Spline Curve : Pyramid Description