Bézier Curves and Surfaces (2)

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Contents

Reparameterizing Bézier Curves

- Bézier Curve $\mathbf{P}(t)$:

  a set of control points $\{\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n\}$ with Bernstein polynomials $\{B_{i,n}(t)\}$ $t \in [0,1]$

$$\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P}_i B_{i,n}(t)$$

- Purpose

  General B-spline curves: piecewise Bézier curves over an arbitrary parametric interval

Defining the Reparameterized Curve

- Given a Bézier curve $\mathbf{P}(t)$, a new parameterization of the curve where $t \in [a,b]$ can be developed as

$$\mathbf{P}_{[a,b]}(t) = \mathbf{P}(\frac{t-a}{b-a})$$

$\mathbf{P}_{[a,b]}(t)$ and $\mathbf{P}(t)$ are exactly the same curve, but traversed through different ranges of t.

Impaction of Parameterization on Bézier Curve Properties

- $\mathbf{P}_{[0,1]}(t)=\mathbf{P}(t)$
- Using the chain rule, the derivative of the curve $\mathbf{P}_{[a,b]}(t)$ at a value t is equal to

$$\frac{1}{b-a}\mathbf{P}'(\frac{t-a}{b-a})$$

- Subdividing the curve $\mathbf{P}_{[a,b]}(t)$ at the point $c\in[a,b]$, is equivalent to subdividing the curve $\mathbf{P}(t)$ at the point $\frac{c-a}{b-a}$

Bézier Control Polygons for a Cubic Curve

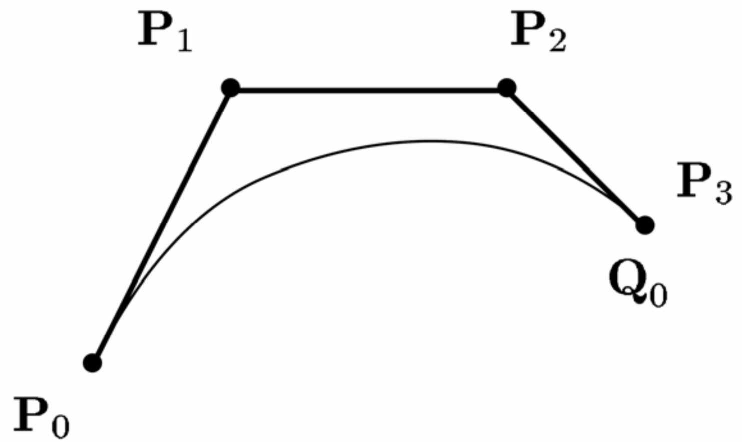
- A Matrix Equation for a Cubic Curve
- Reparameterization using the Matrix Form
- A Specific Example
- An Expanded Example

A Matrix Equation for a Cubic Curve

- A cubic polynomial curve $\mathbf{P}(t)$ can be written as a cubic Bézier curve
- Let $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ be the control points of the curve $\mathbf{P}(t)$

$$\mathbf{P}(t) = \sum_{i=0}^{3} \mathbf{P}_i B_{i,3}(t)$$

its matrix form is



Cubic Bézier curve and its control polygon

A Matrix Equation for a Cubic Curve

$$\mathbf{P}(t) = \sum_{i=0}^{3} \mathbf{P}_i B_i(t)$$

$$= (1-t)^3\mathbf{P}_0 + 3t(1-t)^2\mathbf{P}_1 + 3t^2(1-t)\mathbf{P}_2 + t^3\mathbf{P}_3$$

$$= \begin{bmatrix} (1-t)^3 & 3t(1-t)^2 & 3t^2(1-t) & t^3 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

A Matrix Equation for a Cubic Curve

Where

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Notes:

- The matrix M defines the blending functions for the curve $\mathbf{P}(t)$ - i.e. the cubic Bernstein polynomials.
- There are three equations here, one for each of the x, y and z components of $\mathbf{P}(t)$.

Reparameterization using the Matrix Form

- Let $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ be the control points of the curve $\mathbf{P}(t)$
  - In general, the used parametric interval is $[0,1]$
  - $\mathbf{P}_0=\mathbf{P}(0)$, $\mathbf{P}_1=\mathbf{P}(1)$
- Given an interval $[a,b]$, there exists a unique control polygon $\{\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3\}$ defining a Bézier curve $\mathbf{Q}(t)$, such that $\mathbf{Q}(0)=\mathbf{Q}_0=\mathbf{P}(a)$ and $\mathbf{Q}(1)=\mathbf{Q}_1=\mathbf{P}(b)$

Reparameterization using the Matrix Form

- Purpose: finding the Bézier polygon for the portion of the curve $\mathbf{P}(t)$ where $t \in [a,b]$
- Solution: by reparameterization and by manipulating the matrix representation above

Reparameterization using the Matrix Form

- Defining the new curve as $\mathbf{Q}(t)$, then

$$\mathbf{Q}(t)=\mathbf{P}((b-a)t+a)$$

  - Both $\mathbf{Q}(t)$ and $\mathbf{P}(t)$ are cubic curves, *and represent the same curve.*
  - The difference of $\mathbf{Q}(t)$ and $\mathbf{P}(t)$ is their *parametric domain*
    - $\mathbf{P}(t)$: $t \in [0,1]$
    - $\mathbf{Q}_{[0,1]}(t)=\mathbf{P}_{[a,b]}(t)$: $t \in [a,b]$

Reparameterization using the Matrix Form

$$\mathbf{Q}(t) = \mathbf{P}((b-a)t+a)$$

$$= \begin{bmatrix} 1 & (b-a)t+a & ((b-a)t+a)^2 & ((b-a)t+a)^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} \quad C \quad \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

where the matrix $[C]$ has columns whose entries are the coefficients of $1, t, t^2$ and t^3 respectively in the polynomials $1, (b-a)t+a, ((b-a)t+a)^2$, and $((b-a)t+a)^3$, respectively

Reparameterization using the Matrix Form

- $\mathbf{Q}(t)$ can be written as

$$\begin{aligned}\mathbf{Q}(t) &= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} CM\mathbf{P} \\ &= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M\left(S_{[a,b]}\mathbf{P}\right)\end{aligned}$$

where $S_{[a,b]}$ is equal to $S_{[a,b]} = M^{-1}CM$

- The new control points for the portion of the curve where t ranges from a to b can now be written as ($S_{[a,b]}\mathbf{P}$)

A Specific Example

- $\mathbf{P}(t)$: parameter ranges from 1 to 2
  - It is natural extension of $\mathbf{P}(t)$ from $[0,1]$ to $[1,2]$
  - It is useful to learn how to piece together two Bézier curves: The general B-spline curves are piecewise Bézier curves which are smoothly joined.

Matrix Representation of P(*t*) on [1,2]

$$\mathbf{Q}(t) = \mathbf{P}(t+1)$$

$$= \begin{bmatrix} 1 & (t+1) & (t+1)^2 & (t+1)^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M \left(S_{[1,2]} \mathbf{P} \right)$$

Matrix Representation of P(*t*) on [1,2]

where

$$
\begin{aligned}
S_{[1,2]} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & 0 & 0 \\ 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{4}{3} & \frac{5}{3} & 2 \\ 1 & \frac{5}{3} & \frac{8}{3} & 4 \\ 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -4 & 4 \\ -1 & 6 & -12 & 8 \end{bmatrix}
\end{aligned}
$$

Matrix Representation of $\mathbf{P}(t)$ on [1,2]

- The control polygon for that portion of $\mathbf{P}(t)$ curve where t ranges from 1 to 2 is:

$$\begin{bmatrix} \mathbf{Q}_0 \\ \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \mathbf{Q}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -4 & 4 \\ -1 & 6 & -12 & 8 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{P}_3 \\ -\mathbf{P}_2 + 2\mathbf{P}_3 \\ \mathbf{P}_1 - 4\mathbf{P}_2 + 4\mathbf{P}_3 \\ -\mathbf{P}_0 + 6\mathbf{P}_1 - 12\mathbf{P}_2 + 8\mathbf{P}_3 \end{bmatrix}$$

Geometric Interpretation of the New Control Points $\{Q_0, Q_1, Q_2, Q_3\}$

Defining new temporary points $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$

$\{\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3\}$ can be calculated by a simple geometric process using only the initial control polygon

$\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$

$$\mathbf{Q}_0 = \mathbf{P}_3$$

$$\mathbf{Q}_1 = \mathbf{P}_3 + (\mathbf{P}_3 - \mathbf{P}_2) \tag{2}$$

$$\mathbf{R}_1 = \mathbf{P}_1 + (\mathbf{P}_1 - \mathbf{P}_0) \tag{3}$$

$$\mathbf{R}_2 = \mathbf{P}_2 + (\mathbf{P}_2 - \mathbf{P}_1) \tag{4}$$

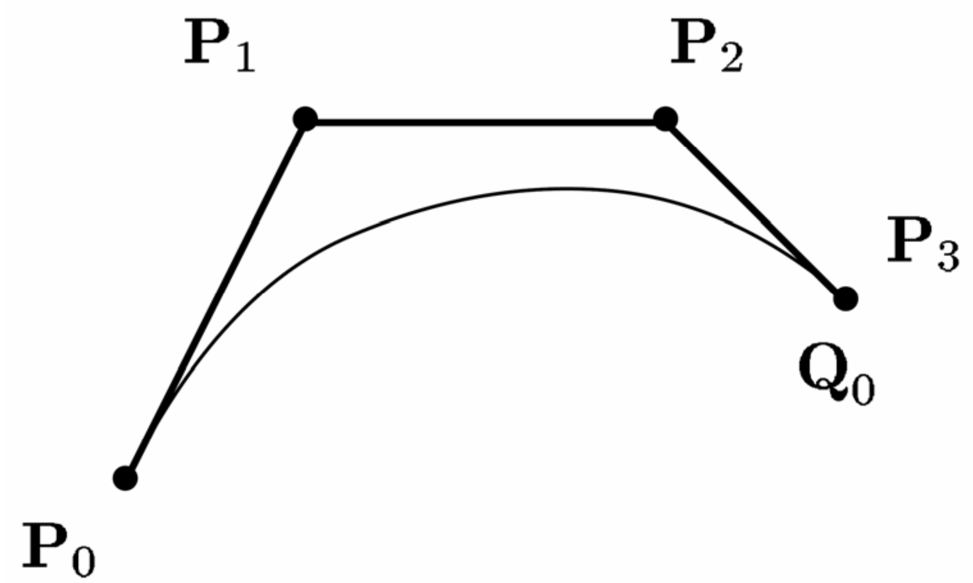
$$\begin{aligned}\mathbf{R}_3 &= \mathbf{R}_2 + (\mathbf{R}_2 - \mathbf{R}_1)\\ &= \mathbf{P}_2 + (\mathbf{P}_2 - \mathbf{P}_1) + (\mathbf{P}_2 + (\mathbf{P}_2 - \mathbf{P}_1) - \mathbf{P}_1 + (\mathbf{P}_1 - \mathbf{P}_0))\\ &= \mathbf{P}_0 - 4\mathbf{P}_1 + 4\mathbf{P}_2 \end{aligned} \tag{5}$$

$$\begin{aligned}\mathbf{Q}_2 &= \mathbf{Q}_1 + (\mathbf{Q}_1 - \mathbf{R}_2)\\ &= \mathbf{P}_3 + (\mathbf{P}_3 - \mathbf{P}_2) + (\mathbf{P}_3 + (\mathbf{P}_3 - \mathbf{P}_2) - \mathbf{P}_2 + (\mathbf{P}_2 - \mathbf{P}_1))\\ &= \mathbf{P}_1 - 4\mathbf{P}_2 + 4\mathbf{P}_3 \end{aligned} \tag{6}$$

$$\begin{aligned}\mathbf{Q}_3 &= \mathbf{Q}_2 + (\mathbf{Q}_2 - \mathbf{R}_3)\\ &= \mathbf{P}_1 - 4\mathbf{P}_2 + 4\mathbf{P}_3 + (\mathbf{P}_1 - 4\mathbf{P}_2 + 4\mathbf{P}_3 - \mathbf{P}_0 - 4\mathbf{P}_1 + 4\mathbf{P}_2)\\ &= \mathbf{P}_0 + 6\mathbf{P}_1 - 12\mathbf{P}_2 + 8\mathbf{P}_3 \end{aligned} \tag{7}$$

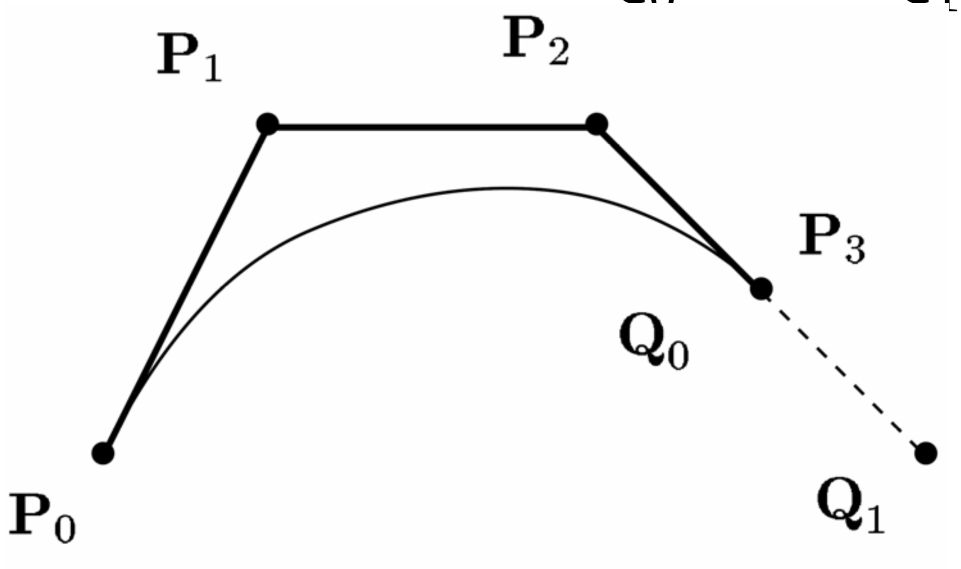
Geometric Interpretation of the Q_0 (1)

- $\mathbf{Q}_0 = \mathbf{P}_0$



Geometric Interpretation of the Q_1 (2)

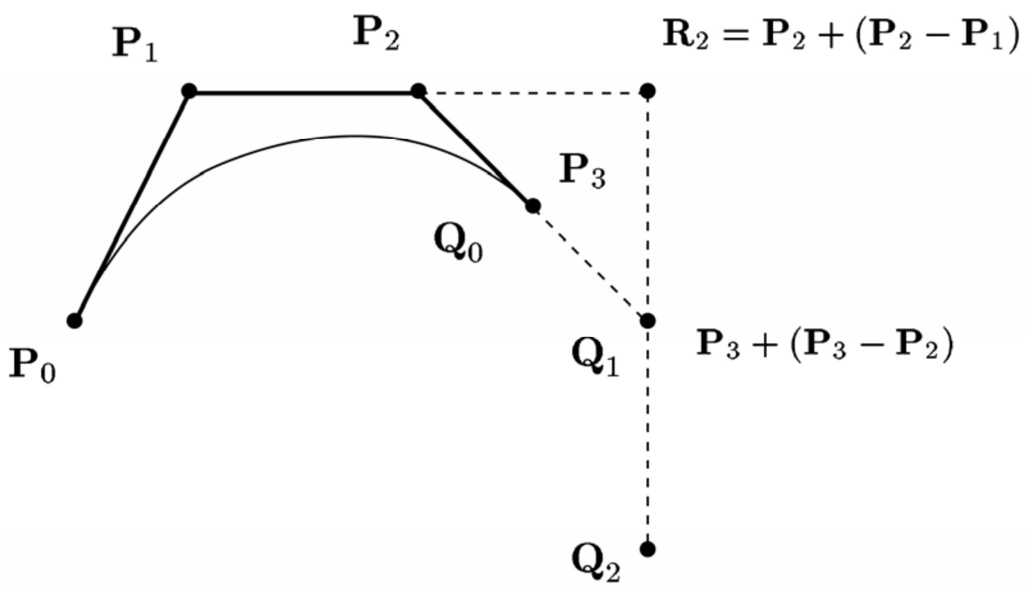
- $\mathbf{Q}_1 = \mathbf{P}_3 + (\mathbf{P}_3 - \mathbf{P}_2)$ lies on the line $\overline{\mathbf{P}_2\mathbf{P}_3}$, where the distance between $\mathbf{P}_2$ and $\mathbf{P}_3$ is equal to that between $\mathbf{Q}_0$ and $\mathbf{Q}_1$.



Geometric Interpretation of the Q_1 (3)

$$\mathbf{R}_2 = \mathbf{P}_2 + (\mathbf{P}_2 - \mathbf{P}_1)$$

$$\begin{aligned}\mathbf{Q}_2 &= \mathbf{Q}_1 + (\mathbf{Q}_1 - \mathbf{R}_2) \\ &= \mathbf{P}_3 + (\mathbf{P}_3 - \mathbf{P}_2) + (\mathbf{P}_3 + (\mathbf{P}_3 - \mathbf{P}_2) - \mathbf{P}_2 + (\mathbf{P}_2 - \mathbf{P}_1)) \\ &= \mathbf{P}_1 - 4\mathbf{P}_2 + 4\mathbf{P}_3\end{aligned}$$



Geometric Interpretation of the Q_1 (4)

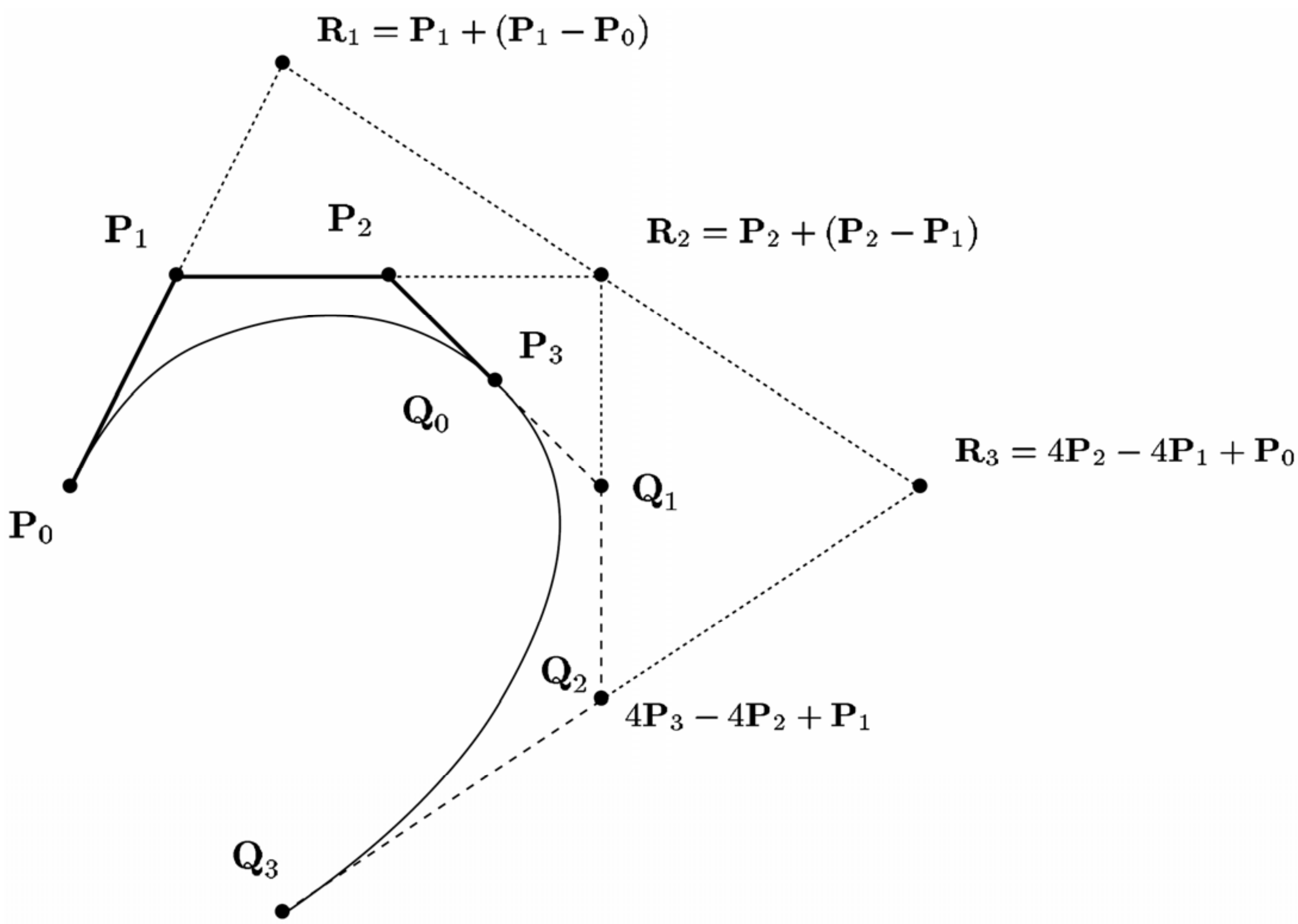
$$\mathbf{R}_1 = \mathbf{P}_1 + (\mathbf{P}_1 - \mathbf{P}_0)$$

$$\mathbf{R}_2 = \mathbf{P}_2 + (\mathbf{P}_2 - \mathbf{P}_1)$$

$$\begin{aligned}
\mathbf{R}_3 &= \mathbf{R}_2 + (\mathbf{R}_2 - \mathbf{R}_1) \\
&= \mathbf{P}_2 + (\mathbf{P}_2 - \mathbf{P}_1) + (\mathbf{P}_2 + (\mathbf{P}_2 - \mathbf{P}_1) - \mathbf{P}_1 + (\mathbf{P}_1 - \mathbf{P}_0)) \\
&= \mathbf{P}_0 - 4\mathbf{P}_1 + 4\mathbf{P}_2
\end{aligned}$$

$$\begin{aligned}
\mathbf{Q}_3 &= \mathbf{Q}_2 + (\mathbf{Q}_2 - \mathbf{R}_3) \\
&= \mathbf{P}_1 - 4\mathbf{P}_2 + 4\mathbf{P}_3 + (\mathbf{P}_1 - 4\mathbf{P}_2 + 4\mathbf{P}_3 - \mathbf{P}_0 - 4\mathbf{P}_1 + 4\mathbf{P}_2) \\
&= \mathbf{P}_0 + 6\mathbf{P}_1 - 12\mathbf{P}_2 + 8\mathbf{P}_3
\end{aligned}$$

Geometric Interpretation of the Q_1 (4)



A Specific Example: Results

- The above geometric construction is the inverse process of the de Casteljau geometric construction.
- These two functions represent the *same curve*.
- Exercise: constructing the control points of $\mathbf{Q}(t)=\mathbf{P}(t)$, $t\in[0,2]$.

Tips: the result control points $\{\mathbf{P}_0, \mathbf{R}_1, \mathbf{R}_2, \mathbf{Q}_3\}$

An Expanded Example

- The example above
  - illustrated there are many Bézier polygons that can represent a cubic curve
  - did not quite illustrate the necessary characteristics of the algorithm
- Considering the cubic curve $\mathbf{P}(t)$ when $t \in [1,b]$
  - $\mathbf{Q}(t)=\mathbf{P}(at+1)$ where $a=b-1$

Matrix Representation of P(t) on [1, b]

$$
\begin{aligned}
\mathbf{Q}(t) &= \mathbf{P}(at+1) \\
&= \begin{bmatrix} 1 & (at+1) & (at+1)^2 & (at+1)^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & a & 2a & 3a \\ 0 & 0 & a^2 & 3a^2 \\ 0 & 0 & 0 & a^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M \left(S_{[1,b]} \mathbf{P} \right)
\end{aligned}
$$

Matrix Representation of P(*t*) on [1, *b*]

where

$$S_{[1,b]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & a & 2a & 3a \\ 0 & 0 & a^2 & 3a^2 \\ 0 & 0 & 0 & a^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & 0 & 0 \\ 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & a & 2a & 3a \\ 0 & 0 & a^2 & 3a^2 \\ 0 & 0 & 0 & a^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -a & (a+1) \\ 0 & a^2 & -2a(a+1) & (a+1)^2 \\ -a^3 & 3a^2(a+1) & -3a(a+1)^2 & (a+1)^3 \end{bmatrix}$$

Matrix Representation of P(*t*) on [1, *b*]

The control polygon of the curve $\mathbf{P}(t)$ where $t \in [1, b]$

$$
\begin{bmatrix} \mathbf{Q}_0 \\ \mathbf{Q}_1 \\ \mathbf{Q}_2 \\ \mathbf{Q}_3 \end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -a & (a+1) \\
0 & a^2 & -2a(a+1) & (a+1)^2 \\
-a^3 & 3a^2(a+1) & -3a(a+1)^2 & (a+1)^3
\end{bmatrix}
\begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}
$$

$$
=
\begin{bmatrix}
\mathbf{P}_3 \\
-a\mathbf{P}_2 + (a+1)\mathbf{P}_3 \\
a^2\mathbf{P}_1 - 2a(a+1)\mathbf{P}_2 + (a+1)^2\mathbf{P}_3 \\
-a^3\mathbf{P}_0 + 3a^2(a+1)\mathbf{P}_1 - 3a(a+1)^2\mathbf{P}_2 + (a+1)^3\mathbf{P}_3
\end{bmatrix}
$$

Geometric Interpretation of the New Control Points $\{Q_0, Q_1, Q_2, Q_3\}$

Defining new temporary points $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$

$\{\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3\}$ can be calculated by a simple geometric process using only the initial control polygon

$\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$

$$\mathbf{Q}_0 = \mathbf{P}_3$$

$$\mathbf{Q}_1 = \mathbf{P}_3 + a(\mathbf{P}_3 - \mathbf{P}_2)$$

$$\mathbf{R}_1 = \mathbf{P}_1 + a(\mathbf{P}_1 - \mathbf{P}_0)$$

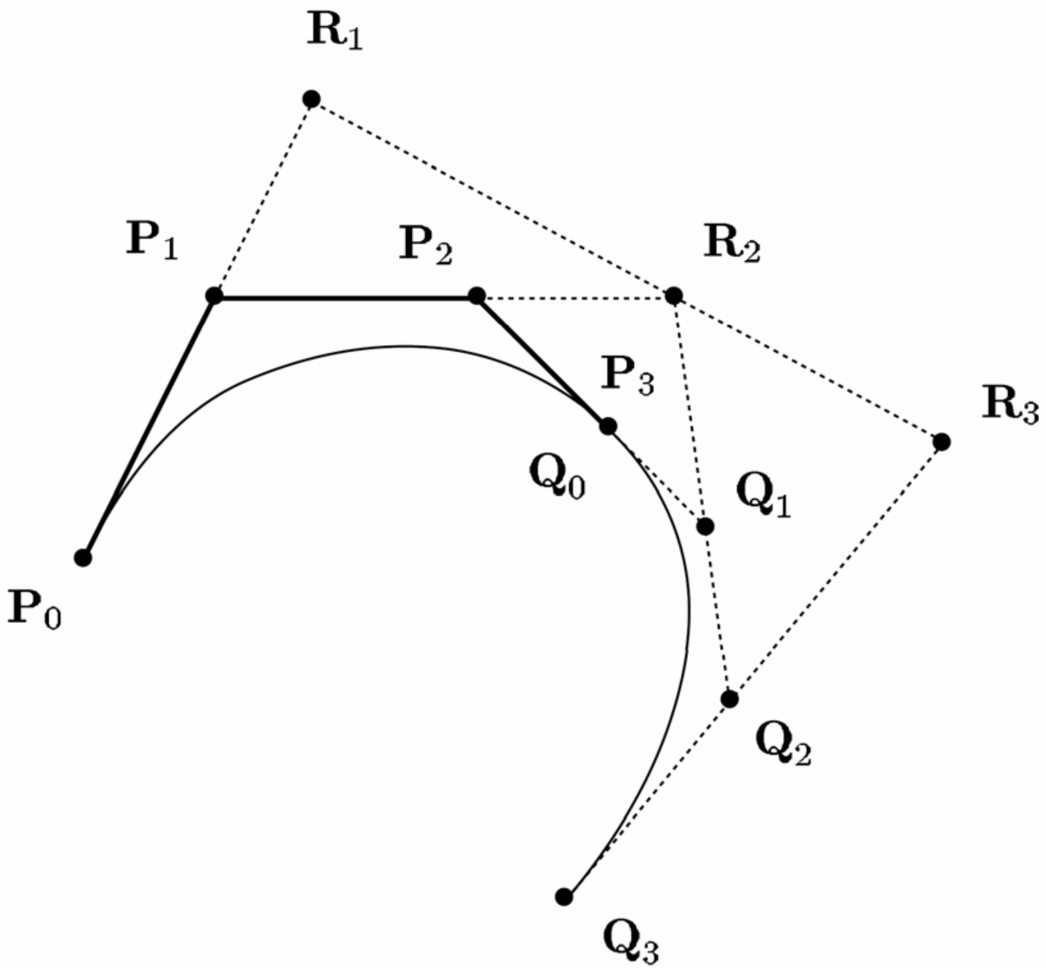
$$\mathbf{R}_2 = \mathbf{P}_2 + a(\mathbf{P}_2 - \mathbf{P}_1)$$

$$\begin{aligned}\mathbf{R}_3 &= \mathbf{R}_2 + a(\mathbf{R}_2 - \mathbf{R}_1)\\ &= \mathbf{P}_2 + a(\mathbf{P}_2 - \mathbf{P}_1) + a(\mathbf{P}_2 + a(\mathbf{P}_2 - \mathbf{P}_1) - \mathbf{P}_1 + a(\mathbf{P}_1 - \mathbf{P}_0))\\ &= a^2\mathbf{P}_0 - 2a(a+1)\mathbf{P}_1 + (a+1)^2\mathbf{P}_2\end{aligned}$$

$$\begin{aligned}\mathbf{Q}_2 &= \mathbf{Q}_1 + a(\mathbf{Q}_1 - \mathbf{R}_2)\\ &= \mathbf{P}_3 + a(\mathbf{P}_3 - \mathbf{P}_2) + a(\mathbf{P}_3 + a(\mathbf{P}_3 - \mathbf{P}_2) - \mathbf{P}_2 + a(\mathbf{P}_2 - \mathbf{P}_1))\\ &= a^2\mathbf{P}_1 - 2a(a+1)\mathbf{P}_2 + (a+1)^2\mathbf{P}_3\end{aligned}$$

$$\begin{aligned}\mathbf{Q}_3 &= \mathbf{Q}_2 + a(\mathbf{Q}_2 - \mathbf{R}_3)\\ &= -a^3\mathbf{P}_0 + 3a^2(a+1)\mathbf{P}_1 - 3a(a+1)^2\mathbf{P}_2 + (a+1)^3\mathbf{P}_3\end{aligned}$$

Geometric Interpretation of the New Control Points $\{Q_0, Q_1, Q_2, Q_3\}$



Geometric Interpretation of the New Control Points $\{Q_0, Q_1, Q_2, Q_3\}$

- Results
  - The important factor here is the a term
  - Each of these points is on an extension of a line of the original control polygon, or the extension of a constructed line
  - The factor a determines how much to extend.

The Equations for a Bézier Curve of Arbitrary Degree

- Overview
  - The Bézier curve representation is one that is utilized most frequently in computer graphics and geometric modeling.
  - The curve is defined geometrically, which means that the parameters have geometric meaning - they are just points in three-dimensional space.
  - It was developed by two competing European engineers in the late 1960s to attempt to draw automotive components.

The Equations for a Bézier Curve of Arbitrary Degree

- Specification of the Bézier Curve of Arbitrary Degree
  - Generalizing the development for the quadratic and cubic Bézier curves
  - Given the set of control points, $\{\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n\}$, defining a Bézier curve of degree n by either *Analytic Definition* or *Geometric Construction*.

The Analytic Definition

$$\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P}_i B_{i,n}(t)$$

where

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

are the Bernstein polynomials of degree n , and t ranges between zero and one $0 \leqslant t \leqslant 1$.

Geometric Definition

$$\mathbf{P}(t) = \mathbf{P}_n^{(n)}(t)$$

where

$$\mathbf{P}_i^{(j)}(t) = \begin{cases} (1-t)\mathbf{P}_{i-1}^{(j-1)}(t) + t\mathbf{P}_i^{(j-1)}(t) & \text{if } j > 0, \\ \mathbf{P}_i & \text{otherwise} \end{cases}$$

where t ranges between zero and one $0 \leqslant t \leqslant 1$

Properties of the Bézier Curve

- $\mathbf{P}_0$ and $\mathbf{P}_n$ are on the curve.
- The curve is continuous and has continuous derivatives of all orders.
- The tangent line to the curve at the point $\mathbf{P}_0$ is the line $\mathbf{P}_0\mathbf{P}_1$. The tangent to the curve at the point $\mathbf{P}_n$ is the line $\mathbf{P}_{n\text{-}1}\mathbf{P}_n$.
- The curve lies within the convex hull of its control points. This is because each successive $\mathbf{P}_i^{(j)}$ is a convex combination of the points $\mathbf{P}_i^{(j-1)}$ and $\mathbf{P}_{i-1}^{(j-1)}$.
- $\mathbf{P}_0,\mathbf{P}_1,\ldots,\mathbf{P}_n$ are all on the curve only if the curve is linear.

Summary of the Bézier Curve

- Given a sequence of $n+1$ control points, one can specify a Bézier curve of degree n defined by these points.
- Two definitions of the curve can be given:
  - An analytic definition specifying the blending of the control points with Bernstein polynomials
  - A geometric definition specifying a recursive generation procedure that calculates successive points on line segments developed from the control point sequence.

Bézier Patches

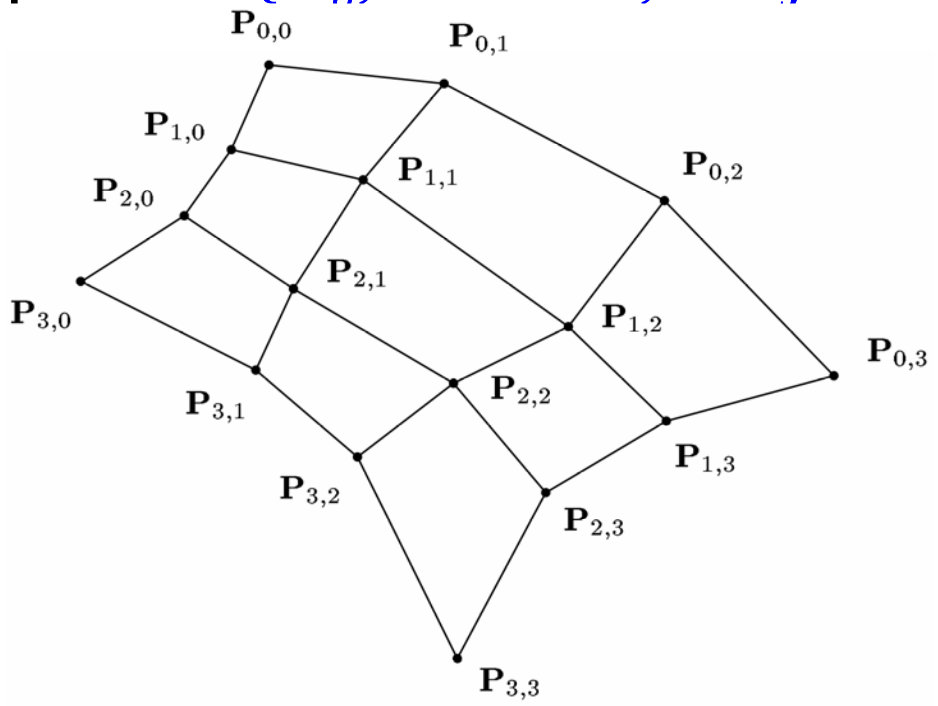
- Overview
  - Pierre Bézier in Renault and Paul de Casteljau in Citroën, initially developed a Bézier curve representation and extended it to a surface patch methodology
    - The extension of Bézier curves to surfaces is called the Bézier patch
  - The Bézier patch is the most commonly used surface representation in computer graphics

Bézier Patches

- Bézier curve and patch
  - The Bézier curve is a function of one variable and takes a sequence of control points.
  - The Bézier patch is a function of two variables with an array of control points.
  - Most of the methods for the patch are direct extensions of those for the curves.

Definition of the Bézier Patch

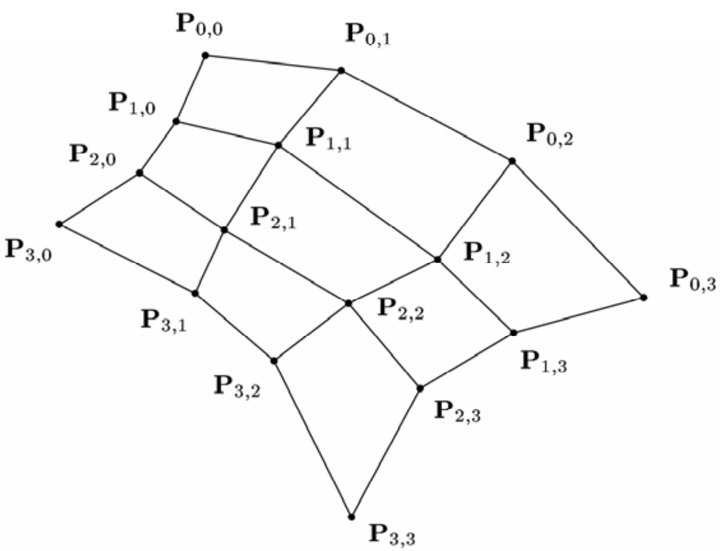
- The patch is constructed from an $n \times m$ array of control points: $\{\mathbf{P}_{ij},\ 0 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant m\}$

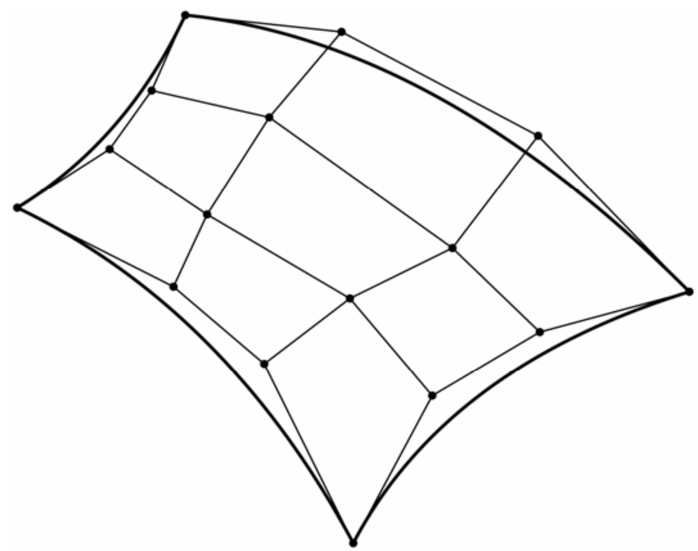


Definition of the Bézier Patch

- The Bézier patch is parameterized by two variables, is given by the equation

$$\mathbf{P}(u,v) = \sum_{j=0}^{m}\sum_{i=0}^{n}\mathbf{P}_{i,j}B_{i,n}(u)B_{j,m}(v)$$





Definition of the Bézier Patch

- It is summations running over all the control points
- The *bi-variate* Bernstein Polynomials serving as the functions that blend the control points together

$$B_{i,n}(u)B_{j,m}(v)$$

Deductions from Definition of the Bézier Patch

- By set v=0, we obtain

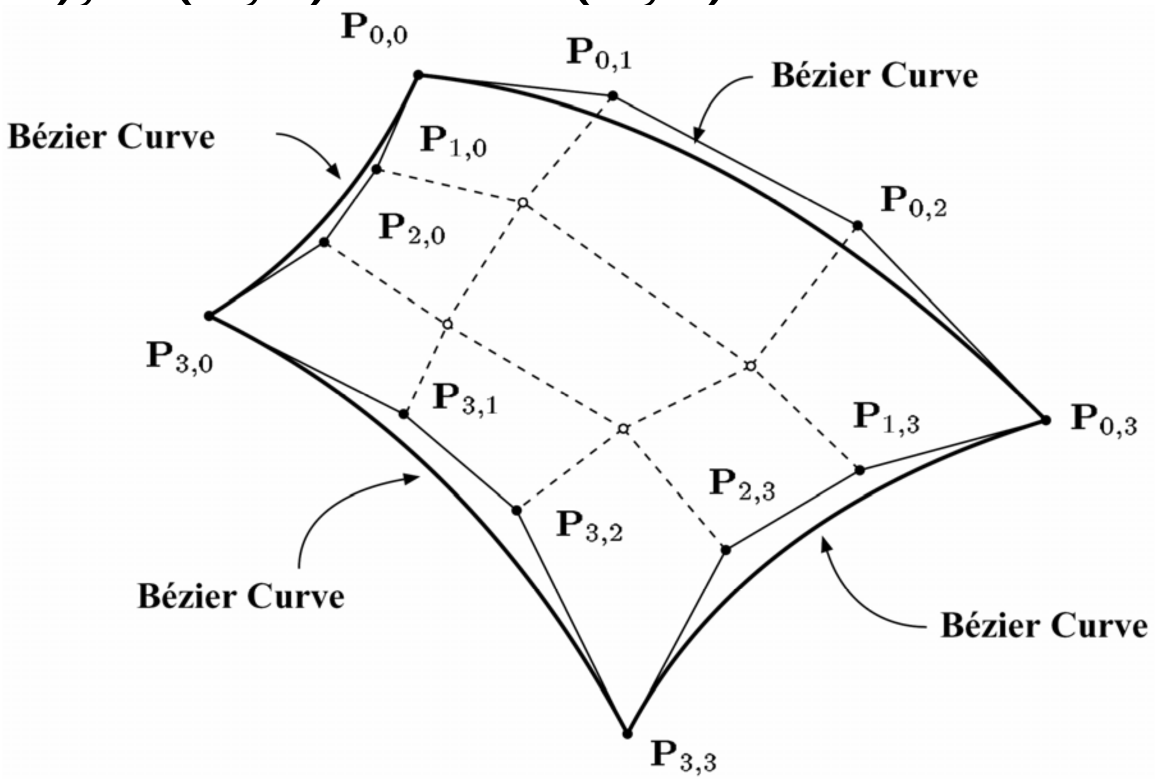
$$\mathbf{P}(u,0) = \sum_{j=0}^{m}\sum_{i=0}^{n}\mathbf{P}_{i,j}B_{i,n}(u)B_{j,m}(0) = \sum_{i=0}^{n}\mathbf{P}_{i,0}B_{i,n}(u)$$

since $B_{0m}(0)=1$ and $B_{jm}(0)=0$ for $j=1,2,\ldots,m$

Result: $\mathbf{P}(u,0)$ is a Bézier Curve

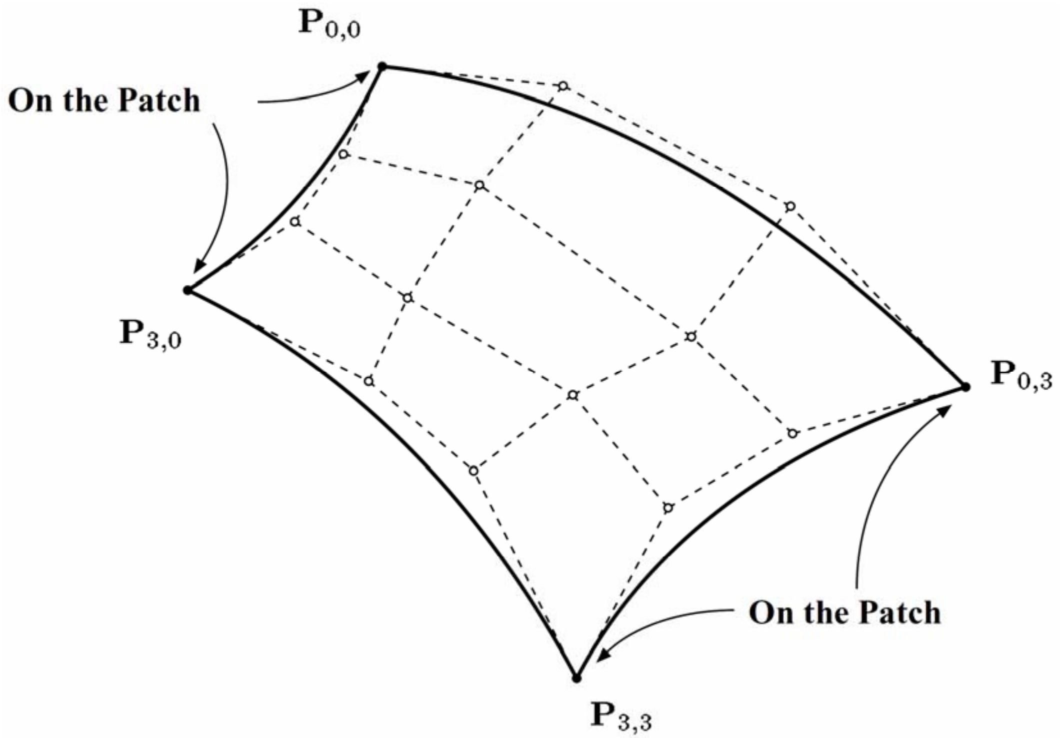
Deductions from Definition of the Bézier Patch

- $\mathbf{P}(u,1)$, $\mathbf{P}(1,v)$ and $\mathbf{P}(0,v)$ are Bézier Curve



Relations between the Bézier Curve and Patch

- The corner ones of control points are actually on the patch



Properties of the Bézier Patch

- The four points $\mathbf{P}_{0,0}$, $\mathbf{P}_{0,m}$, $\mathbf{P}_{n,0}$ and $\mathbf{P}_{n,m}$ are on the patch. The other control points are all on the patch only if the patch is planar.
- The patch is continuous and partial derivatives of all orders exist and are continuous.
- The patch lies within the convex hull of its control points.

Bézier Curves on Bézier Patches

- Overview
  - $\mathbf{P}(0,v)$ and $\mathbf{P}(1,v)$ are Bézier curves lying on the boundary of the Bézier patch.
  - A Bézier patch can be treated as a continuous set of Bézier curves. That is, for any fixed parameter u_0 or v_0 we can define a Bézier curve that lies directly on the surface of the patch.
  - It is a very valuable tool for calculations on the patch

Bézier Curves on Bézier Patches

Grouping factors of the Bézier patch function appropriately

$$\mathbf{P}(u,v) = \sum_{j=0}^{m} \left[\sum_{i=0}^{n} \mathbf{P}_{i,j} B_{i,n}(u) \right] B_{j,m}(v)$$

If we fix $u=u_0$, the internal sum can be calculated (for $j=0, \ldots, m$). This implies that $\mathbf{P}(u_0,v)$ is a Bézier curve on the surface.

Bézier Curves on Bézier Patches

If we define $\mathbf{Q}_j(u)$ to be the value

$$\mathbf{Q}_j(u) = \sum_{i=0}^{n} \mathbf{P}_{i,j} B_{i,n}(u)$$

we can see that

$$\mathbf{P}(u,v) = \sum_{j=0}^{m} \mathbf{Q}_j(u) B_{j,m}(v)$$

That is, the quantities $\mathbf{Q}_j(u)$ form the control points of another Bézier curve, and together for all u and v, they form the surface.

Bézier Curves on Bézier Patches

Therefore, given $u=u_0$, we can calculate the quantities $\mathbf{Q}_0(u_0)$, $\mathbf{Q}_1(u_0)$, ..., $\mathbf{Q}_m(u_0)$, giving m control points to utilize for the curve

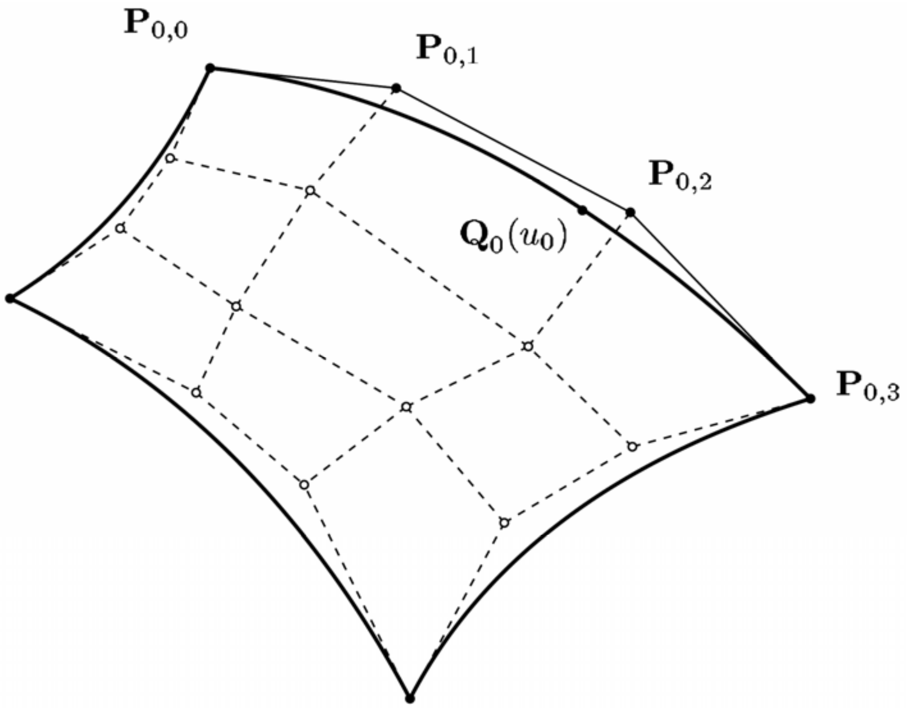
$$\mathbf{Q}(v) = \sum_{j=0}^{m} \mathbf{Q}_j(u_0) B_{j,m}(v)$$

- This curve lies on the patch: $\mathbf{P}(u_0,v)=\mathbf{Q}(v)$,
- $\mathbf{Q}(v_0)$ is the point on the patch at (u_0,v_0).

Result: Calculating a point on the patch can be reduced to finding several points on curves which is parameter independent

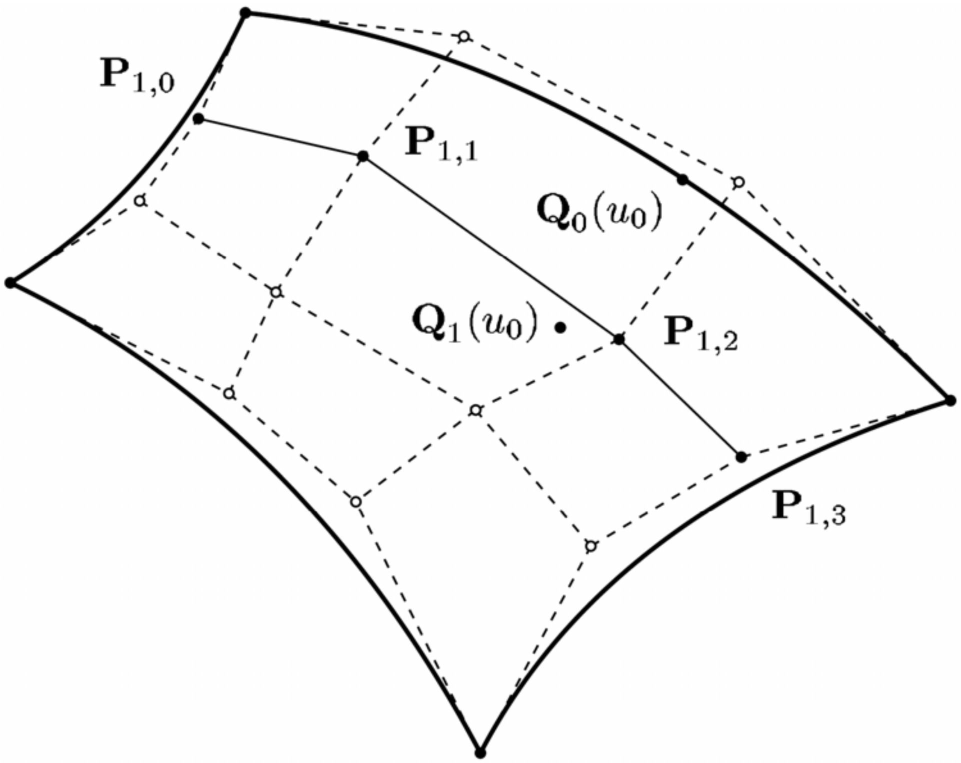
Calculating a Point on a Bi-Cubic Surface: STEP 1

The point $\mathbf{Q}_0(u_0)$, is calculated as a point on the Bézier curve defined by the control points $\mathbf{P}_{0,0}$, $\mathbf{P}_{0,1}$, $\mathbf{P}_{0,2}$ and $\mathbf{P}_{0,3}$.



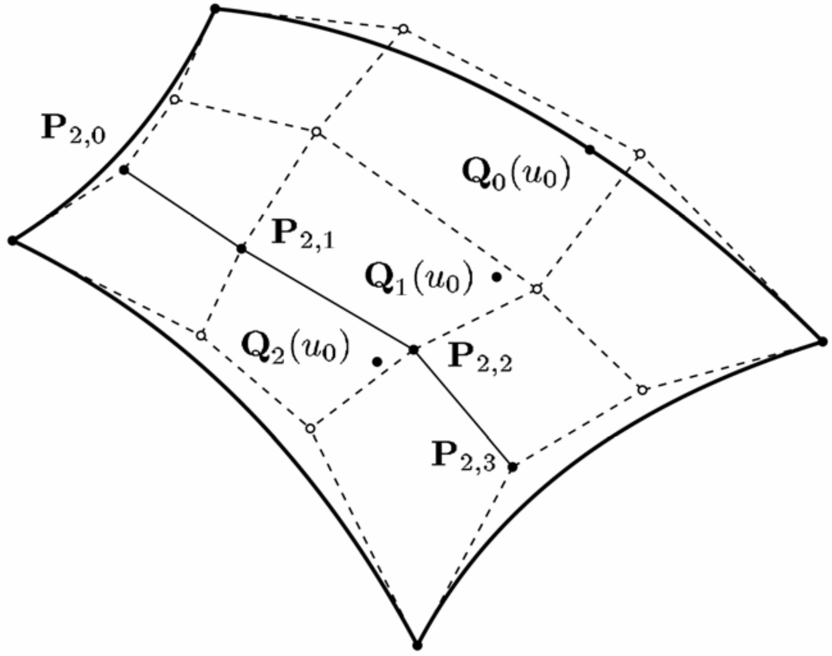
Calculating a Point on a Bi-Cubic Surface: STEP 2

The point $\mathbf{Q}_1(u_0)$, is calculated as a point on the Bézier curve defined by the control points $\mathbf{P}_{1,0}$, $\mathbf{P}_{1,1}$, $\mathbf{P}_{1,2}$ and $\mathbf{P}_{1,3}$



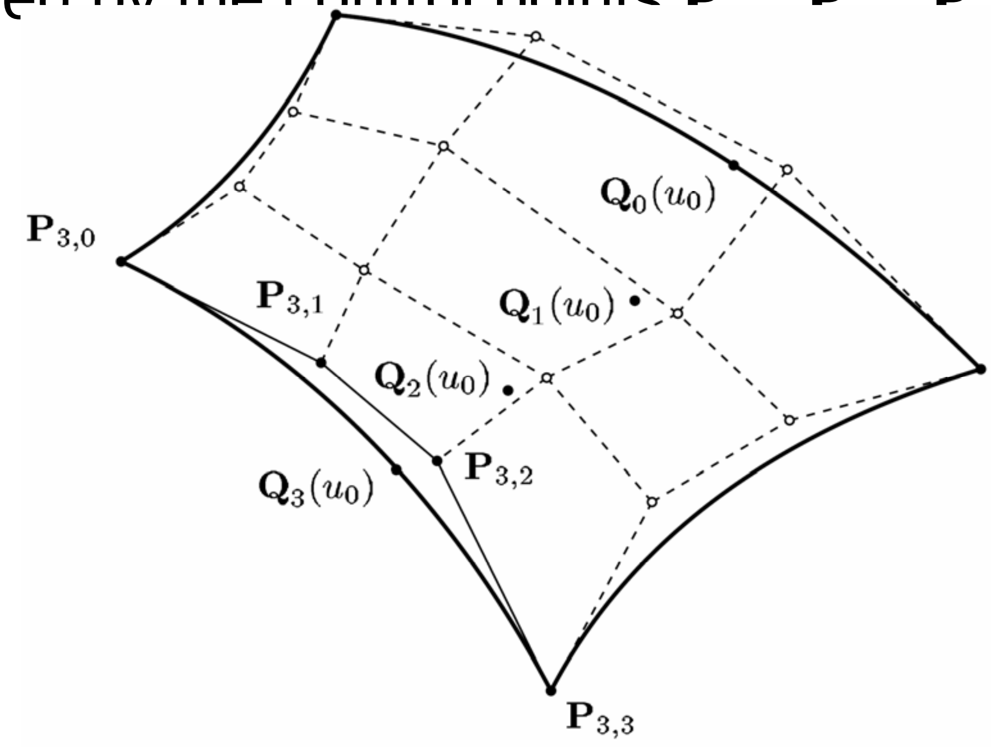
Calculating a Point on a Bi-Cubic Surface: STEP 3

The point $\mathbf{Q}_2(u_0)$, is calculated as a point on the Bézier curve defined by the control points $\mathbf{P}_{2,0}$, $\mathbf{P}_{2,1}$, $\mathbf{P}_{2,2}$ and $\mathbf{P}_{2,3}$.



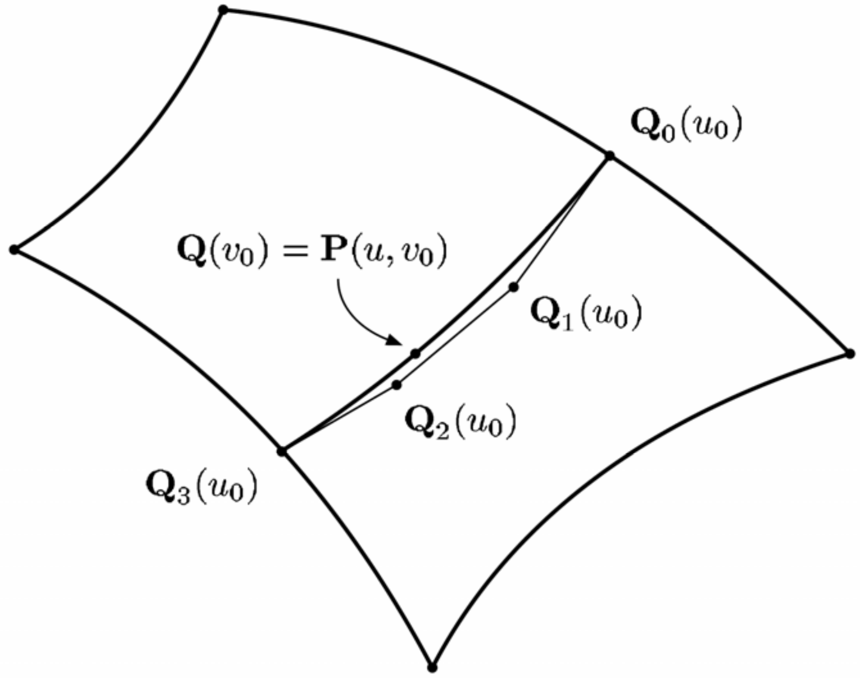
Calculating a Point on a Bi-Cubic Surface:STEP 4

The point $\mathbf{Q}_3(u_0)$, is calculated as a point on the Bézier curve defined by the control points $\mathbf{P}_{3,0}$, $\mathbf{P}_{3,1}$, $\mathbf{P}_{3,2}$ and $\mathbf{P}_{3,3}$.



Calculating a Point on a Bi-Cubic Surface: STEP5

The point $\mathbf{P}(u_0,v_0)$, on the patch, is calculated as a point on the Bézier curve defined by the control points $\mathbf{Q}_0(u_0)$, $\mathbf{Q}_1(u_0)$, $\mathbf{Q}_2(u_0)$, $\mathbf{Q}_3(u_0)$.



Subdivision of Bézier Patches

If we take the analytic equation of a Bézier patch, fix u and group factors appropriately, we obtain

$$\mathbf{P}(u,v) = \sum_{j=0}^{m} \left[\sum_{i=0}^{n} \mathbf{P}_{i,j} B_{i,n}(u) \right] B_{j,m}(v)$$

We notice that portion of the equation inside the brackets is the representation of a Bézier curve.

Subdivision of Bézier Patches

If we let $\mathbf{Q}_j(u)$ be the value inside the brackets,

$$\mathbf{Q}_j(u) = \sum_{i=0}^{n} \mathbf{P}_{i,j} B_{i,n}(u)$$

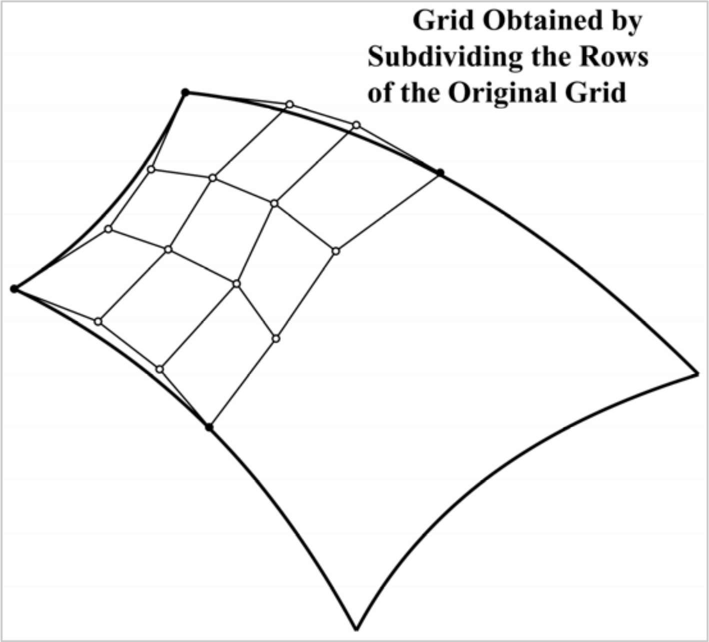
Then

$$\mathbf{P}(u,v) = \sum_{j=0}^{m} \mathbf{Q}_j(u) B_{j,m}(v)$$

That is, the quantities $\mathbf{Q}_j(u)$ form the control points of another Bézier curve, and together for all u and v, they form the surface

Subdivision of Bézier Patches

- If we subdivide each of the *m* rows of the $\mathbf{P}_{i,j}$ matrix, it implies that the $\mathbf{Q}_j$s in the above equation represent only points from the first half of the patch (with respect to u)
- The second half of the patch can be obtained in a similar fashion.
- The first and second half of the patch, with respect to v, can be obtained by subdividing the columns.



The above illustration shows the result of subdividing the rows in the 4×4 case.

A Matrix Representation of the Cubic Bézier Patch

Overview

- The matrix representation of the cubic Bézier patch allows us to specify many operations with Bézier patches
- The matrix operations can be performed quickly on computer systems optimized for geometry operations with matrices

Developing the Matrix Formulation

A cubic Bézier curve can be written in a matrix form similar to that for a Bézier Curve by utilizing the representation of a Bézier patch as a continuous set of Bézier curves

$$
\begin{aligned}
\mathbf{P}(u,v) &= \sum_{j=0}^{3}\sum_{i=0}^{3}\mathbf{P}_{i,j}B_{i,3}(u)B_{j,3}(v) \\
&= \sum_{j=0}^{3}\left[\sum_{i=0}^{3}\mathbf{P}_{i,j}B_{i,3}(u)\right]B_{j,3}(v) \\
&= \sum_{j=0}^{3}\begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{P}_{0,j} \\ \mathbf{P}_{1,j} \\ \mathbf{P}_{2,j} \\ \mathbf{P}_{3,j} \end{bmatrix} B_{j,3}(v) \\
&= \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} & \mathbf{P}_{0,3} \\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \mathbf{P}_{1,3} \\ \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \mathbf{P}_{2,3} \\ \mathbf{P}_{3,0} & \mathbf{P}_{3,1} & \mathbf{P}_{3,2} & \mathbf{P}_{3,3} \end{bmatrix}\begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}
\end{aligned}
$$

Developing the Matrix Formulation

The cubic Bézier patch is frequently written

$$\mathbf{P}(u,v) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} M \begin{bmatrix} \mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} & \mathbf{P}_{0,3} \\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \mathbf{P}_{1,3} \\ \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \mathbf{P}_{2,3} \\ \mathbf{P}_{3,0} & \mathbf{P}_{3,1} & \mathbf{P}_{3,2} & \mathbf{P}_{3,3} \end{bmatrix} M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}$$

where

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Patch Subdivision Using the Matrix Form

- Purpose: subdividing the patch at the point $u=1/2$
- Method: reparameterizing the matrix equation above (by substituting $u/2$ for u) to cover only the first half of the patch, and simplify to obtain.

Patch Subdivision Using the Matrix Form

$$\mathbf{P}(\frac{u}{2}, v) = \begin{bmatrix} 1 & (\frac{u}{2}) & (\frac{u}{2})^2 & (\frac{u}{2})^3 \end{bmatrix} M \begin{bmatrix} \mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} & \mathbf{P}_{0,3} \\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \mathbf{P}_{1,3} \\ \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \mathbf{P}_{2,3} \\ \mathbf{P}_{3,0} & \mathbf{P}_{3,1} & \mathbf{P}_{3,2} & \mathbf{P}_{3,3} \end{bmatrix} M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} M \begin{bmatrix} \mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} & \mathbf{P}_{0,3} \\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \mathbf{P}_{1,3} \\ \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \mathbf{P}_{2,3} \\ \mathbf{P}_{3,0} & \mathbf{P}_{3,1} & \mathbf{P}_{3,2} & \mathbf{P}_{3,3} \end{bmatrix} M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} M S_L \begin{bmatrix} \mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} & \mathbf{P}_{0,3} \\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \mathbf{P}_{1,3} \\ \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \mathbf{P}_{2,3} \\ \mathbf{P}_{3,0} & \mathbf{P}_{3,1} & \mathbf{P}_{3,2} & \mathbf{P}_{3,3} \end{bmatrix} M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}$$

Patch Subdivision Using the Matrix Form

where the matrix S_L is defined as

$$S_L = M^{-1}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} M$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & 0 & 0 \\ 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{6} & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{12} & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}$$

Patch Subdivision Using the Matrix Form

The matrix S_L is identical to the left subdivision matrix for the curve case. So in particular, the subpatch $\mathbf{P}(u/2,v)$ is again a Bézier patch whose control points of this patch is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} & \mathbf{P}_{0,3} \\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \mathbf{P}_{1,3} \\ \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \mathbf{P}_{2,3} \\ \mathbf{P}_{3,0} & \mathbf{P}_{3,1} & \mathbf{P}_{3,2} & \mathbf{P}_{3,3} \end{bmatrix}$$

Calculation of the Second Half of the Patch

For the second half of the patch: First we reparameterize the original curve, and then simplify to obtain

$$\mathbf{P}\left(\frac{1}{2}+\frac{u}{2},v\right)=\left[\begin{array}{cccc}1 & \left(\frac{1}{2}+\frac{u}{2}\right) & \left(\frac{1}{2}+\frac{u}{2}\right)^2 & \left(\frac{1}{2}+\frac{u}{2}\right)^3\end{array}\right] M \left[\begin{array}{cccc}\mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} & \mathbf{P}_{0,3}\\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \mathbf{P}_{1,3}\\ \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \mathbf{P}_{2,3}\\ \mathbf{P}_{3,0} & \mathbf{P}_{3,1} & \mathbf{P}_{3,2} & \mathbf{P}_{3,3}\end{array}\right] M^T \left[\begin{array}{c}1\\ v\\ v^2\\ v^3\end{array}\right]$$

$$=\left[\begin{array}{cccc}1 & u & u^2 & u^3\end{array}\right]\left[\begin{array}{cccc}1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8}\\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{3}{8}\\ 0 & 0 & \frac{1}{4} & \frac{3}{8}\\ 0 & 0 & 0 & \frac{1}{8}\end{array}\right] M \left[\begin{array}{cccc}\mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} & \mathbf{P}_{0,3}\\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \mathbf{P}_{1,3}\\ \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \mathbf{P}_{2,3}\\ \mathbf{P}_{3,0} & \mathbf{P}_{3,1} & \mathbf{P}_{3,2} & \mathbf{P}_{3,3}\end{array}\right] M^T \left[\begin{array}{c}1\\ v\\ v^2\\ v^3\end{array}\right]$$

$$=\left[\begin{array}{cccc}1 & u & u^2 & u^3\end{array}\right] M S_R \left[\begin{array}{cccc}\mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} & \mathbf{P}_{0,3}\\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \mathbf{P}_{1,3}\\ \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \mathbf{P}_{2,3}\\ \mathbf{P}_{3,0} & \mathbf{P}_{3,1} & \mathbf{P}_{3,2} & \mathbf{P}_{3,3}\end{array}\right] M^T \left[\begin{array}{c}1\\ v\\ v^2\\ v^3\end{array}\right]$$

Calculation of the Second Half of the Patch

where

$$S_R = \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- S_R is identical to the right subdivision matrix in the curve case
- S_R can be applied to a set of control points to produce the control points for the second half of the patch

General Subdivision with either Parameter

- The matrix representation of control points for the first and second portions of the patch when subdivision is done with respect to v:

$$PS_L \text{ and } PS_R$$

where P is

$$P = \begin{bmatrix} \mathbf{P}_{0,0} & \mathbf{P}_{0,1} & \mathbf{P}_{0,2} & \mathbf{P}_{0,3} \\ \mathbf{P}_{1,0} & \mathbf{P}_{1,1} & \mathbf{P}_{1,2} & \mathbf{P}_{1,3} \\ \mathbf{P}_{2,0} & \mathbf{P}_{2,1} & \mathbf{P}_{2,2} & \mathbf{P}_{2,3} \\ \mathbf{P}_{3,0} & \mathbf{P}_{3,1} & \mathbf{P}_{3,2} & \mathbf{P}_{3,3} \end{bmatrix}$$

General Subdivision with either Parameter

Combining these two methods, we can see that the arrays below segment the patch into quarters

$S_L P S_L$	$0 \leqslant u \leqslant 1/2,\ 0 \leqslant v \leqslant 1/2$
$S_L P S_R$	$0 \leqslant u \leqslant 1/2,\ 1/2 \leqslant v \leqslant 1$
$S_R P S_L$	$1/2 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1/2$
$S_R P S_R$	$1/2 \leqslant u \leqslant 1,\ 1/2 \leqslant v \leqslant 1$

Advanced Topics on Bézier Curves/Patches

- Triangular Bézier Patches
- Rational Bézier Curves/Surfaces
- Topics on Bézier
  - Degree Elevation
  - Degree Reduction
  - The Variation Diminishing Property
  - Nonparametric Curves/Surfaces: $(t,f(t))=(t(u), f(u))$
  - Integrals
  - Geometric Continuity
  - Conversion between Different Bézier Patches
  - Offset ……

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