Bézier Curves and Surfaces (2)

Hongxin Zhang and Jieqing Feng

2006-12-07

State Key Lab of CAD&CG
Zhejiang University
Contents

- Reparameterizing Bézier Curves
- Bézier Control Polygons for a Cubic Curve
- The Equations for a Bézier Curve of Arbitrary Degree
- Bézier Patches
- Bézier Curves on Bézier Patches
- Subdivision of Bézier Patches
- A Matrix Representation of the Cubic Bézier Patch
- Advanced Topics on Bézier Curves/Patches
- Course Downloaded
Reparameterizing Bézier Curves

• Bézier Curve $\mathbf{P}(t)$:
  
  a set of control points $\{\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n\}$ with
  Bernstein polynomials $\{B_{i,n}(t)\}$ $t \in [0,1]$
  
  $$\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P}_i B_{i,n}(t)$$

• Purpose
  
  General B-spline curves: piecewise Bézier curves over an arbitrary parametric interval
Defining the Reparameterized Curve

• Given a Bézier curve $P(t)$, a new parameterization of the curve where $t \in [a,b]$ can be developed as

$$P_{[a,b]}(t) = P\left(\frac{t - a}{b - a}\right)$$

$P_{[a,b]}(t)$ and $P(t)$ are exactly the same curve, but traversed through different ranges of $t$. 
Impaction of Parameterization on Bézier Curve Properties

- \( P_{[0,1]}(t) = P(t) \)

- Using the chain rule, the derivative of the curve \( P_{[a,b]}(t) \) at a value \( t \) is equal to
  \[
  \frac{1}{b-a} P'(\frac{t - a}{b - a})
  \]

- Subdividing the curve \( P_{[a,b]}(t) \) at the point \( c \in [a,b] \), is equivalent to subdividing the curve \( P(t) \) at the point \( \frac{c-a}{b-a} \)
Bézier Control Polygons for a Cubic Curve

- A Matrix Equation for a Cubic Curve
- Reparameterization using the Matrix Form
- A Specific Example
- An Expanded Example
A Matrix Equation for a Cubic Curve

- A cubic polynomial curve $P(t)$ can be written as a cubic Bézier curve.
- Let $P_0, P_1, P_2, P_3$ be the control points of the curve $P(t)$.

$$P(t) = \sum_{i=0}^{3} P_i B_{i,3}(t)$$

its matrix form is

Cubic Bézier curve and its control polygon
A Matrix Equation for a Cubic Curve

\[ P(t) = \sum_{i=0}^{3} P_i B_i(t) \]

\[ = (1 - t)^3 P_0 + 3t(1 - t)^2 P_1 + 3t^2(1 - t) P_2 + t^3 P_3 \]

\[ = \begin{bmatrix}
(1 - t)^3 & 3t(1 - t)^2 & 3t^2(1 - t) & t^3 \\
\end{bmatrix}
\begin{bmatrix}
P_0 \\
P_1 \\
P_2 \\
P_3 \\
\end{bmatrix} \]

\[ = \begin{bmatrix}
1 & t & t^2 & t^3 \\
\end{bmatrix} M \begin{bmatrix}
P_0 \\
P_1 \\
P_2 \\
P_3 \\
\end{bmatrix} \]
A Matrix Equation for a Cubic Curve

Where

\[ M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \]

Notes:

- The matrix \( M \) defines the blending functions for the curve \( \mathbf{P}(t) \) - i.e. the cubic Bernstein polynomials.
- There are three equations here, one for each of the \( x \), \( y \) and \( z \) components of \( \mathbf{P}(t) \).
Reparameterization using the Matrix Form

- Let $P_0, P_1, P_2, P_3$ be the control points of the curve $P(t)$
  - In general, the used parametric interval is $[0,1]$  
  - $P_0=P(0), P_1=P(1)$
- Given an interval $[a,b]$, there exists a unique control polygon \{\(Q_0, Q_1, Q_2, Q_3\)\} defining a Bézier curve $Q(t)$, such that $Q(0)=Q_0=P(a)$ and $Q(1)=Q_1=P(b)$
Reparameterization using the Matrix Form

- **Purpose**: finding the Bézier polygon for the portion of the curve $P(t)$ where $t \in [a, b]$
- **Solution**: by reparameterization and by manipulating the matrix representation above
Reparameterization using the Matrix Form

• Defining the new curve as \( Q(t) \), then

\[ Q(t) = P((b-a)t+a) \]

- Both \( Q(t) \) and \( P(t) \) are cubic curves, and represent the same curve.

- The difference of \( Q(t) \) and \( P(t) \) is their parametric domain
  - \( P(t): \ t \in [0,1] \)
  - \( Q_{[0,1]}(t) = P_{[a,b]}(t): \ t \in [a,b] \)
Reparameterization using the Matrix Form

\[ Q(t) = P((b-a)t + a) \]

\[ = \begin{bmatrix} 1 & (b-a)t + a & ((b-a)t + a)^2 & ((b-a)t + a)^3 \end{bmatrix} \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} \]

where the matrix \([C]\) has columns whose entries are the coefficients of 1, \(t\), \(t^2\) and \(t^3\) respectively in the polynomials 1, \((b-a)t+a\), \((b-a)t+a)^2\), and \(((b-a)t+a)^3\), respectively
Reparameterization using the Matrix Form

• $Q(t)$ can be written as

$$Q(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} CM \mathbf{P}$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M \left( S_{[a,b]} \mathbf{P} \right)$$

where $S_{[a,b]}$ is equal to $S_{[a,b]} = M^{-1} CM$

• The new control points for the portion of the curve where $t$ ranges from $a$ to $b$ can now be written as $(S_{[a,b]} \mathbf{P})$
A Specific Example

• **$P(t)$**: parameter ranges from 1 to 2
  - It is natural extension of $P(t)$ from [0,1] to [1,2]
  - It is useful to learn how to piece together two Bézier curves: The general B-spline curves are piecewise Bézier curves which are smoothly joined.
Matrix Representation of $P(t)$ on $[1,2]$ 

$$Q(t) = P(t + 1)$$

$$= \begin{bmatrix} 1 & (t + 1) & (t + 1)^2 & (t + 1)^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M(S_{[1,2]} P)$$
Matrix Representation of $P(t)$ on $[1,2]$

where

$$S_{[1,2]} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1
\end{bmatrix}^{-1} \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & \frac{1}{3} & 0 & 0 \\
1 & \frac{2}{3} & \frac{1}{3} & 0 \\
1 & 1 & 1 & 1
\end{bmatrix} \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & \frac{4}{3} & \frac{5}{3} & 2 \\
1 & \frac{5}{3} & \frac{8}{3} & 4 \\
1 & 2 & 4 & 8
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 2 \\
0 & 1 & -4 & 4 \\
-1 & 6 & -12 & 8
\end{bmatrix}$$
The control polygon for that portion of \( P(t) \) curve where \( t \) ranges from 1 to 2 is:

\[
\begin{bmatrix}
Q_0 \\
Q_1 \\
Q_2 \\
Q_3
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 2 \\
0 & 1 & -4 & 4 \\
-1 & 6 & -12 & 8
\end{bmatrix}
\begin{bmatrix}
P_0 \\
P_1 \\
P_2 \\
P_3
\end{bmatrix} =
\begin{bmatrix}
P_3 \\
-P_2 + 2P_3 \\
P_1 - 4P_2 + 4P_3 \\
-P_0 + 6P_1 - 12P_2 + 8P_3
\end{bmatrix}
\]
Geometric Interpretation of the New Control Points \( \{Q_0, Q_1, Q_2, Q_3\} \)

Defining new temporary points \( R_1, R_2, R_3 \)

\( \{Q_0, Q_1, Q_2, Q_3\} \) can be calculated by a simple geometric process using only the initial control polygon \( \{P_0, P_1, P_2, P_3\} \)

- \( Q_0 = P_3 \)
- \( Q_1 = P_3 + (P_3 - P_2) \)  \( (2) \)
- \( R_1 = P_1 + (P_1 - P_0) \)  \( (3) \)
- \( R_2 = P_2 + (P_2 - P_1) \)  \( (4) \)
- \( R_3 = R_2 + (R_2 - R_1) \)
  \[ = P_2 + (P_2 - P_1) + (P_2 + (P_2 - P_1) - P_1 + (P_1 - P_0)) \]
  \[ = P_0 - 4P_1 + 4P_2 \]  \( (5) \)
- \( Q_2 = Q_1 + (Q_1 - R_2) \)
  \[ = P_3 + (P_3 - P_2) + (P_3 + (P_3 - P_2) - P_2 + (P_2 - P_1)) \]
  \[ = P_1 - 4P_2 + 4P_3 \]  \( (6) \)
- \( Q_3 = Q_2 + (Q_2 - R_3) \)
  \[ = P_1 - 4P_2 + 4P_3 + (P_1 - 4P_2 + 4P_3 - P_0 - 4P_1 + 4P_2) \]
  \[ = P_0 + 6P_1 - 12P_2 + 8P_3 \]  \( (7) \)
Geometric Interpretation of the $Q_0$ (1)

- $Q_0 = P_0$
Geometric Interpretation of the $Q_1$ (2)

- $Q_1 = P_3 + (P_3 - P_2)$ lies on the line $\overline{P_2P_3}$, where the distance between $P_2$ and $P_3$ is equal to that between $Q_0$ and $Q_1$. 

![Diagram showing geometric interpretation](image)
Geometric Interpretation of the $Q_1$ (3)

\[ R_2 = P_2 + (P_2 - P_1) \]

\[ Q_2 = Q_1 + (Q_1 - R_2) \]

\[ = P_3 + (P_3 - P_2) + (P_3 + (P_3 - P_2) - P_2 + (P_2 - P_1)) \]

\[ = P_1 - 4P_2 + 4P_3 \]
Geometric Interpretation of the $Q_1$ (4)

\[
R_1 = P_1 + (P_1 - P_0)
\]

\[
R_2 = P_2 + (P_2 - P_1)
\]

\[
R_3 = R_2 + (R_2 - R_1) = P_2 + (P_2 - P_1) + (P_2 + (P_2 - P_1) - P_1 + (P_1 - P_0)) = P_0 - 4P_1 + 4P_2
\]

\[
Q_3 = Q_2 + (Q_2 - R_3) = P_1 - 4P_2 + 4P_3 + (P_1 - 4P_2 + 4P_3 - P_0 - 4P_1 + 4P_2) = P_0 + 6P_1 - 12P_2 + 8P_3
\]
Geometric Interpretation of the $Q_1$ (4)
A Specific Example: Results

- The above geometric construction is the inverse process of the \textit{de Casteljau geometric construction}.
- These two functions represent the \textit{same curve}.
- Exercise: constructing the control points of $Q(t) = P(t)$, $t \in [0, 2]$.

Tips: the result control points $\{P_0, R_1, R_2, Q_3\}$. 
An Expanded Example

• The example above
  ✷ illustrated there are many Bézier polygons that can represent a cubic curve
  ✷ did not quite illustrate the necessary characteristics of the algorithm

• Considering the cubic curve $P(t)$ when $t \in [1, b]$
  ✷ $Q(t) = P(at+1)$ where $a = b-1$
Matrix Representation of $P(t)$ on $[1, b]$

$$Q(t) = P(at + 1)$$

$$= \begin{bmatrix} 1 & (at + 1) & (at + 1)^2 & (at + 1)^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & a & 2a & 3a \\ 0 & 0 & a^2 & 3a^2 \\ 0 & 0 & 0 & a^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M (S_{[1,b]} P)$$
Matrix Representation of \( P(t) \) on \([1, b]\)

where

\[
S_{[1,b]} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & a & 2a & 3a \\
0 & 0 & a^2 & 3a^2 \\
0 & 0 & 0 & a^3
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & \frac{1}{3} & 0 & 0 \\
1 & \frac{2}{3} & \frac{1}{3} & 0 \\
1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & a & 2a & 3a \\
0 & 0 & a^2 & 3a^2 \\
0 & 0 & 0 & a^3
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -a & (a+1) \\
0 & a^2 & -2a(a+1) & (a+1)^2 \\
-a^3 & 3a^2(a+1) & -3a(a+1)^2 & (a+1)^3
\end{bmatrix}
\]
Matrix Representation of $P(t)$ on $[1, b]$

The control polygon of the curve $P(t)$ where $t \in [1, b]$

\[
\begin{bmatrix}
Q_0 \\
Q_1 \\
Q_2 \\
Q_3
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -a & (a+1) \\
0 & a^2 & -2a(a+1) & (a+1)^2 \\
-a^3 & 3a^2(a+1) & -3a(a+1)^2 & (a+1)^3
\end{bmatrix} \begin{bmatrix}
P_0 \\
P_1 \\
P_2 \\
P_3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
P_3 \\
-aP_2 + (a+1)P_3 \\
a^2P_1 - 2a(a+1)P_2 + (a+1)^2P_3 \\
-a^3P_0 + 3a^2(a+1)P_1 - 3a(a+1)^2P_2 + (a+1)^3P_3
\end{bmatrix}
\]
Geometric Interpretation of the New Control Points \{Q_0, Q_1, Q_2, Q_3\}

Defining new temporary points \( R_1, R_2, R_3 \)

\{Q_0, Q_1, Q_2, Q_3\} can be calculated by a simple geometric process using only the initial control polygon \{P_0, P_1, P_2, P_3\}

\[
\begin{align*}
Q_0 &= P_3 \\
Q_1 &= P_3 + a(P_3 - P_2) \\
R_1 &= P_1 + a(P_1 - P_0) \\
R_2 &= P_2 + a(P_2 - P_1) \\
R_3 &= R_2 + a(R_2 - R_1) \\
&= P_2 + a(P_2 - P_1) + a(P_2 + a(P_2 - P_1) - P_1 + a(P_1 - P_0)) \\
&= a^2P_0 - 2a(a + 1)P_1 + (a + 1)^2P_2 \\
Q_2 &= Q_1 + a(Q_1 - R_2) \\
&= P_3 + a(P_3 - P_2) + a(P_3 + a(P_3 - P_2) - P_2 + a(P_2 - P_1)) \\
&= a^2P_1 - 2a(a + 1)P_2 + (a + 1)^2P_3 \\
Q_3 &= Q_2 + a(Q_2 - R_3) \\
&= -a^3P_0 + 3a^2(a + 1)P_1 - 3a(a + 1)^2P_2 + (a + 1)^3P_3
\end{align*}
\]
Geometric Interpretation of the New Control Points \( \{Q_0, Q_1, Q_2, Q_3\} \)
Geometric Interpretation of the New Control Points \( \{ Q_0, Q_1, Q_2, Q_3 \} \)

- Results
  - The important factor here is the \( a \) term
  - Each of these points is on an extension of a line of the original control polygon, or the extension of a constructed line
  - The factor \( a \) determines how much to extend.
The Equations for a Bézier Curve of Arbitrary Degree

• Overview
  - The Bézier curve representation is one that is utilized most frequently in computer graphics and geometric modeling.
  - The curve is defined geometrically, which means that the parameters have geometric meaning - they are just points in three-dimensional space.
  - It was developed by two competing European engineers in the late 1960s to attempt to draw automotive components.
The Equations for a Bézier Curve of Arbitrary Degree

• Specification of the Bézier Curve of Arbitrary Degree
  - Generalizing the development for the quadratic and cubic Bézier curves
  - Given the set of control points, \( \{P_0, P_1, \ldots, P_n\} \), defining a Bézier curve of degree \( n \) by either Analytic Definition or Geometric Construction.
The Analytic Definition

\[ P(t) = \sum_{i=0}^{n} P_i B_{i,n}(t) \]

where

\[ B_{i,n}(t) = \binom{n}{i} t^i (1 - t)^{n-i} \]

are the Bernstein polynomials of degree \( n \), and \( t \) ranges between zero and one \( 0 \leq t \leq 1 \).
Geometric Definition

\[ P(t) = P_{n}^{(n)}(t) \]

where

\[ P_{i}^{(j)}(t) = \begin{cases} 
(1 - t)P_{i - 1}^{(j-1)}(t) + tP_{i}^{(j-1)}(t) & \text{if } j > 0, \\
P_{i} & \text{otherwise}
\end{cases} \]

where \( t \) ranges between zero and one \( 0 \leq t \leq 1 \).
Properties of the Bézier Curve

- $P_0$ and $P_n$ are on the curve.
- The curve is continuous and has continuous derivatives of all orders.
- The tangent line to the curve at the point $P_0$ is the line $P_0P_1$. The tangent to the curve at the point $P_n$ is the line $P_{n-1}P_n$.
- The curve lies within the convex hull of its control points. This is because each successive $P_i^{(j)}$ is a convex combination of the points $P_i^{(j-1)}$ and $P_{i-1}^{(j-1)}$.
- $P_0, P_1, \ldots, P_n$ are all on the curve only if the curve is linear.
Summary of the Bézier Curve

• Given a sequence of \(n+1\) control points, one can specify a Bézier curve of degree \(n\) defined by these points.

• Two definitions of the curve can be given:
  ✷ An analytic definition specifying the blending of the control points with Bernstein polynomials
  ✷ A geometric definition specifying a recursive generation procedure that calculates successive points on line segments developed from the control point sequence.
Bézier Patches

• Overview
  - Pierre Bézier in Renault and Paul de Casteljau in Citroën, initially developed a Bézier curve representation and extended it to a surface patch methodology
    - The extension of Bézier curves to surfaces is called the Bézier patch
  - The Bézier patch is the most commonly used surface representation in computer graphics
Bézier Patches

- Bézier curve and patch
  - The Bézier curve is a function of one variable and takes a sequence of control points.
  - The Bézier patch is a function of two variables with an array of control points.
  - Most of the methods for the patch are direct extensions of those for the curves.
Definition of the Bézier Patch

- The patch is constructed from an $n \times m$ array of control points: $\{\mathbf{P}_{ij}, 0 \leq i \leq n, 0 \leq j \leq m\}$
Definition of the Bézier Patch

- The Bézier patch is parameterized by two variables, is given by the equation

\[ P(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j} B_{i,m}(u) B_{j,n}(v) \]
Definition of the Bézier Patch

- It is summations running over all the control points
- The bi-variate Bernstein Polynomials serving as the functions that blend the control points together

\[ B_{i,n}(u)B_{j,m}(v) \]
Deductions from Definition of the Bézier Patch

• By set \( v=0 \), we obtain

\[
P(u, 0) = \sum_{j=0}^{m} \sum_{i=0}^{n} P_{i,j} B_{i,n}(u) B_{j,m}(0)
\]

\[
= \sum_{i=0}^{n} P_{i,0} B_{i,n}(u)
\]

since \( B_{0m}(0)=1 \) and \( B_{jm}(0)=0 \) for \( j=1,2,\ldots,m \)

Result: \( P(u,0) \) is a Bézier Curve
Deductions from Definition of the Bézier Patch

- $P(u, 1), P(1, v)$ and $P(0, v)$ are Bézier Curve
Relations between the Bézier Curve and Patch

- The corner ones of control points are actually on the patch

![Diagram showing the relationship between the Bézier curve and the patch, with control points labeled as P_{0,0}, P_{0,3}, P_{3,0}, and P_{3,3}.]
Properties of the Bézier Patch

- The four points $P_{0,0}$, $P_{0,m}$, $P_{n,0}$ and $P_{n,m}$ are on the patch. The other control points are all on the patch only if the patch is planar.
- The patch is continuous and partial derivatives of all orders exist and are continuous.
- The patch lies within the convex hull of its control points.
Bézier Curves on Bézier Patches

• Overview

♦ \( P(0,v) \) and \( P(1,v) \) are Bézier curves lying on the boundary of the Bézier patch.

♦ A Bézier patch can be treated as a continuous set of Bézier curves. That is, for any fixed parameter \( u_0 \) or \( v_0 \), we can define a Bézier curve that lies directly on the surface of the patch.

♦ It is a very valuable tool for calculations on the patch.
Grouping factors of the Bézier patch function appropriately

\[
P(u, v) = \sum_{j=0}^{m} \left[ \sum_{i=0}^{n} P_{i,j} B_{i,n}(u) \right] B_{j,m}(v)
\]

If we fix \(u = u_0\), the internal sum can be calculated (for \(j = 0, ..., m\)). This implies that \(P(u_0, v)\) is a Bézier curve on the surface.
If we define $Q_{j}(u)$ to be the value

$$Q_{j}(u) = \sum_{i=0}^{n} P_{i,j} B_{i,n}(u)$$

we can see that

$$P(u, v) = \sum_{j=0}^{m} Q_{j}(u) B_{j,m}(v)$$

That is, the quantities $Q_{j}(u)$ form the control points of another Bézier curve, and together for all $u$ and $v$, they form the surface.
Bézier Curves on Bézier Patches

Therefore, given \( u = u_0 \), we can calculate the quantities \( Q_0(u_0), Q_1(u_0), \ldots, Q_m(u_0) \), giving \( m \) control points to utilize for the curve

\[
Q(v) = \sum_{j=0}^{m} Q_j(u_0) B_{j,m}(v)
\]

- This curve lies on the patch: \( P(u_0,v) = Q(v) \),
- \( Q(v_0) \) is the point on the patch at \( (u_0,v_0) \).

Result: Calculating a point on the patch can be reduced to finding several points on curves which is parameter independent.
Calculating a Point on a Bi-Cubic Surface: STEP 1

The point $Q_0(u_0)$, is calculated as a point on the Bézier curve defined by the control points $P_{0,0}$, $P_{0,1}$, $P_{0,2}$ and $P_{0,3}$.
Calculating a Point on a Bi-Cubic Surface: STEP 2

The point $Q_1(u_0)$, is calculated as a point on the Bézier curve defined by the control points $P_{1,0}$, $P_{1,1}$, $P_{1,2}$ and $P_{1,3}$.
The point $\mathbf{Q}_2(u_0)$, is calculated as a point on the Bézier curve defined by the control points $\mathbf{P}_{2,0}$, $\mathbf{P}_{2,1}$, $\mathbf{P}_{2,2}$ and $\mathbf{P}_{2,3}$. 
Calculating a Point on a Bi-Cubic Surface: STEP 4

The point $Q_3(u_0)$ is calculated as a point on the Bézier curve defined by the control points $P_{3,0}$, $P_{3,1}$, $P_{3,2}$, and $P_{3,3}$. 
The point $P(u_0, v_0)$, on the patch, is calculated as a point on the Bézier curve defined by the control points $Q_0(u_0)$, $Q_1(u_0)$, $Q_2(u_0)$, $Q_3(u_0)$. 
If we take the analytic equation of a Bézier patch, fix $u$ and group factors appropriately, we obtain

$$P(u, v) = \sum_{j=0}^{m} \left[ \sum_{i=0}^{n} P_{i,j} B_{i,n}(u) \right] B_{j,m}(v)$$

We notice that portion of the equation inside the brackets is the representation of a Bézier curve.
Subdivision of Bézier Patches

If we let $Q_j(u)$ be the value inside the brackets,

$$Q_j(u) = \sum_{i=0}^{n} P_{i,j} B_{i,n}(u)$$

Then

$$P(u, v) = \sum_{j=0}^{m} Q_j(u) B_{j,m}(v)$$

That is, the quantities $Q_j(u)$ form the control points of another Bézier curve, and together for all $u$ and $v$, they form the surface.
Subdivision of Bézier Patches

• If we subdivide each of the $m$ rows of the $P_{ij}$ matrix, it implies that the $Q_j$'s in the above equation represent only points from the first half of the patch (with respect to $u$).

• The second half of the patch can be obtained in a similar fashion.

• The first and second half of the patch, with respect to $v$, can be obtained by subdividing the columns.

The above illustration shows the result of subdividing the rows in the $4 \times 4$ case.
A Matrix Representation of the Cubic Bézier Patch

- Overview
- Developing the Matrix Formulation
- Patch Subdivision Using the Matrix Form
- Calculation of the Second Half of the Patch
- General Subdivision with either Parameter
Overview

- The matrix representation of the cubic Bézier patch allows us to specify many operations with Bézier patches.
- The matrix operations can be performed quickly on computer systems optimized for geometry operations with matrices.
A cubic Bézier curve can be written in a matrix form similar to that for a Bézier Curve by utilizing the representation of a Bézier patch as a continuous set of Bézier curves.

\[
P(u, v) = \sum_{j=0}^{3} \left( \sum_{i=0}^{3} P_{i,j} B_{i,3}(u) B_{j,3}(v) \right)
\]

\[
= \sum_{j=0}^{3} \left[ \sum_{i=0}^{3} P_{i,j} B_{i,3}(u) \right] B_{j,3}(v)
\]

\[
- \sum_{j=0}^{3} \left[ \begin{array}{cccc}
1 & u & u^2 & u^3 \\
1 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1
\end{array} \right] \begin{bmatrix}
P_{0,j} \\
P_{1,j} \\
P_{2,j} \\
P_{3,j}
\end{bmatrix}
\]

\[
= \left[ \begin{array}{cccc}
1 & u & u^2 & u^3 \\
1 & 0 & 0 & 0 \\
-3 & 3 & 0 & 0 \\
3 & -6 & 3 & 0 \\
-1 & 3 & -3 & 1
\end{array} \right] \begin{bmatrix}
P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\
P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\
P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\
P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3}
\end{bmatrix} \begin{bmatrix}
1 & -3 & 3 & -1 \\
0 & 3 & -6 & 3 \\
0 & 0 & 3 & -3 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 \\
v \\
v^2 \\
v^3
\end{bmatrix}
\]
Developing the Matrix Formulation

The cubic Bézier patch is frequently written

\[
P(u, v) = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} M \begin{bmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} \end{bmatrix} M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}
\]

where

\[
M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}
\]
Patch Subdivision Using the Matrix Form

• Purpose: subdividing the patch at the point $u=1/2$
• Method: reparameterizing the matrix equation above (by substituting $u/2$ for $u$) to cover only the first half of the patch, and simplify to obtain.
Patch Subdivision Using the Matrix Form

\[ P\left(\frac{u}{2}, v\right) = \begin{bmatrix} 1 & \left(\frac{u}{2}\right) & \left(\frac{u}{2}\right)^2 & \left(\frac{u}{2}\right)^3 \end{bmatrix} M \begin{bmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} \end{bmatrix} M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix} \]

\[ = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} M \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} M \begin{bmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} \end{bmatrix} M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix} \]

\[ = \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} M S_L \begin{bmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} \end{bmatrix} M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix} \]
Patch Subdivision Using the Matrix Form

where the matrix $S_L$ is defined as

$$S_L = M^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} M$$
Patch Subdivision Using the Matrix Form

The matrix $S_L$ is identical to the left subdivision matrix for the curve case. So in particular, the subpatch $P(u/2,v)$ is again a Bézier patch whose control points of this patch is

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\
\frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8}
\end{bmatrix}
\begin{bmatrix}
P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\
P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\
P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\
P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3}
\end{bmatrix}
\]
Calculation of the Second Half of the Patch

For the second half of the patch: First we reparameterize the original curve, and then simplify to obtain

\[
P\left(\frac{1}{2} + \frac{u}{2}, v\right) = \begin{bmatrix} 1 \ (\frac{1}{2} + \frac{u}{2}) \ (\frac{1}{2} + \frac{u}{2})^2 \ (\frac{1}{2} + \frac{u}{2})^3 \end{bmatrix} M \begin{bmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} \end{bmatrix} M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} M \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{3}{8} \\ 0 & 0 & \frac{1}{4} & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} M \begin{bmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} \end{bmatrix} M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} M S_R \begin{bmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} \end{bmatrix} M^T \begin{bmatrix} 1 \\ v \\ v^2 \\ v^3 \end{bmatrix}
\]
Calculation of the Second Half of the Patch

where

\[ S_R = \begin{bmatrix}
\frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\
0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
0 & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & 1
\end{bmatrix} \]

- \( S_R \) is identical to the right subdivision matrix in the curve case
- \( S_R \) can be applied to a set of control points to produce the control points for the second half of the patch
General Subdivision with either Parameter

- The matrix representation of control points for the first and second portions of the patch when subdivision is done with respect to $\nu$:

$$PS_L \text{ and } PS_R$$

where $P$ is

$$P = \begin{bmatrix}
P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\
P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\
P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\
P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3}
\end{bmatrix}$$
General Subdivision with either Parameter

Combining these two methods, we can see that the arrays below segment the patch into quarters

\[ S_LPS_L \quad 0 \leq u \leq 1/2, \quad 0 \leq v \leq 1/2 \]
\[ S_LPS_R \quad 0 \leq u \leq 1/2, \quad 1/2 \leq v \leq 1 \]
\[ S_RPS_L \quad 1/2 \leq u \leq 1, \quad 0 \leq v \leq 1/2 \]
\[ S_RPS_R \quad 1/2 \leq u \leq 1, \quad 1/2 \leq v \leq 1 \]
Advanced Topics on Bézier Curves/Patches

• Triangular Bézier Patches
• Rational Bézier Curves/Surfaces
• Topics on Bézier
  ✷ Degree Elevation
  ✷ Degree Reduction
  ✷ The Variation Diminishing Property
  ✷ Nonparametric Curves/Surfaces: \((t, f(t)) = (t(u), f(u))\)
  ✷ Integrals
  ✷ Geometric Continuity
  ✷ Conversion between Different Bézier Patches
  ✷ Offset ……
http://www.cad.zju.edu.cn/home/jqfeng/GM/GM03.zip