Bézier Curves and Surfaces (1)

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Contents

Bernstein Polynomials

Introduction of Polynomials

- Polynomial: $p(t)=a_n t^n+a_{n-1}t^{n-1}+\ldots+a_1 t+a_0$ are the linear combination of power basis $\{(1,t,t^2,\ldots,t^n)\}$
- Polynomials are incredibly useful mathematical tools in Science and Engineering
  - Simply defined
  - Calculated quickly on computer systems
  - Represent a tremendous variety of functions
  - Differentiated and integrated easily
  - Pieced together to form spline curves that can approximate any function to any accuracy desired

Introduction of Polynomials

- The set of polynomials of degree less than or equal to n forms a vector space
  - Polynomials can be added together
  - Polynomials can be multiplied by a scalar
  - All the vector space properties hold
- The set of functions $\{(1, t, t^2, \ldots, t^n)\}$ form a basis for above vector space
  - Any polynomial of degree less than or equal to n can be uniquely written as a linear combinations of these functions

Introduction of Polynomials

- Notes
  - The *power basis* is only one of an infinite number of bases for the space of polynomials
  - The *Bernstein basis* is another of the commonly used bases for the space of polynomials

Definition of Bernstein Polynomial

- The n+1 Bernstein polynomials of degree n are defined by

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

for $i=0,1,\ldots,n$. where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$

- Note: $B_{i,n}(t)=0$ if $i<0$ or $i>n$

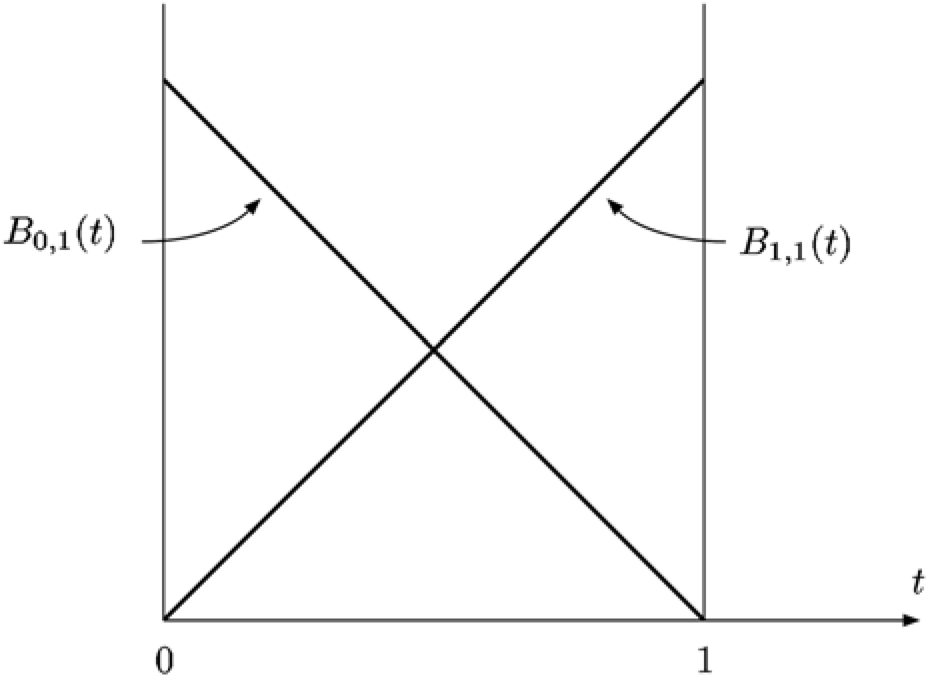
Example of Bernstein Polynomials (1)

The Bernstein Polynomials of degree 1 are

$$B_{0,1}(t)=1-t$$

$$B_{1,1}(t)=t$$

When $0 \leqslant t \leqslant 1$, they can be plotted as



Example of Bernstein Polynomials (2)

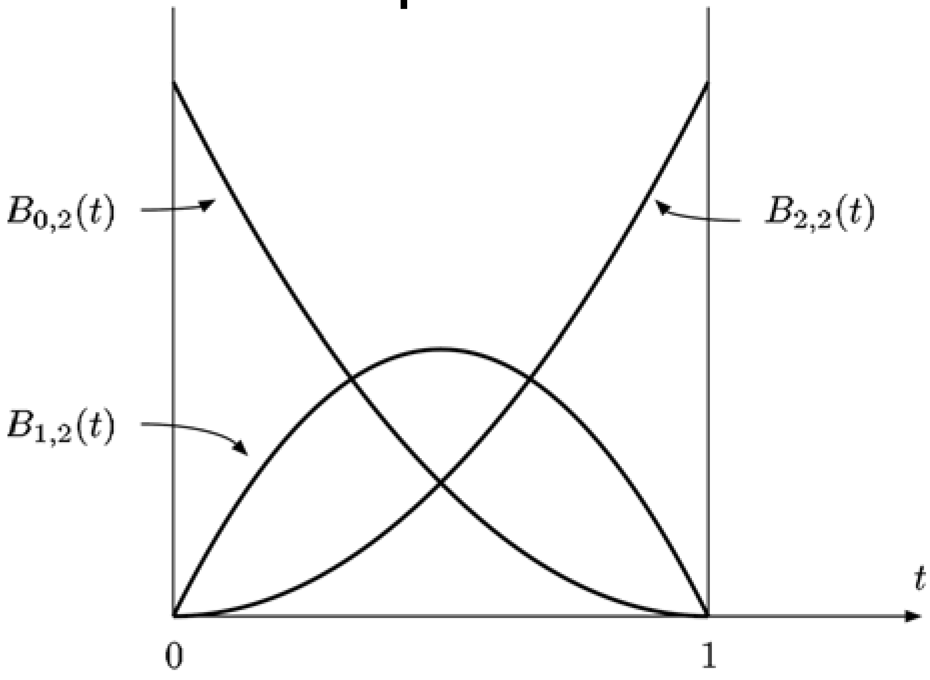
The Bernstein Polynomials of degree 2 are

$$B_{0,2}(t)=(1-t)^2$$

$$B_{1,2}(t)=2t(1-t)$$

$$B_{2,2}(t)=t^2$$

When $0 \leqslant t \leqslant 1$, they can be plotted as

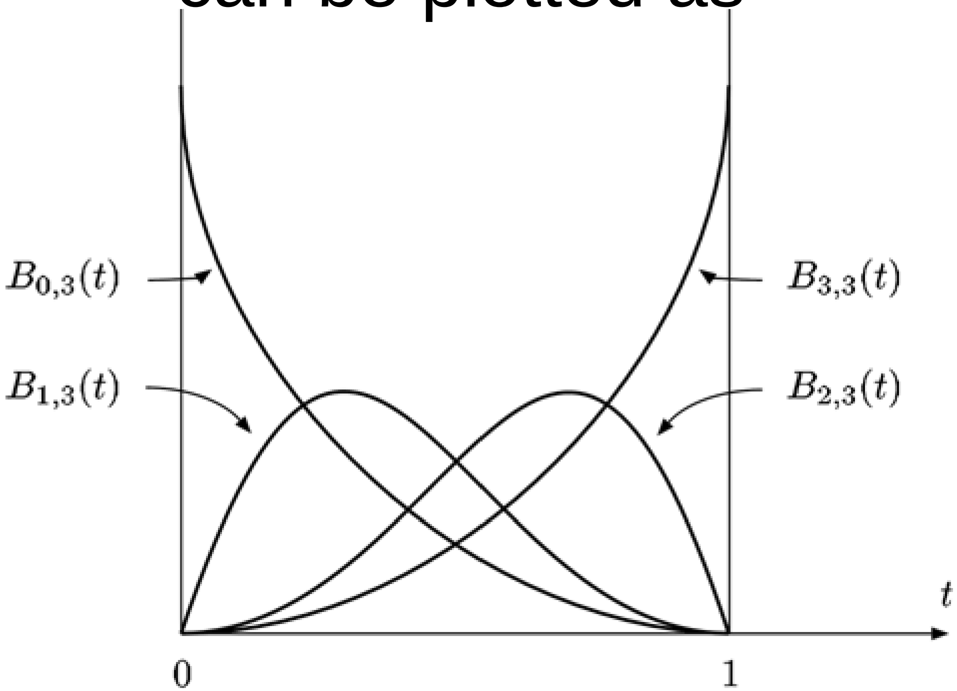


Example of Bernstein Polynomials (3)

The Bernstein Polynomials of degree 3 are

$$B_{0,3}(t)=(1-t)^3$$
$$B_{1,3}(t)=3t(1-t)^2$$
$$B_{2,3}(t)= 3t^2(1-t)$$
$$B_{2,3}(t)=t^3$$

When $0 \leqslant t \leqslant 1$, they can be plotted as



A Recursive Definition of the Bernstein Polynomials

- The Bernstein polynomials of degree n can be defined by blending together two Bernstein polynomials of degree n-1

$$B_{k,n}(t)=(1-t)\,B_{k,n-1}(t)+tB_{k-1,n-1}(t)$$

  - The above statement can be proved by utilizing definition of the Bernstein polynomials

A Recursive Definition of the Bernstein Polynomials

Proof

$$
\begin{aligned}
&(1-t)B_{k,n-1}(t) + tB_{k-1,n-1}(t) \\
&= (1-t)\binom{n-1}{k}t^k(1-t)^{n-1-k} + t\binom{n-1}{k-1}t^{k-1}(1-t)^{n-1-(k-1)} \\
&= \binom{n-1}{k}t^k(1-t)^{n-k} + \binom{n-1}{k-1}t^k(1-t)^{n-k} \\
&= \left[\binom{n-1}{k} + \binom{n-1}{k-1}\right]t^k(1-t)^{n-k} \\
&= \binom{n}{k}t^k(1-t)^{n-k} \\
&= B_{k,n}(t)
\end{aligned}
$$

Properties of Bernstein Polynomial

- Non-Negative
- Partition of Unity
- Symmetry
- Degree Raising
- Linear Precision
- Derivatives

Non-Negative

- A function $f(t)$ is non-negative over an interval $[a,b]$ if $f(t) \geqslant 0$ for $t \in [a,b]$
- The property can be proved easily from the definition of Bernstein Polynomials

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

- The Bernstein Polynomials are positive when $0 < t < 1$.

Partition of Unity

- A set of functions $f_i(t)$ is said to partition of unity if they sum to one for all values of t.
- $\sum_i B_{i,n}(t) = 1$, for all $t \in [0,1]$.

  Proof: $1 = 1^n = [(1-t)+t]^n = \sum_i B_{i,n}(t)$

Partition of Unity

- For any set of points $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$, and for any t, the expression

$$\mathbf{P}(t) = \mathbf{P}_0 B_{0,n}(t) + \mathbf{P}_1 B_{1,n}(t) + \ldots + \mathbf{P}_n B_{n,n}(t)$$

is an affine combination of the set of points $\{\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n\}$ and if $0 \leqslant t \leqslant 1$, it is a convex combination of the points

Symmetry

- $B_{i,n}(t) = B_{n-i,n}(1-t)$
- Proof: from definition……

Degree Raising

- Any of the lower-degree Bernstein polynomials (degree $< n$) can be expressed as a linear combination of Bernstein polynomials of degree n.
- Any Bernstein polynomial of degree n-1 can be written as a linear combination of Bernstein polynomials of degree n.

$$B_{i,n}(t) = \frac{n-i+1}{n+1} B_{i,n+1}(t) + \frac{i+1}{n+1} B_{i+1,n+1}(t)$$

Degree Raising

Proof: $B_{i,n}(t)=(1\text{-}t)B_{i,n}(t)+tB_{i,n}(t)$

$$
\begin{aligned}
tB_{i,n}(t) &= \binom{n}{i} t^{i+1}(1-t)^{n-i} \\
&= \binom{n}{i} t^{i+1}(1-t)^{(n+1)-(i+1)} \\
&= \frac{\binom{n}{i}}{\binom{n+1}{i+i}} B_{i+1,n+1}(t) \\
&= \frac{i+1}{n+1} B_{i+1,n+1}(t)
\end{aligned}
$$

$$
\begin{aligned}
(1-t)B_{i,n}(t) &= \binom{n}{i} t^{i}(1-t)^{n+1-i} \\
&= \frac{\binom{n}{i}}{\binom{n+1}{i}} B_{i,n+1}(t) \\
&= \frac{n-i+1}{n+1} B_{i,n+1}(t)
\end{aligned}
$$

Degree Raising

- Any Bernstein polynomial of degree n can be written as a linear combination of Bernstein polynomials of degree $n+r$ ($r>0$).

$$B_{i,n}(t) = \sum_{j=i}^{i+r} \frac{\binom{n}{i}\binom{r}{j-i}}{\binom{n+r}{j}} B_{j,n+r}(t)$$

Linear Precision

- The monomial t can be expressed as the weighted sum of Bernstein polynomials of degree n with coefficients evenly spaced in the interval [0,1].

$$t = \sum_{i=0}^{n} \frac{i}{n} B_{i,n}(t)$$

- Proof: Definition and some algebraic operations

Derivatives

- Derivatives of the nth degree Bernstein polynomials are Bernstein polynomials of degree n-1

$$\frac{d}{dt}B_{k,n}(t) = n(B_{k-1,n-1}(t) - B_{k,n-1}(t))$$

for $0 \leqslant k \leqslant n$.

Derivatives

Proof:

$$
\begin{aligned}
\frac{d}{dt} B_{k,n}(t) &= \frac{d}{dt} \binom{n}{k} t^k (1-t)^{n-k} \\
&= \frac{kn!}{k!(n-k)!} t^{k-1} (1-t)^{n-k} + \frac{(n-k)n!}{k!(n-k)!} t^k (1-t)^{n-k-1} \\
&= \frac{n(n-1)!}{(k-1)!(n-k)!} t^{k-1} (1-t)^{n-k} + \frac{n(n-1)!}{k!(n-k-1)!} t^k (1-t)^{n-k-1} \\
&= n \left(\frac{(n-1)!}{(k-1)!(n-k)!} t^{k-1} (1-t)^{n-k} + \frac{(n-1)!}{k!(n-k-1)!} t^k (1-t)^{n-k-1} \right) \\
&= n \left(B_{k-1,n-1}(t) - B_{k,n-1}(t) \right)
\end{aligned}
$$

Conversion Between Bernstein Basis and Power Basis

- Conversion from the Bernstein Basis to the Power Basis
- Conversion from the Power Basis to the Bernstein Basis
- The Bernstein Polynomials as a Basis of Polynomial Space

Conversion from the Bernstein Basis to the Power Basis

Since the power basis $\{(1, t, t^2, \ldots, t^n)\}$ forms a basis for the space of polynomials of degree less than or equal to n, any Bernstein polynomial of degree n can be written in terms of the power basis.

$$
\begin{aligned}
B_{k,n}(t) &= \binom{n}{k} t^k (1-t)^{n-k} \\
&= \binom{n}{k} t^k \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} t^i \\
&= \sum_{i=0}^{n-k} (-1)^i \binom{n}{k} \binom{n-k}{i} t^{i+k} \\
&= \sum_{i=k}^{n} (-1)^{i-k} \binom{n}{k} \binom{n-k}{i-k} t^i \\
&= \sum_{i=k}^{n} (-1)^{i-k} \binom{n}{i} \binom{i}{k} t^i
\end{aligned}
$$

Conversion from the Power Basis to the Bernstein Basis

To show that each power basis element can be written as a linear combination of Bernstein Polynomials, we use the degree elevation formulas and induction to calculate:

$$
\begin{aligned}
t^k &= t(t^{k-1}) \\
&= t \sum_{i=k-1}^{n} \frac{\binom{i}{k-1}}{\binom{n}{k-1}} B_{i,n-1}(t) \\
&= \sum_{i=k}^{n} \frac{\binom{i-1}{k-1}}{\binom{n-1}{k-1}} t B_{i-1,n-1}(t) \\
&= \sum_{i=k-1}^{n-1} \frac{\binom{i}{k-1}}{\binom{n}{k-1}} \frac{i}{n} B_{i,n}(t) \\
&= \sum_{i=k-1}^{n-1} \frac{\binom{i}{k}}{\binom{n}{k}} B_{i,n}(t),
\end{aligned}
$$

The Bernstein Polynomials as a Basis of Polynomial Space

- The Bernstein polynomials of degree n form a basis for the space of polynomials of degree less than or equal to n.
  - They span the space of polynomials of degree $\leqslant n$: any polynomial of degree less than or equal to n can be written as a linear combination of the Bernstein polynomials
  - They are linearly independent

Linearly Independent

If there exist constants $c_0, c_1, \ldots, c_n$ *s.t.* the identity $c_0 B_{0,n}(t) + c_1 B_{1,n}(t) + \ldots + c_n B_{n,n}(t) = 0$ holds for all t, then all c_i's must be zero.

$$
\begin{aligned}
0 &= c_0 B_{0,n}(t) + c_1 B_{1,n}(t) + \cdots + c_n B_{n,n}(t) \\
&= c_0 \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{i}{0} t^i + c_1 \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} \binom{i}{1} t^i + \cdots + c_n \sum_{i=n}^{n} (-1)^{i-n} \binom{n}{i} \binom{i}{n} t^i \\
&= c_0 + \left[\sum_{i=0}^{1} c_i \binom{n}{1} \binom{1}{1} \right] t^1 + \cdots + \left[\sum_{i=0}^{n} c_i \binom{n}{n} \binom{n}{n} \right] t^n
\end{aligned}
$$

Linearly Independent

$$c_0 = 0 \quad \rightarrow \quad c_0 = 0$$

$$\sum_{i=0}^{1} c_i \binom{n}{1} \binom{1}{1} = 0 \quad \rightarrow \quad c_1 = 1$$

$$\vdots$$

$$\sum_{i=0}^{n} c_i \binom{n}{n} \binom{n}{n} = 0 \quad \rightarrow \quad c_n = 0$$

A Matrix Representation for Bernstein Polynomials

- Given a polynomial written as a linear combination of the Bernstein basis functions

$$B(t) = c_0 B_{0,n}(t) + c_1 B_{1,n}(t) + \ldots + c_n B_{n,n}(t)$$

$$B(t) = \begin{bmatrix} B_{0,n}(t) & B_{1,n}(t) & \cdots & B_{n,n}(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

A Matrix Representation for Bernstein Polynomials

$$B(t) = \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \end{bmatrix} \begin{bmatrix} b_{0,0} & 0 & 0 & \cdots & 0 \\ b_{1,0} & b_{1,1} & 0 & \cdots & 0 \\ b_{2,0} & b_{2,1} & b_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n,0} & b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

- The $b_{i,j}$ are the coefficients of the power basis that are used to determine the respective Bernstein polynomials.
- The matrix in this case is lower triangular.

Examples of Matrix Representation

$$B(t) = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

$$B(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

A Divide-and-Conquer Method for Drawing a Bézier Curve

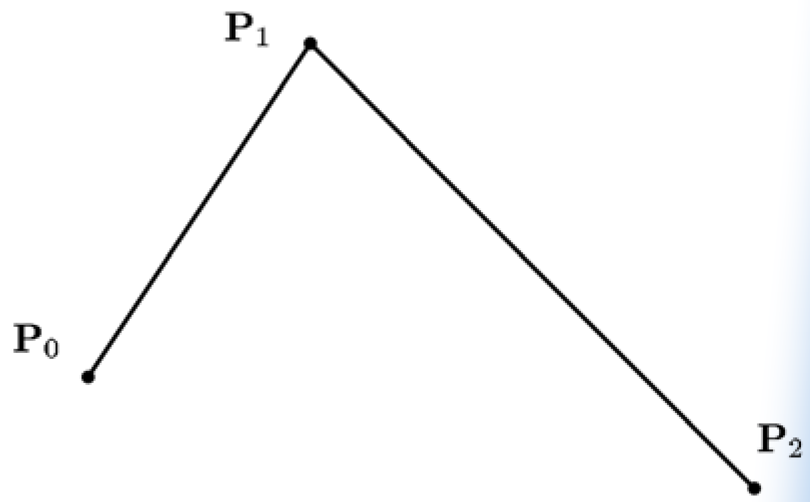
In the late 1960s, two European engineers independently developed a mathematical curve formulation which was extremely useful for modeling and design and also easily adaptable to use on a computer system.

- P. de Casteljau at Citroën
- P. Bézier at Rénault (1910~1999)
- A. R. Forrest

P. Bézier

The Subdivision Procedure

1. The curve is defined by using three control points $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$. Whereas these points can be arbitrarily placed in three-dimensional space
2. The curve will pass through the points $\mathbf{P}_0$ and $\mathbf{P}_2$ and will lie within the triangle $\triangle \mathbf{P}_0\mathbf{P}_1\ \mathbf{P}_2$
3. $\mathbf{P}_1$ will be a control point that serves as a ``handle'' or a ``influence'' on the curve



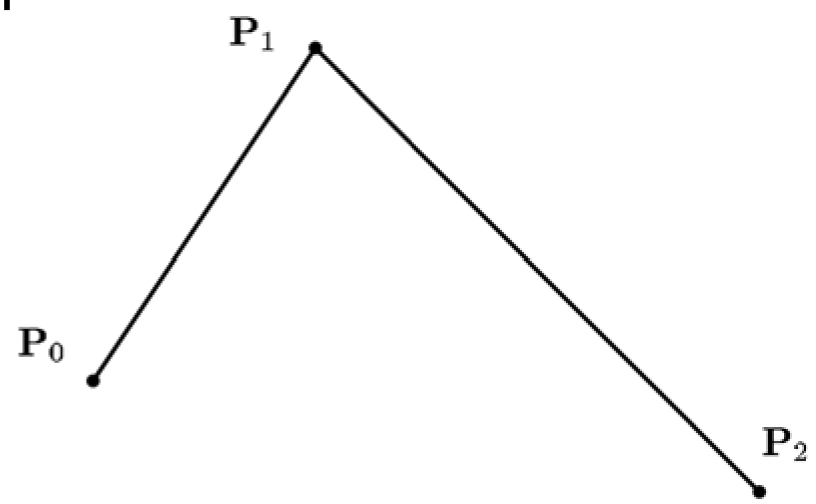
The Subdivision Procedure

4. Our general procedure will split the curve into two segments, each of which is again specified by three control points.

5. With this procedure, we can recursively generate many small segments of the curve, which can be eventually approximated by straight lines when the curve is to be drawn

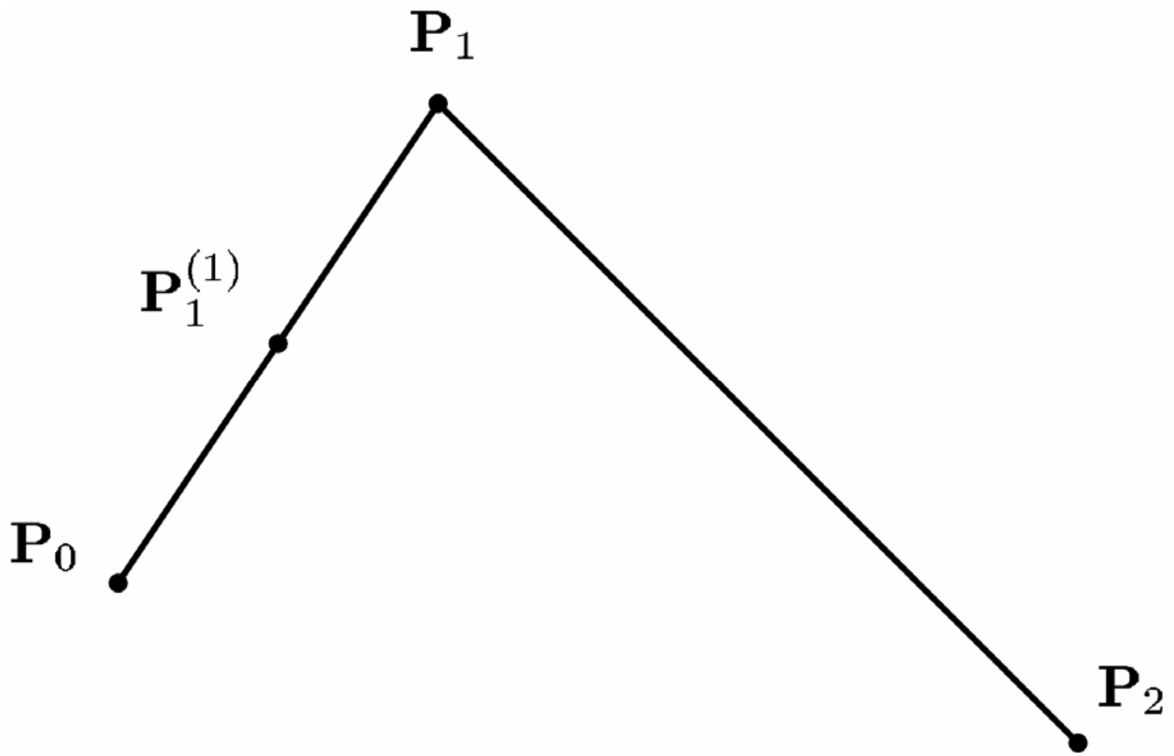
Note:

The most complicated mathematics being the calculation of midpoints of the lines connecting control points.



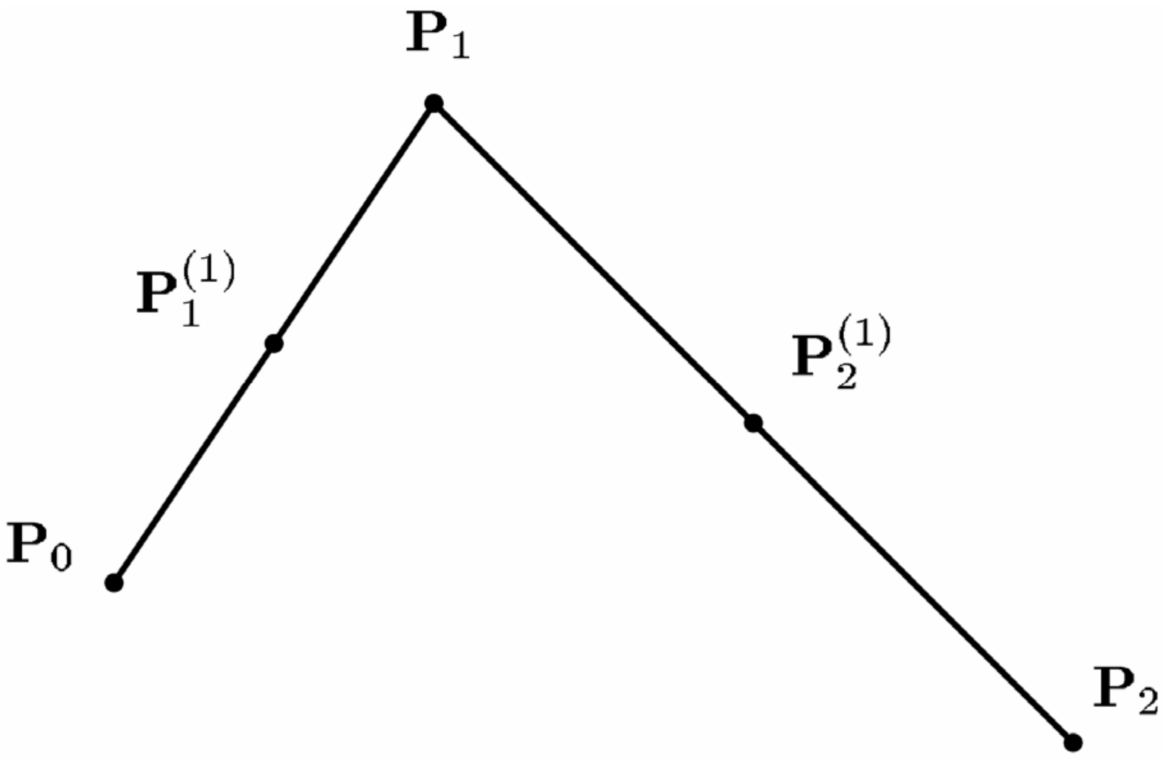
The Basic Subdivision Procedure (1)

First, let $\mathbf{P}_1^{(1)}$ be the midpoint of the segment $\overline{\mathbf{P}_0\mathbf{P}_1}$



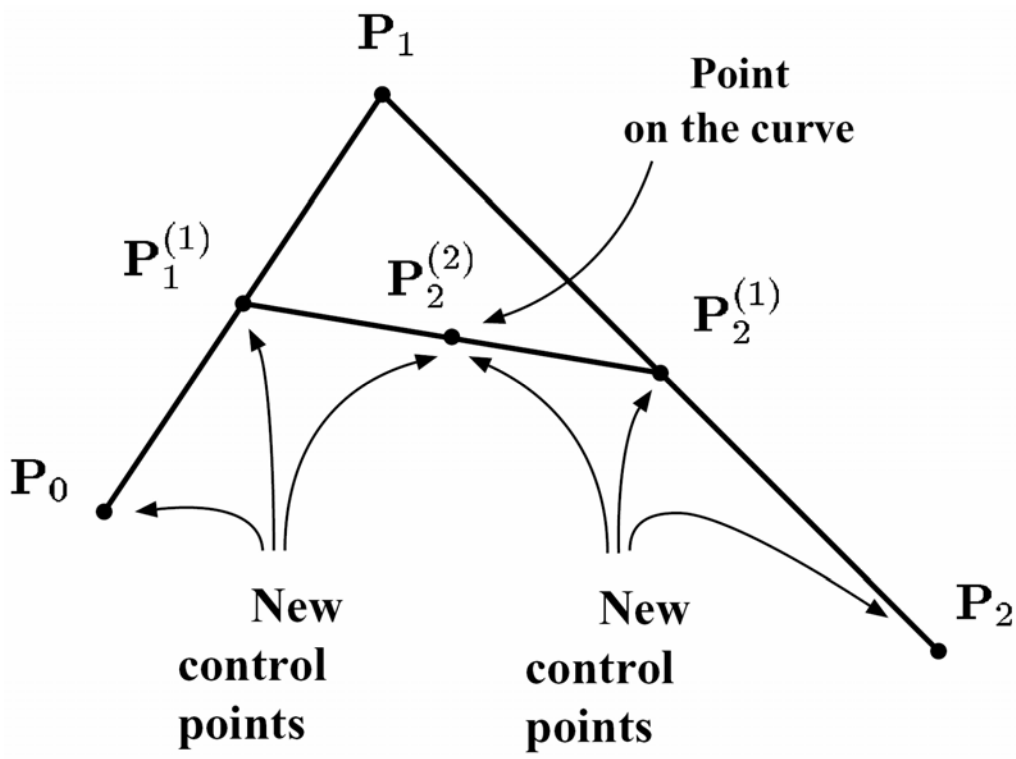
The Basic Subdivision Procedure (2)

let $\mathbf{P}_2^{(1)}$ be the midpoint of segment $\overline{\mathbf{P}_1\mathbf{P}_2}$



The Basic Subdivision Procedure (3)

finally let $\mathbf{P}_2^{(2)}$ be the midpoint of the segment $\overline{\mathbf{P}_1^{(1)}\mathbf{P}_2^{(1)}}$



The Basic Subdivision Procedure

- We define $\mathbf{P}_2^{(2)}$ to be a point on the curve
- The two new sets control points

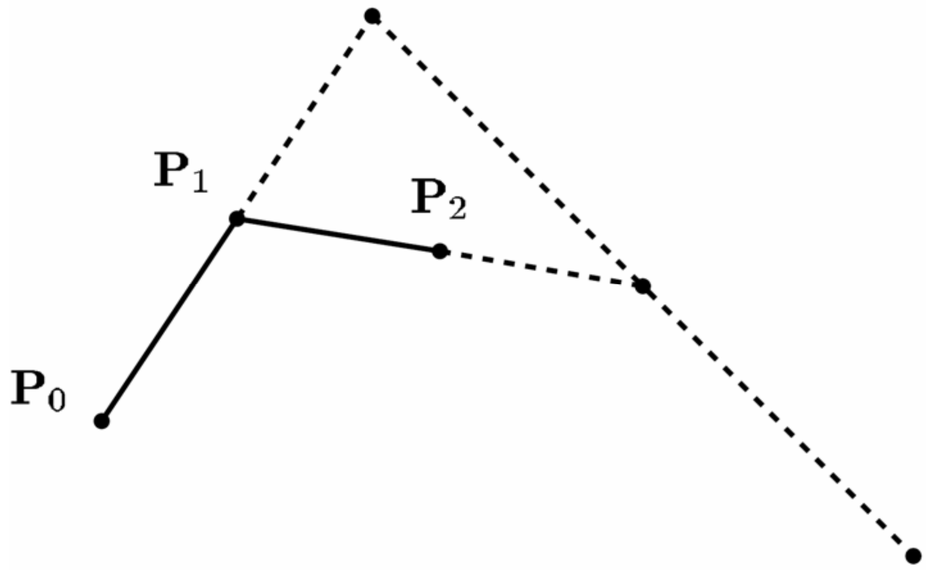
$$\left\{\mathbf{P}_0, \mathbf{P}_1^{(1)}, \mathbf{P}_2^{(2)}\right\} \quad \left\{\mathbf{P}_2^{(2)}, \mathbf{P}_2^{(1)}, \mathbf{P}_2\right\}$$

can be use to define the first and second portions of the subdivided curve.

- Result: an additional point on the curve + two new sets of three control points

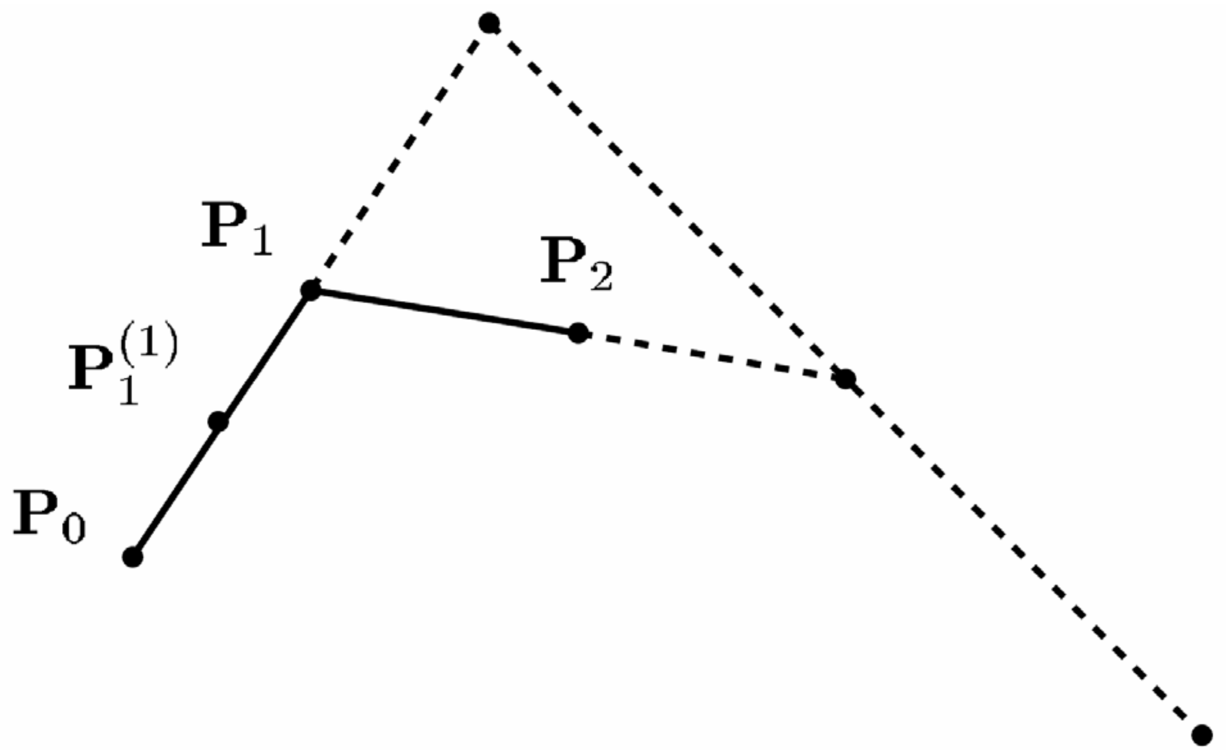
Continuing the Subdivision (Left)

- Performing the procedure again, we use the control points $\left\{\mathbf{P}_0, \mathbf{P}_1^{(1)}, \mathbf{P}_2^{(2)}\right\}$ and relabeling them for convenience as $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$



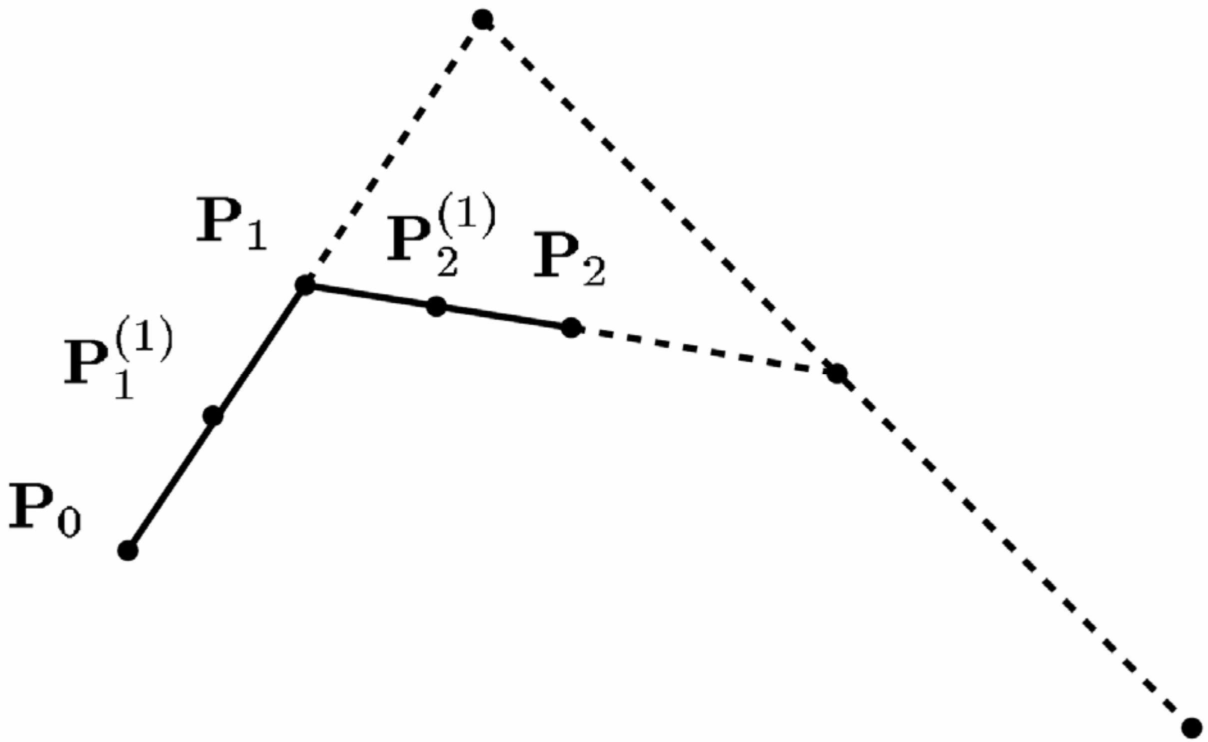
Continuing the Subdivision (Left-1)

First, let $\mathbf{P}_1^{(1)}$ be the midpoint of the segment $\overline{\mathbf{P}_0\mathbf{P}_1}$



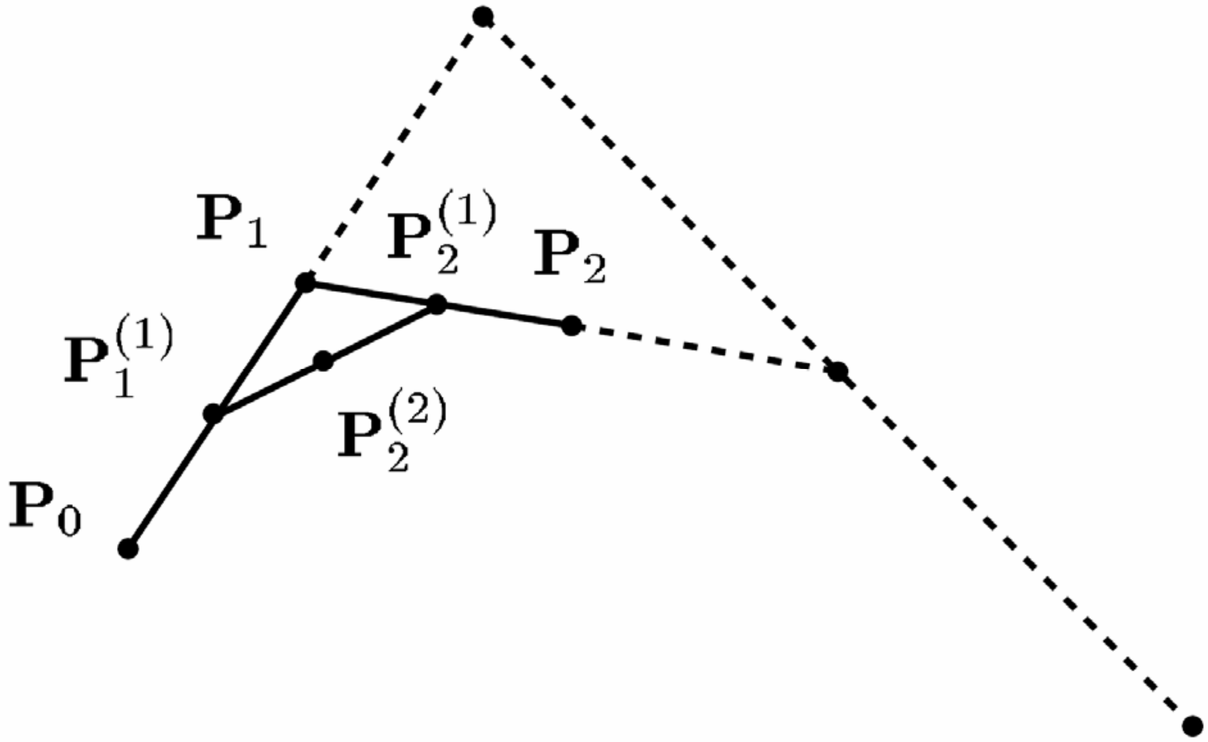
Continuing the Subdivision (Left-2)

let $\mathbf{P}_2^{(1)}$ be the midpoint of segment $\overline{\mathbf{P}_1\mathbf{P}_2}$



Continuing the Subdivision (Left-3)

finally let $\mathbf{P}_2^{(2)}$ be the midpoint of the segment $\overline{\mathbf{P}_1^{(1)}\mathbf{P}_2^{(1)}}$

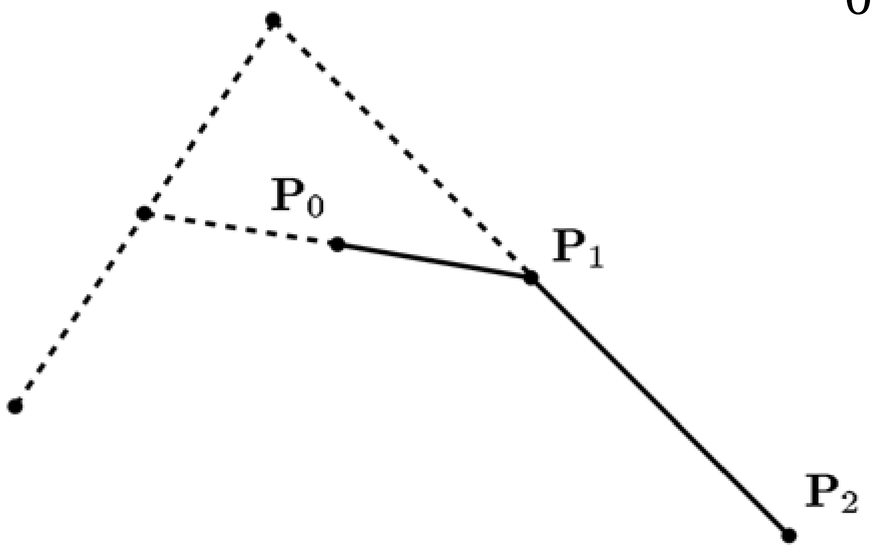


Continuing the Subdivision (Left)

- We now define $\mathbf{P}_2^{(2)}$ to be a point on the curve. This process produces another point on the curve, and creates two new sets of control points as was the case before.

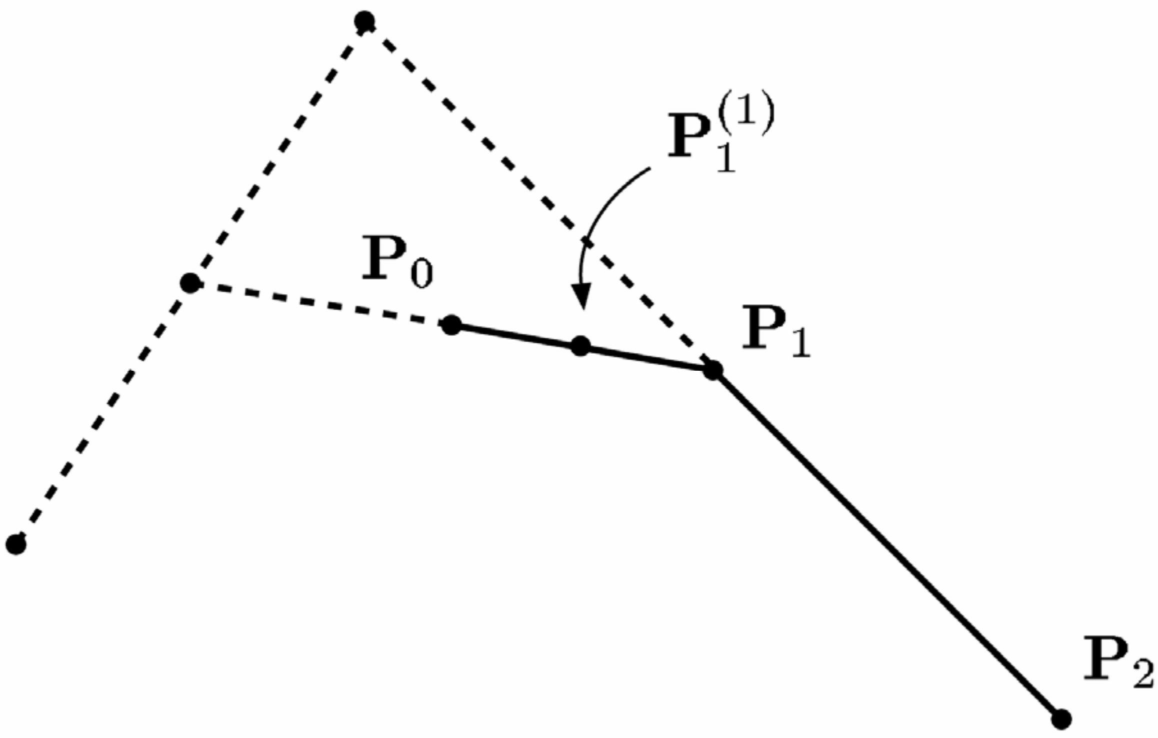
Continuing the Subdivision (Right)

- Next, we consider the control points $\left\{\mathbf{P}_2^{(2)}, \mathbf{P}_2^{(1)}, \mathbf{P}_2\right\}$ generated in the first subdivision and relabel them as $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$



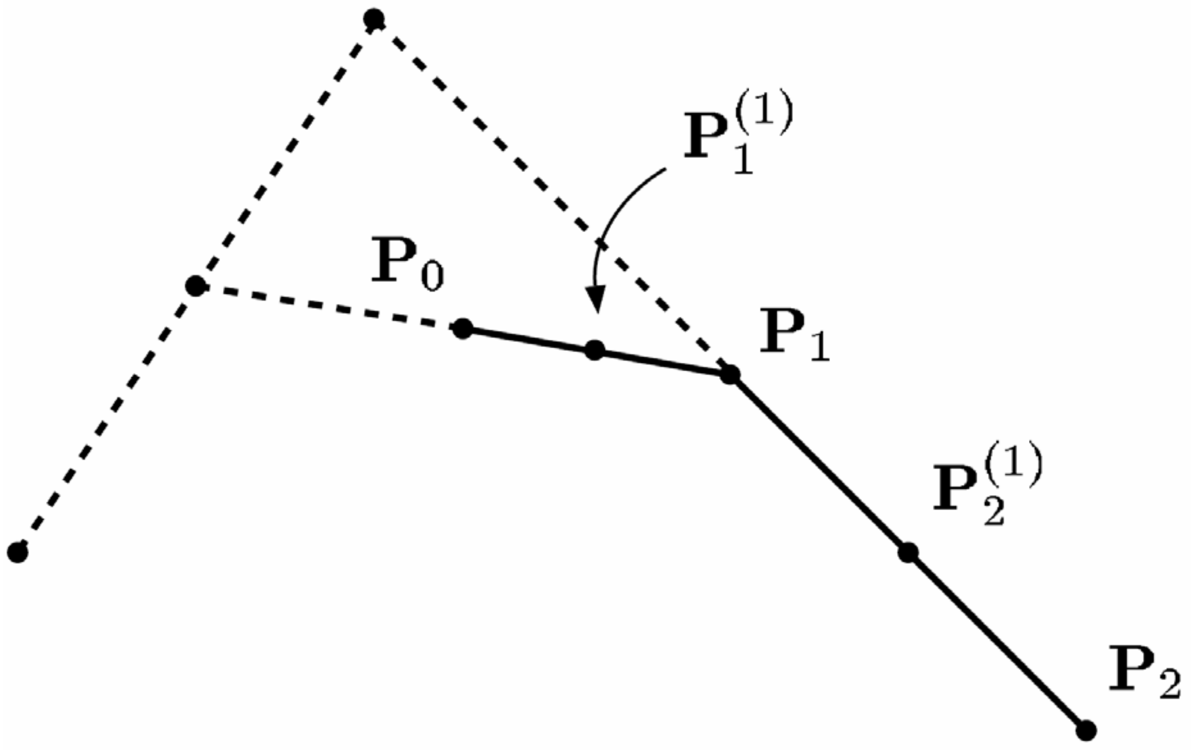
Continuing the Subdivision (Right-1)

First, let $\mathbf{P}_1^{(1)}$ be the midpoint of the segment $\overline{\mathbf{P}_0\mathbf{P}_1}$



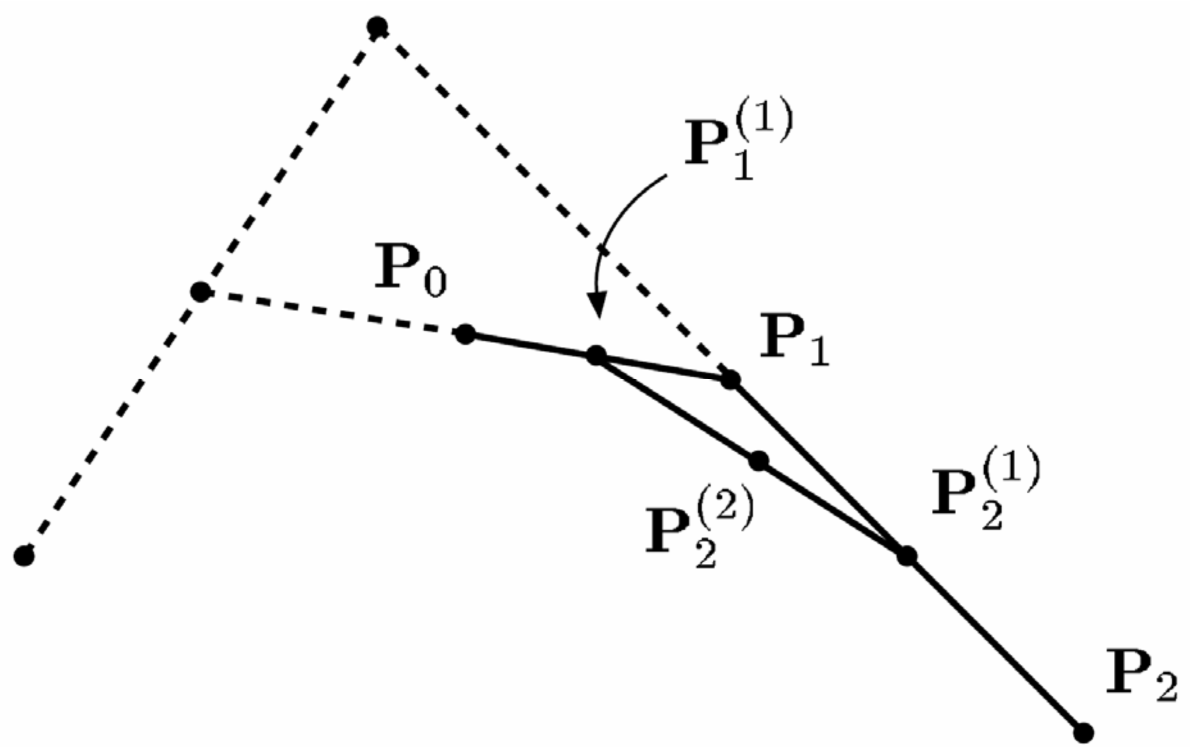
Continuing the Subdivision (Right-2)

let $\mathbf{P}_2^{(1)}$ be the midpoint of segment $\overline{\mathbf{P}_1\mathbf{P}_2}$



Continuing the Subdivision (Right-3)

finally let $\mathbf{P}_2^{(2)}$ be the midpoint of the segment $\overline{\mathbf{P}_1^{(1)}\mathbf{P}_2^{(1)}}$



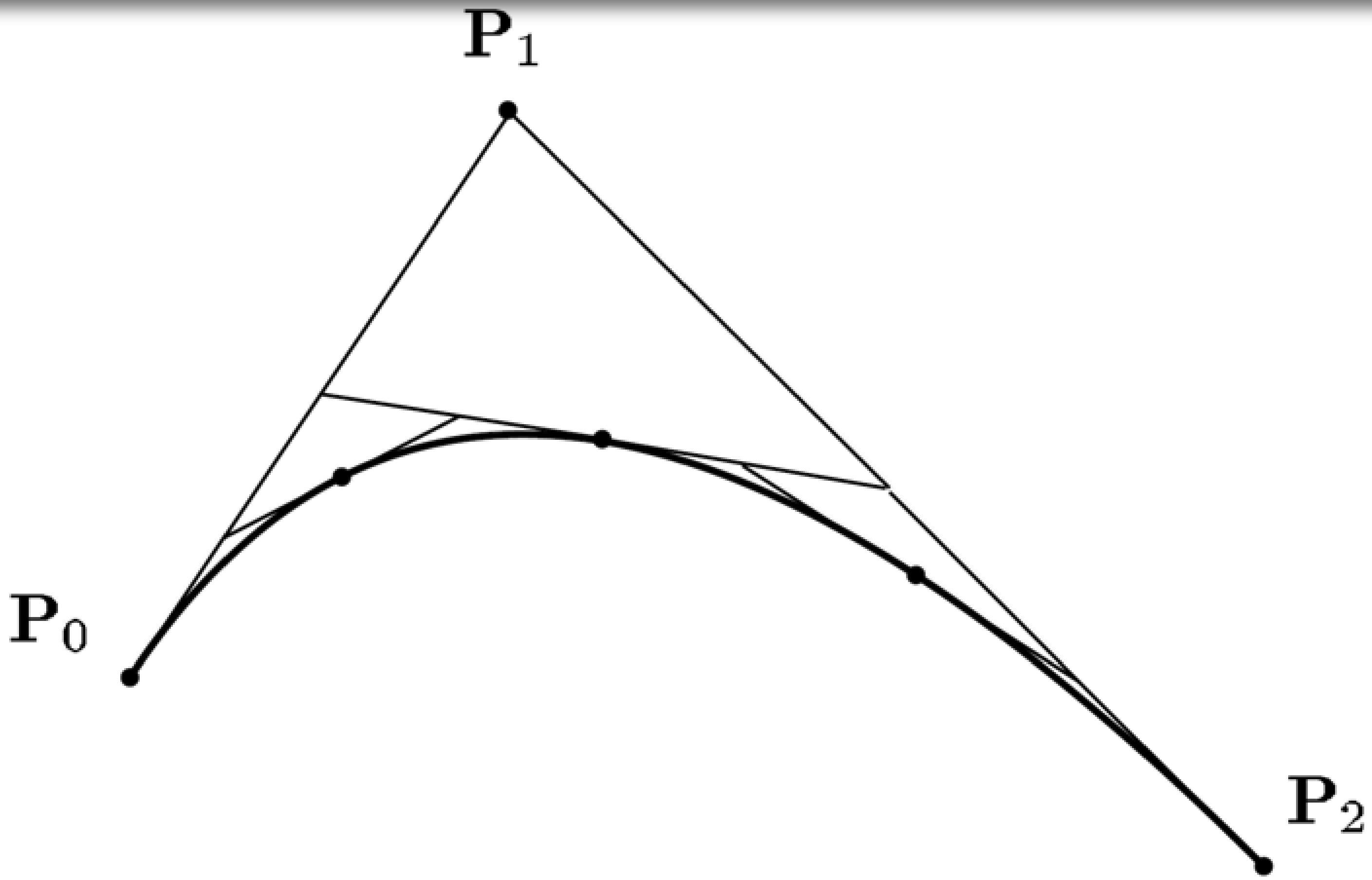
Continuing the Subdivision (Right)

- We now have $\mathbf{P}_2^{(2)}$ on the curve .

The Subdivision Algorithm

- Three points have now been generated on the curve and four subcurves have been generated.
- At each step the process creates both a point on the curve and two new sets of control points.
- This effectively subdivides the curve into two new curve segments, each of which can be handled separately.

The Subdivision Algorithm



Summary

- It is a *geometric* method, as it uses only the midpoint formula as it's fundamental tool.
- It uses the basic computer science paradigm of (sub)divide and conquer to calculate points on the curve.
- The curve can be ``drawn'' using computer graphics by calculating a somewhat-dense set of points, and connecting them with straight lines.
- The curve drawn by this method is a quadratic Bézier curve.

Quadratic Bezier Curves

- Development of the Quadratic Bézier Curve
- Developing the Equation of the Curve
- Properties of the Quadratic Curve
- Summarizing the Development of the Curve

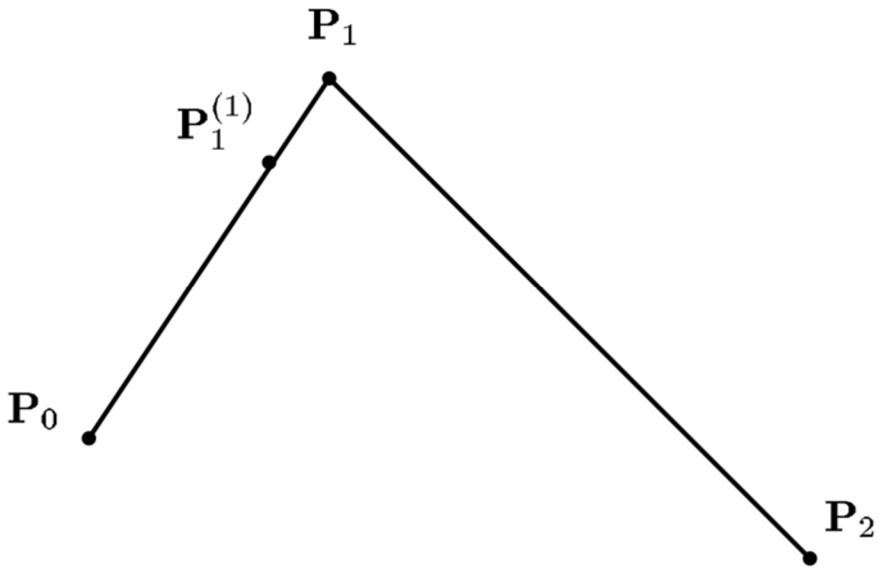
Development of the Quadratic Bézier Curve

- Given three control points $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$, we develop a divide procedure that is based upon a parameter t , which is a number between 0 and 1 (the illustrations utilize the value 0.75).

Development of the Quadratic Bézier Curve (1)

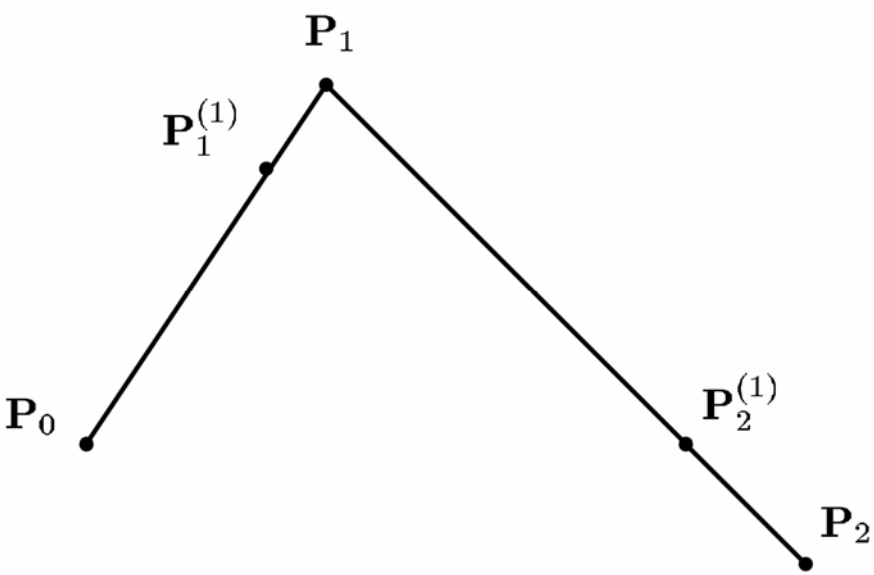
- Let $\mathbf{P}_1^{(1)}$ be the point on the segment $\overline{\mathbf{P}_0\mathbf{P}_1}$ defined by

$$\mathbf{P}_1^{(1)} = (1-t)\mathbf{P}_0 + t\mathbf{P}_1 = \mathbf{P}_0 + t(\mathbf{P}_1 - \mathbf{P}_0)$$



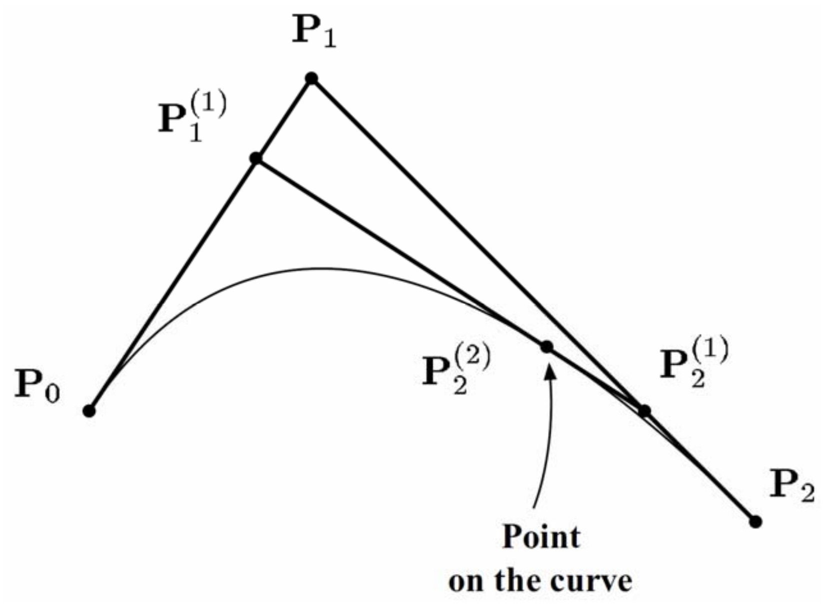
Development of the Quadratic Bézier Curve (2)

- Let $\mathbf{P}_2^{(1)}$ be the point on the segment $\overline{\mathbf{P}_1\mathbf{P}_2}$ defined by $$\mathbf{P}_2^{(1)} = (1-t)\mathbf{P}_1 + t\mathbf{P}_2$$



Development of the Quadratic Bézier Curve (3)

- Let $\mathbf{P}_2^{(2)}$ be the point on the segment $\overline{\mathbf{P}_1^1\mathbf{P}_2^2}$ defined by $\mathbf{P}_2^{(2)} = (1-t)\mathbf{P}_1^{(1)} + t\mathbf{P}_2^{(1)}$



Development of the Quadratic Bézier Curve (4)

- Define $\mathbf{P}(t) = \mathbf{P}_2^{(2)}$

- Note

1. It is a geometric mean to define points on the curve.
2. It is identical to the divide-and-conquer method in the case t=1/2.

Developing the Equation of the Curve

- There is a parameter t involved in the above steps
  - $\mathbf{P}_1^{(1)}$, $\mathbf{P}_2^{(1)}$ and $\mathbf{P}_2^{(2)}$ is really a function of the parameter t.
  - $\mathbf{P}_2^{(2)}$ can be equated with $\mathbf{P}(t)$ since it is a point on the curve that corresponds to the parameter value t.

Developing the Equation of the Curve

$$
\begin{aligned}
\mathbf{P}(t) &= \mathbf{P}_2^{(2)}(t) \\
&= (1-t)\mathbf{P}_1^{(1)}(t) + t\mathbf{P}_2^{(1)}(t)
\end{aligned}
$$

where

$$
\mathbf{P}_1^{(1)}(t) = (1-t)\mathbf{P}_0 + t\mathbf{P}_1
$$

$$
\mathbf{P}_2^{(1)}(t) = (1-t)\mathbf{P}_1 + t\mathbf{P}_2
$$

Substituting these two equations back into the original, we have

Developing the Equation of the Curve

$$
\begin{aligned}
\mathbf{P}(t) &= (1-t)\mathbf{P}_1^{(1)}(t) + t\mathbf{P}_2^{(1)}(t) \\
&= (1-t)\left[(1-t)\mathbf{P}_0 + t\mathbf{P}_1\right] + t\left[(1-t)\mathbf{P}_1 + t\mathbf{P}_2\right] \\
&= (1-t)^2\mathbf{P}_0 + (1-t)t\mathbf{P}_1 + t(1-t)\mathbf{P}_1 + t^2\mathbf{P}_2 \\
&= (1-t)^2\mathbf{P}_0 + 2t(1-t)\mathbf{P}_1 + t^2\mathbf{P}_2
\end{aligned}
$$

- This is quadratic polynomial (as it is a linear combination of quadratic polynomials), and therefore it is a parabolic segment.
- The quadratic Bézier curve is simply a parabolic curve.

Properties of the Quadratic Curve (1)

1. $\mathbf{P}(0)=\mathbf{P}_0$ and $\mathbf{P}(1)=\mathbf{P}_2$, so the curve passes through the control points $\mathbf{P}_0$ and $\mathbf{P}_2$.
2. The curve $\mathbf{P}(t)$ is continuous and has continuous derivatives of all orders. (This is automatic for a polynomial)

Properties of the Quadratic Curve (2)

3. We can differentiate $\mathbf{P}(t)$ with respect to t and obtain

$$\begin{aligned}\frac{d}{dt}\mathbf{P}(t) &= -2(1-t)\mathbf{P}_0 + \left[-2t + 2(1-t)\right]\mathbf{P}_1 + 2t\mathbf{P}_2 \\ &= 2\left[(1-t)\left(\mathbf{P}_1 - \mathbf{P}_0\right) + t\left(\mathbf{P}_2 - \mathbf{P}_1\right)\right]\end{aligned}$$

Thus

$$\frac{d}{dt}\mathbf{P}(0) = 2(\mathbf{P}_1 - \mathbf{P}_0)$$

$$\frac{d}{dt}\mathbf{P}(1) = 2(\mathbf{P}_2 - \mathbf{P}_1)$$

Properties of the Quadratic Curve (3)

4. The functions $(1-t)^2$, $2t(1-t)$ and t^2 that are used to "blend" the control points $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$ together are the degree-2 Bernstein Polynomials. They are all non-negative functions and sum to one.

5. The curve is contained within the triangle $\triangle\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2$. Since
   - $\mathbf{P}(t)$ is a convex combination of the points $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$.
   - The convex hull of a triangle is the triangle itself.

Properties of the Quadratic Curve (4)

6. If the points $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$ are colinear, then the curve is a straight line.

7. The process of calculating one $\mathbf{P}(t)$ subdivides the control points into two sets $\left\{\mathbf{P}_0, \mathbf{P}_1^{(1)}, \mathbf{P}_2^{(2)}\right\}$ and $\left\{\mathbf{P}_2^{(2)}, \mathbf{P}_2^{(1)}, \mathbf{P}_2\right\}$, each of which can be used to define another curve, as in our subdivision process above.

Properties of the Quadratic Curve (4)

8. All the points, generated from the divide-and-conquer method, lie on this curve.

Summarizing the Development of the Curve

- We now have two methods by which we can generate points on the curve
  - The first of which is geometrically based - points are found on the curve by selecting successive points on line segments.
  - The other is an analytic formula, which expresses the curve in functional notation.

The Geometrical Construction Method

- Given points $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$, we can construct a curve $\mathbf{P}(t)$ by the following construction

where $$\mathbf{P}(t) = \mathbf{P}_2^{(2)}$$

$$\mathbf{P}_i^{(j)}(t) = \begin{cases} (1-t)\mathbf{P}_{i-1}^{(j-1)}(t) + t\mathbf{P}_i^{(j-1)}(t) & \text{if } j > 0 \\ \mathbf{P}_i & \text{if } j = 0 \end{cases}$$

for $t \in [0,1]$

The Analytical Formula

- Given points $\mathbf{P}_0$, $\mathbf{P}_1$ and $\mathbf{P}_2$, we can construct a curve $\mathbf{P}(t)$ by the following

$$\mathbf{P}(t) = (1-t)^2\mathbf{P}_0 + 2t(1-t)\mathbf{P}_1 + t^2\mathbf{P}_2$$

for $t \in [0,1]$

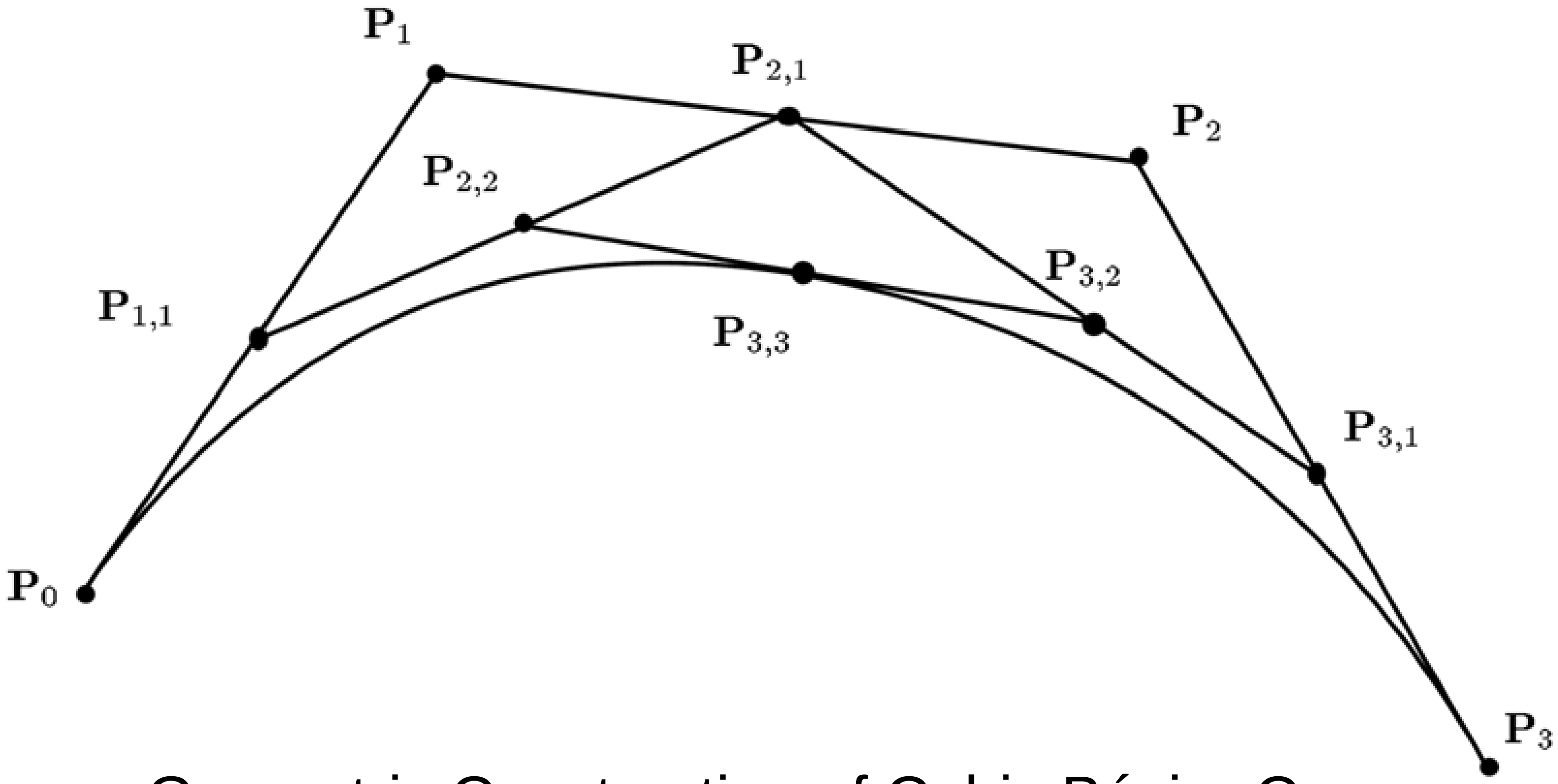
Cubic Bézier Curve – Geometric Construction

Defining cubic Bézier curve trough geometric construction:

Given four control points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, one can generate a curve $\mathbf{P}(t)$ at the parameter t as following

- let $\mathbf{P}_1^{(1)}(t) = t\mathbf{P}_1 + (1-t)\mathbf{P}_0$
- let $\mathbf{P}_2^{(1)}(t) = t\mathbf{P}_2 + (1-t)\mathbf{P}_1$
- let $\mathbf{P}_3^{(1)}(t) = t\mathbf{P}_3 + (1-t)\mathbf{P}_2$
- let $\mathbf{P}_2^{(2)}(t) = t\mathbf{P}_2^{(1)}(t) + (1-t)\mathbf{P}_1^{(1)}(t)$
- let $\mathbf{P}_3^{(2)}(t) = t\mathbf{P}_3^{(1)}(t) + (1-t)\mathbf{P}_2^{(1)}(t)$
- let $\mathbf{P}_3^{(3)}(t) = t\mathbf{P}_3^{(2)}(t) + (1-t)\mathbf{P}_2^{(2)}(t)$
- $\mathbf{P}_3^{(3)}(t)$ is defined to be $\mathbf{P}(t)$

Cubic Bézier Curve – Geometric Construction



Geometric Construction of Cubic Bézier Curve

Cubic Bézier Curve – Geometric Construction

Given four control points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, one can generate a curve $\mathbf{P}(t)$ ($t \in [0,1]$) by

$$\mathbf{P}(t) = \mathbf{P}_3^{(3)}(t)$$

where

$$\mathbf{P}_i^{(j)}(t) = \begin{cases} (1-t)\mathbf{P}_{i-1}^{(j-1)}(t) + t\mathbf{P}_i^{(j-1)}(t) & \text{if } j > 0 \\ \mathbf{P}_i & \text{otherwise} \end{cases}$$

Cubic Bézier Curve –Analytical Construction

Expanding above geometric construction about parameter t

$$
\begin{aligned}
\mathbf{P}(t) &= \mathbf{P}_3^{(3)}(t) \\
&= t\mathbf{P}_3^{(2)}(t) + (1-t)\mathbf{P}_2^{(2)}(t) \\
&= t\left[t\mathbf{P}_3^{(1)}(t) + (1-t)\mathbf{P}_2^{(1)}(t)\right] \\
&\quad + \ (1-t)\left[t\mathbf{P}_2^{(1)}(t) + (1-t)\mathbf{P}_1^{(1)}(t)\right] \\
&= t^2\mathbf{P}_3^{(1)}(t) + 2t(1-t)\mathbf{P}_2^{(1)}(t) + (1-t)^2\mathbf{P}_1^{(1)}(t) \\
&= t^2\left[t\mathbf{P}_3 + (1-t)\mathbf{P}_2\right] + 2t(1-t)\left[t\mathbf{P}_2 + (1-t)\mathbf{P}_1\right] \\
&\quad + \ (1-t)^2\left[t\mathbf{P}_1 + (1-t)\mathbf{P}_0\right] \\
&= t^3\mathbf{P}_3 + 3t^2(1-t)\mathbf{P}_2 + 3t(1-t)^2\mathbf{P}_1 + (1-t)^3\mathbf{P}_0
\end{aligned}
$$

Cubic Bézier Curve –Analytical Construction

Given four control points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, we define the Bézier curve to be

$$\mathbf{P}(t) = \sum_{i=0}^{3} \mathbf{P}_i B_{i,3}(t)$$

where

$$B_{0,3}(t) = (1-t)^3 \qquad B_{2,3}(t) = 3t^2(1-t)$$

$$B_{1,3}(t) = 3t(1-t)^2 \qquad B_{3,3}(t) = t^3$$

the Bernstein polynomials of degree three

Properties of the Cubic Bézier Curve

1. $\mathbf{P}_0$ and $\mathbf{P}_3$ are on the curve.
2. The curve is continuous, infinitely differentiable, and the second derivatives are continuous (automatic for a polynomial curve).
3. The tangent line to the curve at the point $\mathbf{P}_0$ is the line $\overline{\mathbf{P}_0\mathbf{P}_1}$. The tangent to the curve at the point $\mathbf{P}_3$ is the line $\overline{\mathbf{P}_2\mathbf{P}_3}$.

Properties of the Cubic Bézier Curve

- The curve lies within the convex hull of its control points.
- Both $\mathbf{P}_1$ and $\mathbf{P}_2$ are on the curve only if the curve is linear.

A Matrix Representation for Cubic Bezier Curves

Overview

- Purposes of matrix representation
  - Fast computation of matrices multiplication
  - Generating different Bézier control polygons for the cubic curve

Developing the Matrix Equation

- A cubic Bézier Curve can be written in a matrix form
  1. Expanding the analytic definition of the curve into its Bernstein polynomial coefficients,
  2. Then writing these coefficients in a matrix form using the polynomial power basis.

Developing the Matrix Equation

$$
\begin{aligned}
\mathbf{P}(t) &= \sum_{i=0}^{3} \mathbf{P}_i B_i(t) \\
&= (1-t)^3\mathbf{P}_0 + 3t(1-t)^2\mathbf{P}_1 + 3t^2(1-t)\mathbf{P}_2 + t^3\mathbf{P}_3 \\
&= \begin{bmatrix} (1-t)^3 & 3t(1-t)^2 & 3t^2(1-t) & t^3 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}
\end{aligned}
$$

Developing the Matrix Equation

$$\begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

- The matrix M defines the blending functions for the curve $\mathbf{P}(t)$ - i.e. the cubic Bernstein polynomials
- Utilizing equipment that is designed for fast 4×4 matrix calculations, this formulation can be used to quickly calculate points on the curve.

Subdivision Using the Matrix Form

- Suppose we wish to generate the control polygon for the portion of the curve $\mathbf{P}(t)$ where t ranges between 0 and $1/2$
  - Clearly this new curve is a cubic polynomial, and traces the desired portion of $\mathbf{P}$ as t ranges between 0 and 1
  - For the control points of the new curve $\mathbf{Q}(t)$
    - Geometric Construction
    - Using matrix form of the curve $\mathbf{P}$

Subdivision Using the Matrix Form

$$\mathbf{Q}(t) = \mathbf{P}(\frac{t}{2})$$

$$= \begin{bmatrix} 1 & (\frac{t}{2}) & (\frac{t}{2})^2 & (\frac{t}{2})^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} MS_{[0,\frac{1}{2}]} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

Subdivision Using the Matrix Form

$$
\begin{aligned}
S_{[0,\frac{1}{2}]} &= M^{-1}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} M \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & 0 & 0 \\ 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{6} & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{12} & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}
\end{aligned}
$$

Subdivision Using the Matrix Form

- The control points of the Bézier curve $\mathbf{Q}(t)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_0 \\ \frac{1}{2}\mathbf{P}_1 + \frac{1}{2}\mathbf{P}_0 \\ \frac{1}{4}\mathbf{P}_2 + \frac{1}{2}\mathbf{P}_1 + \frac{1}{4}\mathbf{P}_0 \\ \frac{1}{8}\mathbf{P}_3 + \frac{3}{8}\mathbf{P}_2 + \frac{3}{8}\mathbf{P}_1 + \frac{1}{8}\mathbf{P}_0 \end{bmatrix}$$

Subdivision Using the Matrix Form

- Similarly, the Bézier control polygon for the second half of the curve $t \in [1/2, 1]$ can be obtained as following

Subdivision Using the Matrix Form

$$
\begin{aligned}
\mathbf{Q}(t) &= \mathbf{P}\left(\frac{1}{2}+\frac{t}{2}\right) \\
&= \begin{bmatrix} 1 & (\frac{1}{2}+\frac{t}{2}) & (\frac{1}{2}+\frac{t}{2})^2 & (\frac{1}{2}+\frac{t}{2})^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & \frac{1}{4} & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} MS_{[\frac{1}{2},1]} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}
\end{aligned}
$$

Subdivision Using the Matrix Form

$$S_{[\frac{1}{2},1]} = \begin{bmatrix} \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix can be applied to the original Bézier control points to produce Bézier control points for the second half of the curve.

Generating a Sequence of Bézier Control Polygons

- Using matrix calculations similar to those above, we can generate an iterative scheme to generate a sequence of points on the curve

- If we consider the portion of the cubic curve where $\mathbf{P}(t)$ ranges between 1 and 2 , we generate the Bézier control points of $\mathbf{Q}(t)$ by reparameterization of the original curve - namely by replacing t by $t+1$

Generating a Sequence of Bézier Control Polygons

$$\mathbf{Q}(t) = \mathbf{P}(t+1)$$

$$= \begin{bmatrix} 1 & (t+1) & (t+1)^2 & (t+1)^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} MS_{[1,2]} \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix}$$

Generating a Sequence of Bézier Control Polygons

$$S_{[1,2]} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -4 & 4 \\ -1 & 6 & -12 & 8 \end{bmatrix}$$

Generating a Sequence of Bézier Control Polygons

- Now, using a combination of $S_{[0,1/2]}$, $S_{[1/2,1]}$ and $S_{[1,\ 2]}$, we can produce Bézier control polygons along the curve similar to methods developed with divided differences.

$$S_{[1,2]}S_{[0,\frac{1}{2}]} = S_{[\frac{1}{2},1]}$$

Generating a Sequence of Bézier Control Polygons

$$S_{[1,2]}S_{[0,\frac{1}{2}]} = S_{[\frac{1}{2},1]}$$

- Applying $S_{[0,1/2]}$ to obtain a Bézier control polygon for the first half of the curve
- Applying $S_{[1, 2]}$ to this control polygon to obtain the Bézier control polygon for the second half of the curve

Generating a Sequence of Bézier Control Polygons

- Consider $S^i_{[1,2]}S^k_{[0,\frac{1}{2}]}$: applying $S_{[0,1/2]}$ k times and then $S_{[1,2]}$ i times
  - obtain the Bézier control polygon for the portion of the curve where t ranges in the interval $[i/2^k, (i+1)/2^k]$
- By repeatedly applying $S_{[1,2]}$, we move our control polygons along the curve