Preliminary Mathematics of Geometric Modeling (4)

Hongxin Zhang and Jieqing Feng

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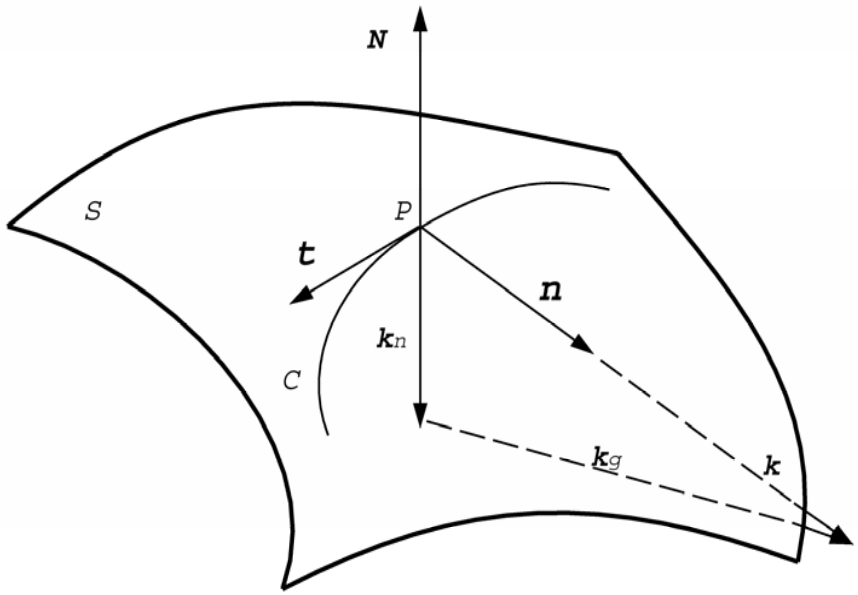
State Key Lab of CAD&CG
Zhejiang University

Differential Geometry of Surfaces

- Tangent plane and surface normal
- First fundamental form *I* (metric)
- Second fundamental form *II* (curvature)
- Principal curvatures
- Gaussian and mean curvatures
  - Explicit surfaces
  - Implicit surfaces
- Euler's theorem

Principal curvatures

Principal curvature: the extrema of normal curvature



Definition of normal curvature

Principal curvatures

The normal curvature is determined by $\lambda = \frac{dv}{du}$

$$\kappa_n = \frac{II}{I} = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2}$$

The extrema of the κ_n is arrived at $\frac{d\kappa_n}{d\lambda} = 0$

$$(E + 2F\lambda + G\lambda^2)(N\lambda + M) - (L + 2M\lambda + N\lambda^2)(G\lambda + F) = 0$$

Principal curvatures

Hence

$$\kappa_n = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} = \frac{M + N\lambda}{F + G\lambda}$$

Extreme case

Since

$$E + 2F\lambda + G\lambda^2 = (E + F\lambda) + \lambda(F + G\lambda)$$
$$L + 2M\lambda + N\lambda^2 = (L + M\lambda) + \lambda(M + N\lambda)$$

Principal curvatures

$$(E + 2F\lambda + G\lambda^2)(N\lambda + M) - (L + 2M\lambda + N\lambda^2)(G\lambda + F) = 0$$

$$(E + F\lambda)(M + N\lambda) = (L + M\lambda)(F + G\lambda)$$

Hence

$$\kappa_n = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} = \frac{M + N\lambda}{F + G\lambda} = \frac{L + M\lambda}{E + F\lambda}$$

Extreme case

The extreme values of κ_n satisfy:

$$(L - \kappa_n E)du + (M - \kappa_n F)dv = 0$$

$$(M - \kappa_n F)du + (N - \kappa_n G)dv = 0$$

Principal curvatures

The homogeneous linear system of equations for *du*, *dv* have a nontrivial solution iff

$$\begin{vmatrix} L - \kappa_n E & M - \kappa_n F \\ M - \kappa_n F & N - \kappa_n G \end{vmatrix} = 0$$

expanding

$$(EG - F^2)\kappa_n^2 - (EN + GL - 2FM)\kappa_n + (LN - M^2) = 0$$

Principal curvatures

Discriminant D of above quadratic equation in κ_n

$$D = 4\left(\frac{EG-F^2}{E^2}\right)(EM-FL)^2 + \left(EN-GL-\frac{2F}{E}(EM-FL)\right)^2 \geq 0$$

- $D > 0$: two extreme normal curvatures (max/min)

  iff

- $D = 0 \Leftrightarrow \quad EM - FL = 0$ and $EN - GL = 0$

  $\Leftrightarrow \quad \exists$ const. k, s.t. $L = kE, \quad M = kF, \quad N = kG$

  - Umbilic point: the normal curvature is the same in all directions.

Principal curvatures

Let

$$K = \frac{LN - M^2}{EG - F^2}$$

Gaussian (Gauss) curvature

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

Mean curvature

Then

$$(EG - F^2)\kappa_n^2 - (EN + GL - 2FM)\kappa_n + (LN - M^2) = 0$$

$$\kappa_n^2 - 2H\kappa_n + K = 0$$

Principal curvatures

Solving

$$\kappa_n^2 - 2H\kappa_n + K = 0$$

We have

$$\kappa_{max} = H + \sqrt{H^2 - K}$$
$$\kappa_{min} = H - \sqrt{H^2 - K}$$

- κ_{max} and κ_{min} are maximum and minimum principle curvature
- The directions in the tangent plane for which κ_n takes maximum and minimum values are called *principal directions*

Principal curvatures

Computation of principal directions

$$\kappa_n = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} = \frac{M + N\lambda}{F + G\lambda} = \frac{L + M\lambda}{E + F\lambda}$$

$$\lambda = -\frac{M - \kappa_n F}{N - \kappa_n G} \qquad \lambda = -\frac{L - \kappa_n E}{M - \kappa_n F}$$

- κ_n takes either κ_{max} or κ_{min}

Principal curvatures

$$\kappa_{max} = H + \sqrt{H^2 - K}$$
$$\kappa_{min} = H - \sqrt{H^2 - K}$$

- When $D=0$ or $H^2=K$, κ_n is a double root

$$\kappa_{max} = \kappa_{min} = H$$

  - The point is an umbilical point, which is locally a part of sphere with radius of curvature
- When $K=H=0$, the point is an flat/planar point

Principal directions

$$(E+2F\lambda+G\lambda^2)(N\lambda+M)-(L+2M\lambda+N\lambda^2)(G\lambda+F)=0$$

removing cubic term about λ

$$(FN-GM)\lambda^2+(EN-GL)\lambda+(EM-FL)=0$$

- The discriminant is same with that of the principal curvature as follows

$$D=4\left(\frac{EG-F^2}{E^2}\right)(EM-FL)^2+\left(EN-GL-\frac{2F}{E}(EM-FL)\right)^2 \geq 0$$

D=0 → $L=kE, \quad M=kF, \quad N=kG$

FN=*GM*, *EN*=*GL* and *EM*=*FL* →

Umbilical point: the principal directions are not defined!

Principal directions

When discriminant $D>0$, the principal directions λ_{max} and λ_{min} satisfy

$$(FN - GM)\lambda_{max}^2 + (EN - GL)\lambda_{max} + (EM - FL) = 0$$
$$(FN - GM)\lambda_{min}^2 + (EN - GL)\lambda_{min} + (EM - FL) = 0$$

$$\lambda_{max} + \lambda_{min} = -\frac{EN - GL}{FN - GM}$$
$$\lambda_{max}\lambda_{min} = \frac{EM - FL}{FN - GM},$$

Principal directions

Recall the orthogonal condition for two tangent vectors

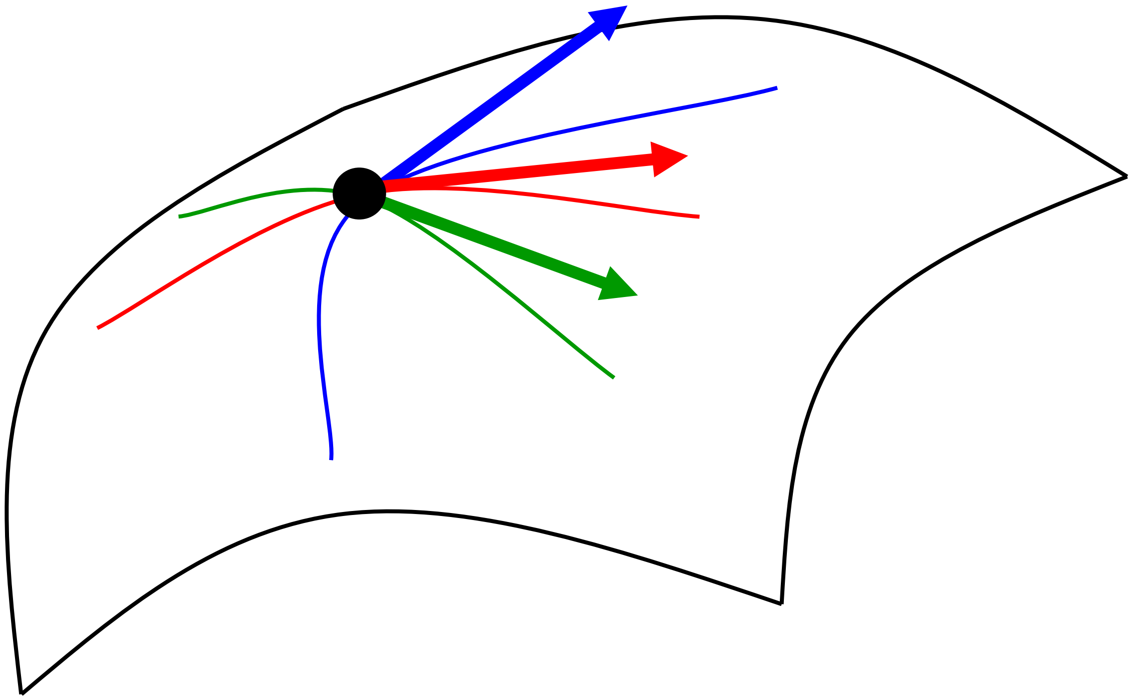
$$Edu_1du_2 + F(du_1dv_2 + dv_1du_2) + Gdv_1dv_2 = 0$$

Substitute above principal directions into it

$$E + F(\lambda_{max} + \lambda_{min}) + G\lambda_{max}\lambda_{min}$$
$$= \frac{1}{FN - GM}[E(FN - GM) - F(EN - GL) + G(EM - FL)]$$
$$= 0$$

The two principal directions are orthogonal !

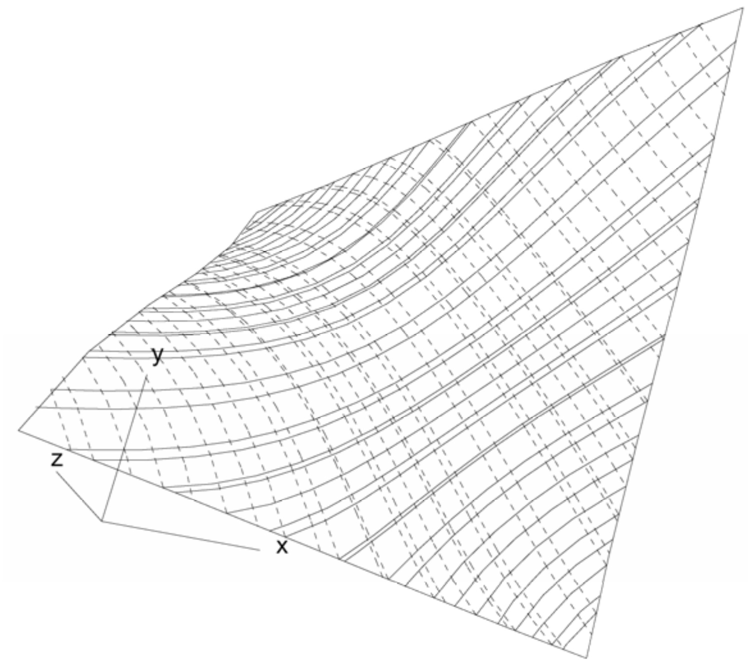
Principal curvatures and directions



Directions at one points on the surface

An example of line of curvature

- *Line of curvature*: a curve on a surface whose tangent at each point is in a principal direction



Example of lines of curvature:
Solid lines: maximum principal direction
Dashed lines: minimum principal direction

Lines of curvature

- Surface parametrization: orthogonal net of lines
  - Lines of curvature
  - The sufficient and necessary condition for the parametric lines to be lines of curvature

$$F = M = 0$$

  - Recall that the arc length parameter for curve

Example: principal curvature

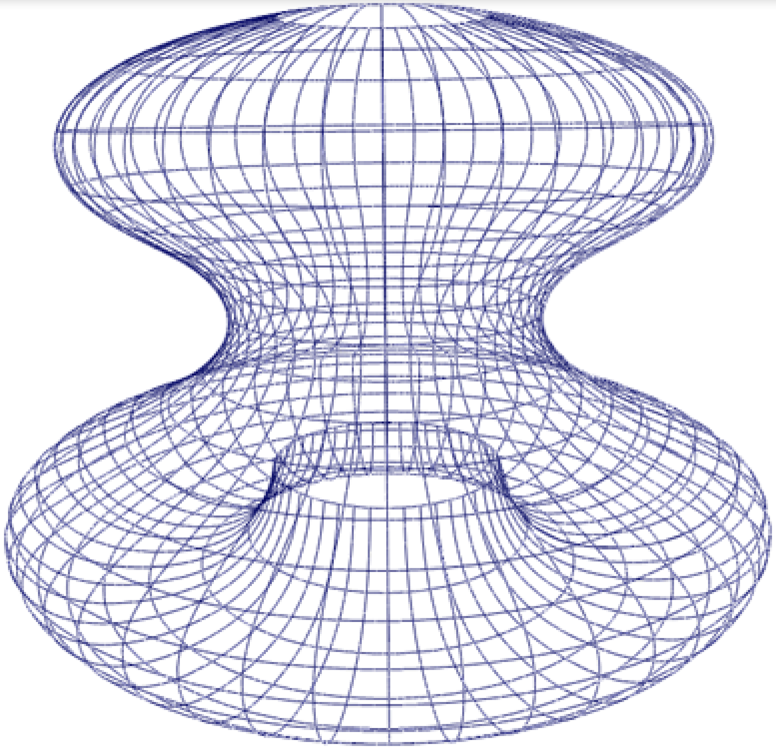
Revolution surface

- Meridian curve

  $x=f(t),\ z=g(t)$

  rotate along z-axis
- Rotation angle: θ
- Parallels curve: circles
- Surface equation:

$$\mathbf{r} = (f(t)\cos\theta, f(t)\sin\theta, g(t))^T$$



Revolution surface

Example: principal curvature

$$\mathbf{r} = (f(t)\cos\theta, f(t)\sin\theta, g(t))^T$$

$$\mathbf{r}_t = (\dot{f}(t)\cos\theta, \dot{f}(t)\sin\theta, \dot{g}(t))^T \qquad \mathbf{r}_\theta = (-f(t)\sin\theta, f(t)\cos\theta, 0)^T$$

Thus $E = \dot{f}^2(t) + \dot{g}^2(t), \qquad F = 0, \qquad G = f^2(t)$

The meridians and parallels are orthogonal

Finally $L = \dfrac{-\ddot{f}\dot{g} + \dot{f}\ddot{g}}{\sqrt{\dot{f}^2(t) + \dot{g}^2(t)}}, \qquad M = 0, \qquad N = \dfrac{f\dot{g}}{\sqrt{\dot{f}^2(t) + \dot{g}^2(t)}}$

The meridians and parallels of a surface of revolution are the lines of curvature.

Gaussian and mean curvatures

- Gaussian curvature: the product of the two principal curvatures

$$K = \kappa_{max}\kappa_{min}$$

- Mean curvature: the average of the two principal curvatures

$$H = \frac{\kappa_{max} + \kappa_{min}}{2}$$

Gaussian and mean curvatures

- Sign of K is same as $LN\text{-}M^2$: $EG\text{-}F^2>0$

$$K = \frac{LN-M^2}{EG-F^2}$$

  - $K>0$: elliptic, κ_{max} and κ_{min} same sign
  - $K<0$: hyperbolic, κ_{max} and κ_{min} different sign
  - $K=0$ and $H\neq 0$: parabolic, one of κ_{max} and κ_{min} is zero
  - $K=H=0$: flat / planar, $\kappa_{max}=\kappa_{min}=0$

Computation of Gaussian and mean Curvatures

- Explicit surfaces: $z=h(x,y)$
- Implicit surfaces: $f(x,y,z)=0$

Explicit surfaces

- Explicit surface

$$z=h(x,y)$$

can be converted into a parametric form

$$\mathbf{r}=(u,v,h(u,v))$$

where $u=x$, $v=y$

- Monge form: the parametric form of explicit surface

Explicit surfaces

Normal

$$\mathbf{N} = \frac{(-h_x, -h_y, 1)^T}{\sqrt{1 + h_x^2 + h_y^2}}$$

The first fundamental form coefficients

$$E = 1 + h_x^2, \quad F = h_x h_y, \quad G = 1 + h_y^2$$

The second fundamental form coefficients

$$L = \frac{h_{xx}}{\sqrt{1 + h_x^2 + h_y^2}} \quad M = \frac{h_{xy}}{\sqrt{1 + h_x^2 + h_y^2}} \quad N = \frac{h_{yy}}{\sqrt{1 + h_x^2 + h_y^2}}$$

Explicit surfaces

Finally

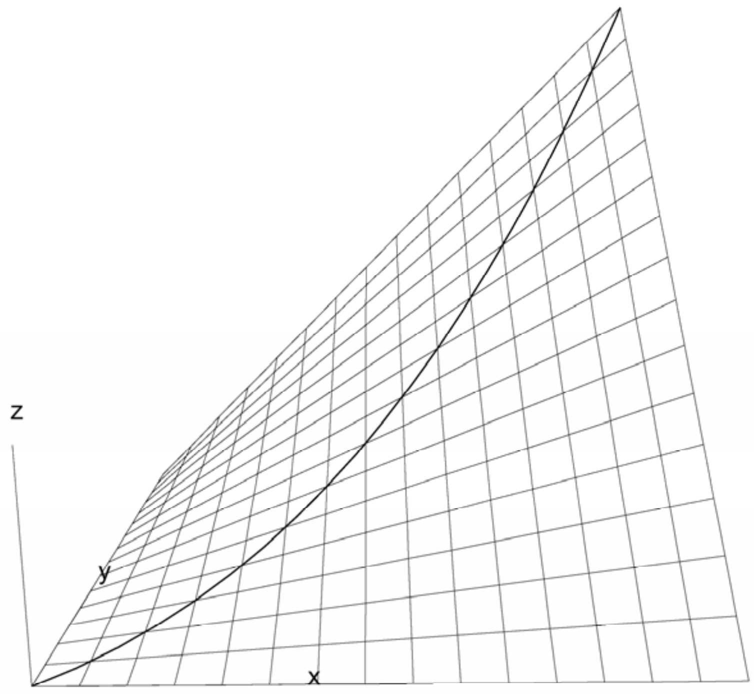
$$K = \frac{LN - M^2}{EG - F^2} = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2}$$

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

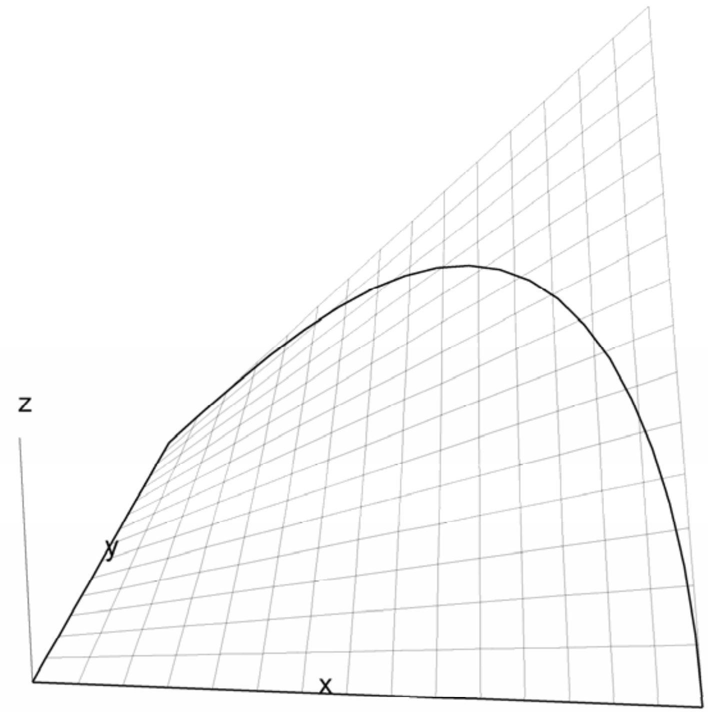
$$= \frac{(1 + h_x^2)h_{yy} - 2h_x h_y h_{xy} + (1 + h_y^2)h_{xx}}{2(1 + h_x^2 + h_y^2)^{3/2}}$$

Example of explicit surface

Hyperbolic paraboloid: $z=xy$



(a)



(b)

Example of explicit surface

Derivatives:

$$h_x = y, \quad h_y = x, \quad h_{xx} = 0, \quad h_{xy} = 1, \quad h_{yy} = 0$$

Normal:

$$\mathbf{N} = \frac{(-y, -x, 1)^T}{\sqrt{x^2 + y^2 + 1}}$$

First fundamental form coefficients:

$$E = 1 + y^2, \quad F = xy, \quad G = 1 + x^2$$

Example of explicit surface

The second fundamental coefficients:

$$L = 0, \quad M = \frac{1}{\sqrt{x^2 + y^2 + 1}}, \quad N = 0$$

Gaussian and mean curvatures: $K<0$, hyperbolic point

$$K = -\frac{1}{(x^2 + y^2 + 1)^2}, \qquad H = -\frac{xy}{(x^2 + y^2 + 1)^{\frac{3}{2}}}$$

Example of explicit surface

$L=N=0$ and $M\neq 0$:

The surface intersects its tangent plane at the iso-parametric lines

Principal curvatures:

$$\kappa_{max} = \frac{-xy + \sqrt{(x^2+1)(y^2+1)}}{(x^2+y^2+1)^{\frac{3}{2}}} > 0$$

$$\kappa_{min} = \frac{-xy - \sqrt{(x^2+1)(y^2+1)}}{(x^2+y^2+1)^{\frac{3}{2}}} < 0$$

Implicit surfaces

- Differential geometry: local geometric properties
- The problems of implicit surfaces can be converted as those of explicit surfaces by using inverse function theorem.
  - if $f(x,y,z)=0$ and $f_z \neq 0$,
    then z can be expressed as function of x and y

    $$z=h(x,y)$$
  - It is similar for $f_x \neq 0$ or $f_y \neq 0$.

Implicit surfaces

Computing the partial derivatives $z=h(x,y)$

$$f(x,y,z)=0 \quad \text{or} \quad f(x,y,h(x,y))=0$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}=0 \qquad \frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial y}=0$$

$z=h(x,y)$

$$h_x=-\frac{f_x}{f_z} \qquad h_y=-\frac{f_y}{f_z}$$

Implicit surfaces

Computing the partial derivatives $z=h(x,y)$

$$f(x,y,z)=0 \quad \text{or} \quad f(x,y,h(x,y))=0$$

$$h_{xx} = \frac{2f_x f_z f_{xz} - f_x^2 f_{zz} - f_z^2 f_{xx}}{f_z^3}$$

$$h_{xy} = \frac{f_x f_z f_{yz} + f_y f_z f_{xz} - f_x f_y f_{zz} - f_z^2 f_{xy}}{f_z^3}$$

$$h_{yy} = \frac{2f_y f_z f_{yz} - f_y^2 f_{zz} - f_z^2 f_{yy}}{f_z^3}$$

Explicit surface

Implicit quadric surface

General form

$$ax^2 + by^2 + cz^2 + dxy + eyz + hxz + kx + ly + mz + n = 0$$

Standard form after transformations

$$f(x, y, z) = \zeta \frac{x^2}{a^2} + \eta \frac{y^2}{b^2} + \xi \frac{z^2}{c^2} - \delta = 0$$

- ζ, η, ξ = -1,0,1; δ = 0,1

Classification of implicit quadrics

Implicit Quadrics	ζ	η	ξ	δ
Ellipsoid	1	1	1	1
Hyperboloid of One Sheet	1	1	-1	1
	1	-1	1	1
	-1	1	1	1
Hyperboloid of Two Sheets	1	-1	-1	1
	-1	1	-1	1
	-1	-1	1	1
Elliptic Cone	1	1	-1	0
	1	-1	1	0
	-1	1	1	0
Elliptic Cylinder	1	1	0	1
	1	0	1	1
	0	1	1	1
Hyperbolic Cylinder	1	-1	0	1
	-1	1	0	1
	1	0	-1	1
	-1	0	1	1
	0	1	-1	1
	0	-1	1	1

Implicit quadric surface

Gaussian curvature

$$K(x,y,z) = \frac{\zeta\eta\xi\delta}{a^2b^2c^2(\zeta^2\frac{x^2}{a^4} + \eta^2\frac{y^2}{b^4} + \xi^2\frac{z^2}{c^4})^2}$$

Mean curvature

$$H(x,y,z) =$$
$$-\frac{\zeta^2b^2c^2(\xi b^2 + \eta c^2)x^2 + \eta^2a^2c^2(\xi a^2 + \zeta c^2)y^2 + \xi^2a^2b^2(\eta a^2 + \zeta b^2)z^2}{2a^4b^4c^4(\zeta^2\frac{x^2}{a^4} + \eta^2\frac{y^2}{b^4} + \xi^2\frac{z^2}{c^4})^{\frac{3}{2}}}$$

Principal curvatures $\kappa(x,y,z) = H \pm \sqrt{H^2 - K}$

Hyperbolic cylinder

When $\zeta=\delta=1$, $\eta=-1$, $\xi=0$

$$f(x,y)=\frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$$

we have

$$K=0, \qquad H=\frac{b^2x^2-a^2y^2}{2a^4b^4(\frac{x^2}{a^4}+\frac{y^2}{b^4})^{\frac{3}{2}}},$$

$$\kappa_{max}=\frac{b^2x^2-a^2y^2}{a^4b^4(\frac{x^2}{a^4}+\frac{y^2}{b^4})^{\frac{3}{2}}}, \qquad \kappa_{min}=0$$

Ellipsoid

When $\zeta=\eta=\xi=\delta=1$

$$f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$

We have

$$K=\frac{1}{a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^2}$$

$$H=\frac{x^2+y^2+z^2-a^2-b^2-c^2}{2a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{3}{2}}}$$

Ellipsoid

Principal curvatures

$$\kappa = \frac{x^2+y^2+z^2-a^2-b^2-c^2}{2a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{3}{2}}}$$
$$\pm \frac{\sqrt{(x^2+y^2+z^2-a^2-b^2-c^2)^2-4a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)}}{2a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{3}{2}}}$$

Special ellipsoid: sphere

When $a=b=c=R$, ellipsoid is a sphere

$$K=1/R^2,\ \ H=-1/R$$

Since $H^2-K=0$ for all points on sphere

We have

A sphere is made of entirely nonflat umbilics

Elliptic cone

When $\zeta=\eta=1$, $\xi=-1$, $\delta=0$

$$f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

We have

$$K=0,\quad H=-\frac{x^2+y^2+z^2}{2a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{3}{2}}},$$

$$\kappa_{max}=0,\kappa_{min}=-\frac{x^2+y^2+z^2}{a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{\frac{3}{2}}}$$

Euler's theorem

Euler's theorem

- The normal curvatures of a surface in an arbitrary direction (in the tangent plane) at point P can be expressed in terms of principal curvatures κ_1 and κ_2 at point P

$$\kappa_n = \kappa_1 \cos^2 \Phi + \kappa_2 \sin^2 \Phi$$

where Φ is the angle between the arbitrary direction and the principal direction κ_1

Euler's theorem: simple proof

Assuming the isoparametric curves of the surface are lines of curvature, then

$$F=M=0$$

Normal curvature is

$$\kappa_n = \frac{L du^2 + N dv^2}{E du^2 + G dv^2}$$

The principal curvatures are

$$\kappa_1 = \frac{L}{E}, \qquad \kappa_2 = \frac{N}{G}$$

Euler's theorem: simple proof

The angle Φ between dv/du and the principle direction corresponding to κ_1 ($dv_1=0$, du_1) can be evaluated (according to first fundamental form)

$$\cos\Phi = E\frac{du}{ds}\frac{du_1}{ds_1}$$

Since $ds_1 = \sqrt{Edu_1^2}$ and $ds = \sqrt{Edu^2 + Gdv^2}$

$$\cos\Phi = \sqrt{E}\frac{du}{ds}, \qquad \sin\Phi = \sqrt{G}\frac{dv}{ds}$$

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