# Preliminary Mathematics of Geometric Modeling (3)

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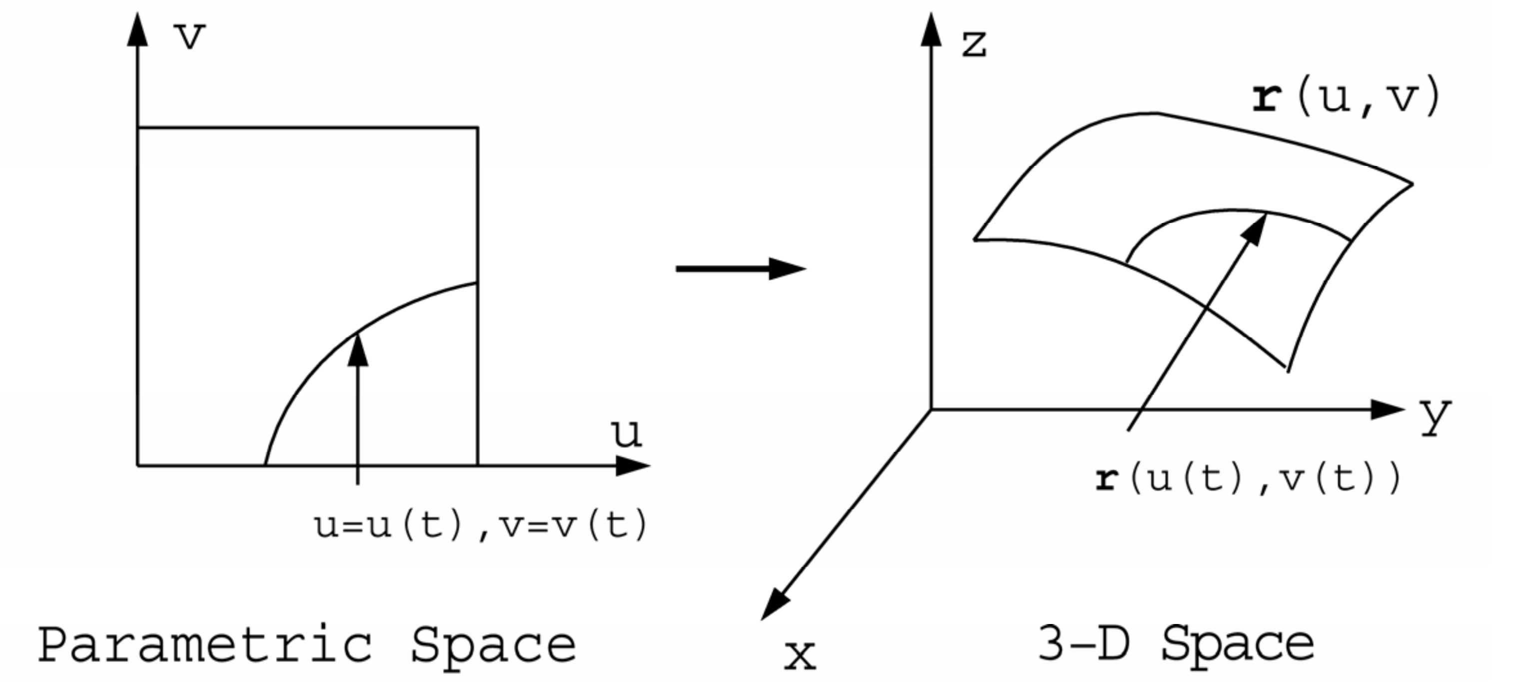
# Differential Geometry of Surfaces

- Tangent plane and surface normal
- First fundamental form I (metric)
- Second fundamental form II (curvature)
- Principal curvatures
- Gaussian and mean curvatures
  - Explicit surfaces
  - Implicit surfaces
- Euler's theorem and Dupin's indicatrix

# Tangent vector on the surface

A parametric surface $\mathbf{r}=\mathbf{r}(u,v)$

A curve $u=u(t)$, $v=v(t)$ in the parametric domain



# Tangent vector on the surface

The tangent vector of the curve on the surface respect to the parameter $t$ :

$$\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$$

$$\boxed{\dot{\mathbf{r}}(t) = \mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v}}$$

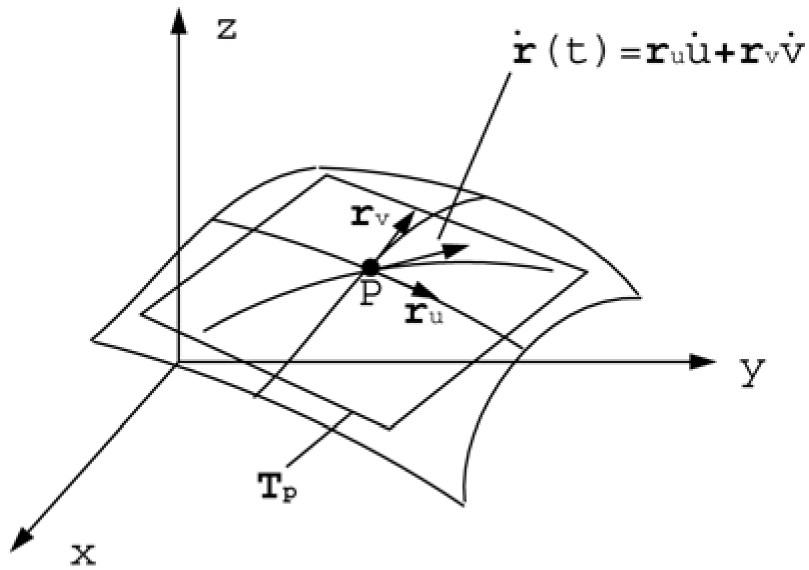
where

$$\dot{u} = \dot{u}(t)$$

$$\dot{v} = \dot{v}(t)$$

# Tangent plane on the surface

The *tangent plane* at point $P$ can be considered as a union of the tangent vectors for all $\mathbf{r}(t)$ through $P$




The tangent plane at a point on a surface

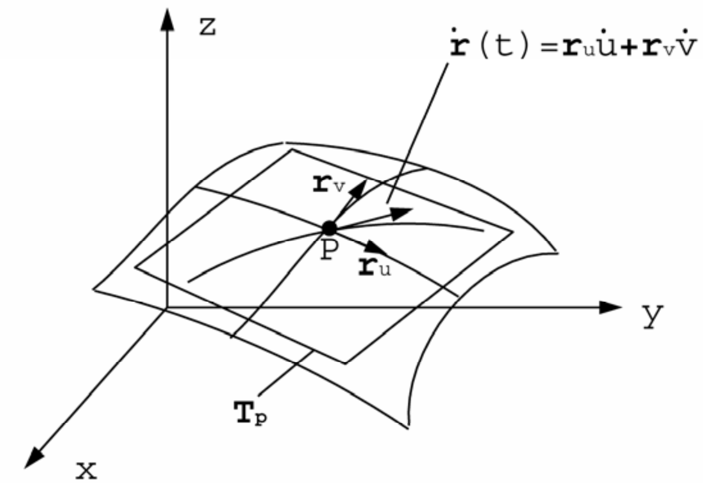
# Tangent plane on the surface

Suppose: $P = \mathbf{r}(u_p, v_p)$

The equation of tangent plane at $\mathbf{r}(u_p, v_p)$ :

$$\boxed{\mathbf{T}_p(\mu,\nu) = \mathbf{r}(u_p, v_p) + \mu\mathbf{r}_u(u_p, v_p) + \nu\mathbf{r}_v(u_p, v_p)}$$

Where $\mu$, $\nu$ are parameter



# Surface normal

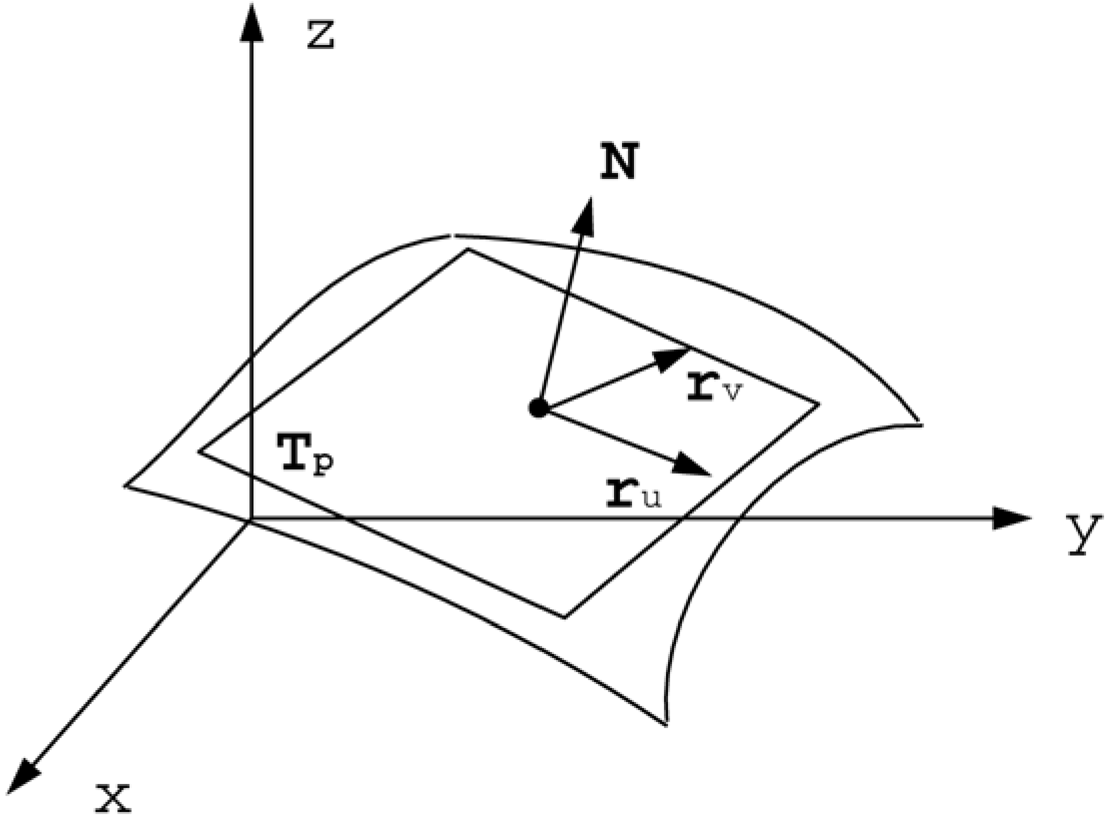
The *surface normal* *vector* is perpendicular to the tangent plane. The unit normal is

$$\boxed{\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}}$$

The implicit form of tangent surface is

$$\boxed{(\mathbf{r} - \mathbf{r}(u_p, v_p)) \cdot \mathbf{N}(u_p, v_p) = 0}$$

# Surface normal




The normal to the point on a surface

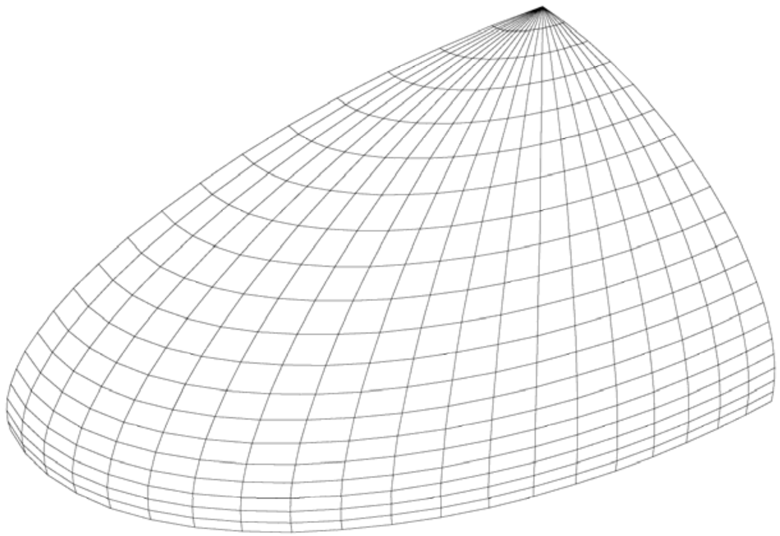
# Regular point on the surface

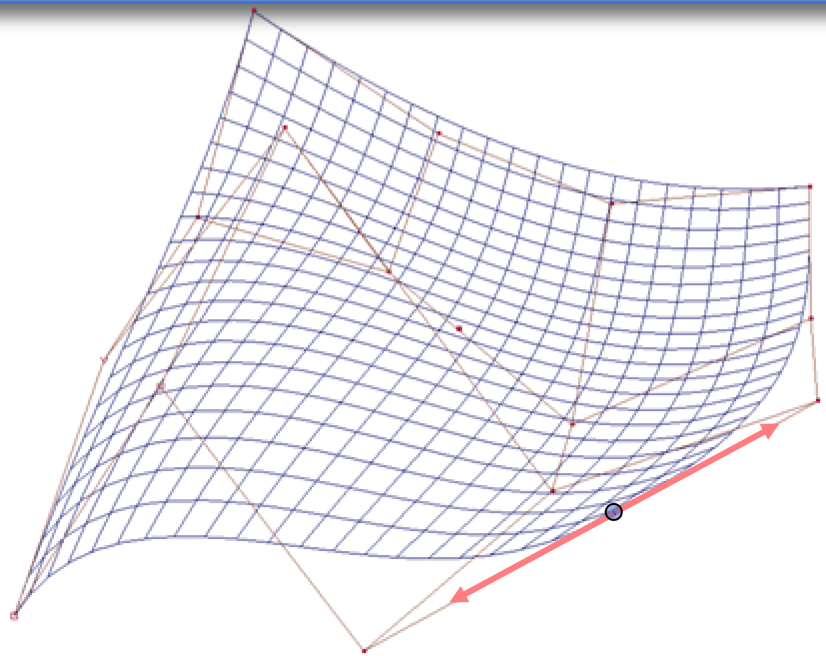
Definition: *A regular (ordinary) point $P$ on a parametric surface is defined as a point where $\mathbf{r}_u \times \mathbf{r}_v \neq 0$. A point which is not a regular point is called a singular point.*

Notes for

- Regular point:
  1. $\mathbf{r}_u \neq 0$ and $\mathbf{r}_v \neq 0$
  2. $\mathbf{r}_u$ is not parallel to $\boldsymbol{r}_v$
- Singular point:
  1. Normal may exist at the singular point

# Examples of singular point



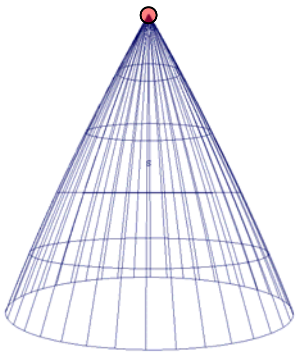


$\mathbf{r}_u = 0$ or $\mathbf{r}_v = 0$

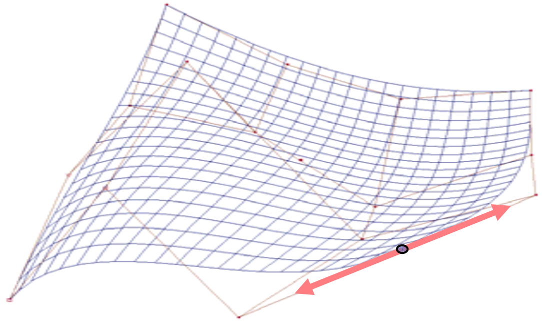
$\mathbf{r}_u \,/\!/\, \mathbf{r}_v$

# Types of singular points

- Essential singularities: specific features of the surface geometry
  - Apex of cone

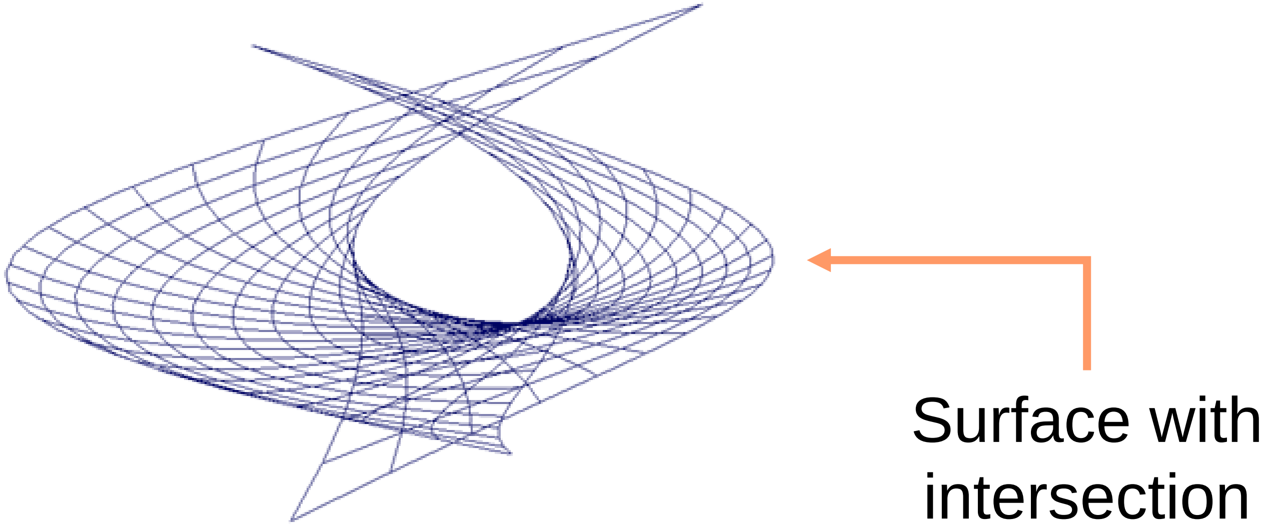


- Artificial singularities
  - parametrization



# Regular surface

- Existence of a tangent plane everywhere on the surface
- Without self-intersection



# Example: elliptic cone

Parametric form: $\mathbf{r}=(at\cos\theta, bt\sin\theta, ct)^T$

Where $0 \leqslant \theta \leqslant 2\pi$, $0 \leqslant t \leqslant l$, $a,b,c$ are *constants*

$$\mathbf{r}_\theta = (-at\sin\theta, bt\cos\theta, 0)^T \qquad \mathbf{r}_t = (a\cos\theta, b\sin\theta, c)^T$$

$$\begin{aligned} |\mathbf{r}_\theta \times \mathbf{r}_t| &= |bct\cos\theta\mathbf{e}_x + act\sin\theta\mathbf{e}_y - abt\mathbf{e}_z| \\ &= \sqrt{t^2(b^2c^2\cos^2\theta + a^2c^2\sin^2\theta + a^2b^2)} \end{aligned}$$

The apex of the cone ($t=0$) is singular

# Normal of implicit surface

Implicit surface: $f(x,y,z)=0$

Considering the two parametric curves on the surfaces

$$\mathbf{r}_1=(x_1(t_1), y_1(t_1), z_1(t_1))$$
$$\mathbf{r}_2=(x_2(t_2), y_2(t_2), z_2(t_2))$$

The $\mathbf{r}_1$ and $\mathbf{r}_2$ intersect at point $P$

By substituting $\mathbf{r}_1$ and $\mathbf{r}_2$ into $f$, we have

# Normal of implicit surface

$$f(x_1(t_1), y_1(t_1), z(t_1)) = 0 \qquad f(x_2(t_2), y_2(t_2), z(t_2)) = 0$$

By differentiation with $t_1$ and $t_2$ respectively

$$f_x \frac{dx_1}{dt_1} + f_y \frac{dy_1}{dt_1} + f_z \frac{dz_1}{dt_1} = 0 \quad f_x \frac{dx_2}{dt_2} + f_y \frac{dy_2}{dt_2} + f_z \frac{dz_2}{dt_2} = 0$$

After simplification, we can deduce:

$$f_x : f_y : f_z =$$
$$\frac{dz_2}{dt_2}\frac{dy_1}{dt_1} - \frac{dz_1}{dt_1}\frac{dy_2}{dt_2} : \frac{dz_1}{dt_1}\frac{dx_2}{dt_2} - \frac{dz_2}{dt_2}\frac{dx_1}{dt_1} : \frac{dx_1}{dt_1}\frac{dy_2}{dt_2} - \frac{dx_2}{dt_2}\frac{dy_1}{dt_1}$$

# Normal of implicit surface

As we know

$$\frac{d\mathbf{r}_1(t_1)}{dt_1} \times \frac{d\mathbf{r}_2(t_2)}{dt_2}$$

$$= \left( \frac{dz_2}{dt_2}\frac{dy_1}{dt_1} - \frac{dz_1}{dt_1}\frac{dy_2}{dt_2}, \frac{dz_1}{dt_1}\frac{dx_2}{dt_2} - \frac{dz_2}{dt_2}\frac{dx_1}{dt_1}, \frac{dx_1}{dt_1}\frac{dy_2}{dt_2} - \frac{dx_2}{dt_2}\frac{dy_1}{dt_1} \right)^T$$

Thus the normal is the gradient of $f$, i.e.

$$\nabla f = (f_x, f_y, f_z)^T \,//\, \frac{d\mathbf{r}_1(t_1)}{dt_1} \times \frac{d\mathbf{r}_2(t_2)}{dt_2}$$

# Normal of implicit surface

Unit normal of the implicit surface

$$\mathbf{N} = \frac{(f_x, f_y, f_z)^T}{\sqrt{f_x^2 + f_y^2 + f_z^2}} = \frac{\nabla f}{|\nabla f|}$$

provided that $|\nabla f| \neq 0$

# Tangent plane of implicit surface

The tangent plane of point $P(x_p,y_p,z_p)$ on the implicit surface $f(x,y,z)=0$ is

$$\nabla f \cdot (\mathbf{r} - P) = 0$$

i.e.

$$\boxed{f_x(x-x_p)+f_y(y-y_p)+f_z(z-z_p)=0}$$

$\mathbf{r}=(x,y,z)$

# Example: elliptic cone

Elliptic cone in implicit form

$$f(x, y, z) = (\frac{x}{a})^2 + (\frac{y}{b})^2 - (\frac{z}{c})^2 = 0$$

The gradient (normal) is

$$\nabla f = (\frac{2x}{a^2}, \frac{2y}{b^2}, -\frac{2z}{c^2})^T$$

subject to $(x, y, z) \in f(x, y, z) = 0$

# First fundamental form / (metric)

The differential arc length of parametric curve on the parametric surface

- Parametric surface $\mathbf{r}=\mathbf{r}(u,v)$
- Parametric curve defined the in parametric domain $u=u(t),\ v=v(t)$
- The differential arc length of parametric curve

$$\boxed{ds = \left|\frac{d\mathbf{r}}{dt}\right| dt = |\dot{\mathbf{r}}| dt = \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}} dt}$$

# First fundamental form / (metric)

$$ds = \left|\frac{d\mathbf{r}}{dt}\right| dt = \left|\mathbf{r}_u \frac{du}{dt} + \mathbf{r}_v \frac{dv}{dt}\right| dt$$

$$= \sqrt{(\mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v}) \cdot (\mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v})} dt$$

$$= \sqrt{E du^2 + 2F du dv + G dv^2} \,,$$

where $E = \mathbf{r}_u \cdot \mathbf{r}_u, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v$

# First fundamental form

- First fundamental form

$$\boxed{I = ds^2 = d\mathbf{r} \cdot d\mathbf{r} = E du^2 + 2F du dv + G dv^2}$$

  - $E$, $F$, $G$ : first fundamental form coefficients
  - $E$, $F$, $G$ are important for intrinsic properties
  - Alternative representation

$$\boxed{I = \frac{1}{E}(E\,du + F\,dv)^2 + \frac{EG - F^2}{E} dv^2}$$

# First fundamental form

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2$$

Thus

$$\begin{aligned}(\mathbf{r}_u \times \mathbf{r}_v)^2 &= (\mathbf{r}_u \times \mathbf{r}_v) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \\ &= (\mathbf{r}_u \cdot \mathbf{r}_u)(\mathbf{r}_v \cdot \mathbf{r}_v) - (\mathbf{r}_u \cdot \mathbf{r}_v)^2 \\ &= EG - F^2 > 0\end{aligned}$$

# First fundamental form

- $I \geq 0$ for arbitrary surface
  - $I > 0$: positive definite provided that the surface is regular
  - $I = 0$ iff $du = 0$ and $dv = 0$

# Example: First fundamental form

Hyperbolic paraboloid:

$$\mathbf{r}(u,v)=(u,v,uv)^T$$

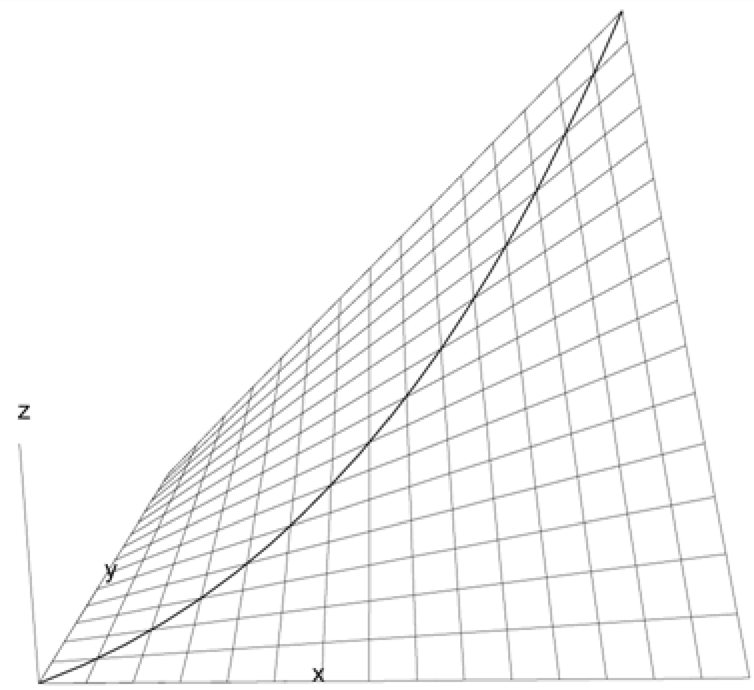
$$0 \le u,v \le 1$$

Curve:

$$u=t,\ v=t.$$

$$0 \le t \le 1$$

Aim: arc length of the curve on the surface




Hyperbolic paraboloid arc length along $u=t$, $v=t$

# Example: First fundamental form

First fundamental form coefficients

$$\mathbf{r}_u = (1, 0, v)^T, \quad \mathbf{r}_v = (0, 1, u)^T$$

$$\boxed{\begin{aligned} E &= \mathbf{r}_u \cdot \mathbf{r}_u = 1 + v^2 \\ F &= \mathbf{r}_u \cdot \mathbf{r}_v = uv \\ G &= \mathbf{r}_v \cdot \mathbf{r}_v = 1 + u^2 \end{aligned}}$$

# Example: First fundamental form

First fundamental form coefficients along the curve

$$E = 1 + t^2, \quad F = t^2, \quad G = 1 + t^2$$

The differential arc length of the curve

$$ds = \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}dt = 2\sqrt{t^2 + \frac{1}{2}}dt$$

# Example: First fundamental form

The arc length of the curve

$$s = 2\int_0^1 \sqrt{t^2 + \frac{1}{2}}dt$$

$$= \left[t\sqrt{t^2 + \frac{1}{2}} + \frac{1}{2}log\left(t + \sqrt{t^2 + \frac{1}{2}}\right)\right]_0^1$$

$$= \sqrt{\frac{3}{2}} + \frac{1}{2}log(\sqrt{2} + \sqrt{3})$$

# Application of first fundamental form: angle between curves on surface

- Two curves on a parametric surface

$$\mathbf{r}_1 = \mathbf{r}(u_1(t), v_1(t)) \qquad \mathbf{r}_2 = \mathbf{r}(u_2(t), v_2(t))$$

  - Angle between $\mathbf{r}_1$ and $\mathbf{r}_2$ is the angle between their tangent vectors
  - Angle between two vectors $\mathbf{a}$ and $\mathbf{b}$

$$\cos\omega = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

where

$$\mathbf{a}=\mathbf{r}_u du_1+\mathbf{r}_v dv_1$$
$$\mathbf{b}=\mathbf{r}_u du_2+\mathbf{r}_v dv_2$$

# Application of first fundamental form: angle between curves on surface

- Angle between curves on the surface

$$\cos\omega = \frac{Edu_1du_2 + F(du_1dv_2 + dv_1du_2) + Gdv_1dv_2}{\sqrt{Edu_1^2 + 2Fdu_1dv_1 + Gdv_1^2}\sqrt{Edu_2^2 + 2Fdu_2dv_2 + Gdv_2^2}}$$
$$= E\frac{du_1}{ds_1}\frac{du_2}{ds_2} + F\left(\frac{du_1}{ds_1}\frac{dv_2}{ds_2} + \frac{dv_1}{ds_1}\frac{du_2}{ds_2}\right) + G\frac{dv_1}{ds_1}\frac{dv_2}{ds_2}\,.$$

  - $\mathbf{r}_1$ and $\mathbf{r}_2$ is orthogonal ($\cos(\pi/2)=0$) if

$$Edu_1du_2 + F(du_1dv_2 + dv_1du_2) + Gdv_1dv_2 = 0$$

# Application of first fundamental form: angle between curves on surface

- Special case: iso-parametric curves

$$\mathbf{r}_1\text{: } u_1(t)=t,\ v_1(t)=0$$

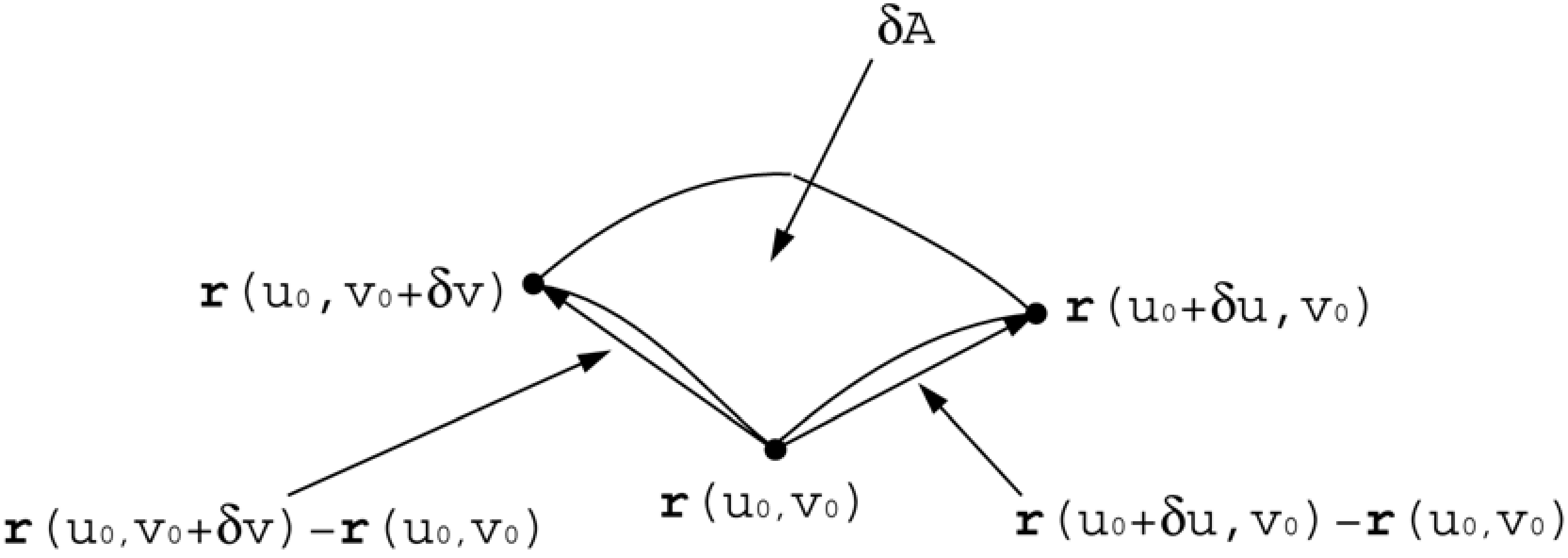
$$\mathbf{r}_2\text{: } u_2(t)=0,\ v_2(t)=t$$

$$\cos\omega = \frac{\mathbf{r}_u \cdot \mathbf{r}_v}{|\mathbf{r}_u||\mathbf{r}_v|} = \frac{\mathbf{r}_u \cdot \mathbf{r}_v}{\sqrt{\mathbf{r}_u \cdot \mathbf{r}_u}\sqrt{\mathbf{r}_v \cdot \mathbf{r}_v}} = \frac{F}{\sqrt{EG}}$$

The iso-parametric curves are orthogonal if $F=0$

# Application of first fundamental form: area of the surface patch

The area bounded by four vertices $\mathbf{r}(u,v)$, $\mathbf{r}(u+\delta u,v)$, $\mathbf{r}(u+\delta u,v)$, $\mathbf{r}(u+\delta u,v+\delta v)$




Area of small surface patch

# Application of first fundamental form: area of the surface patch

The area bounded by four vertices $\mathbf{r}(u,v)$, $\mathbf{r}(u+\delta u,v)$, $\mathbf{r}(u+\delta u,v)$, $\mathbf{r}(u+\delta u,v+\delta v)$

$$\delta A = |\mathbf{r}_u \delta u \times \mathbf{r}_v \delta v| = \sqrt{EG - F^2}\delta u \delta v$$

Recall

In differential form

$$dA = \sqrt{EG - F^2} du dv$$

# Example: area of surface patch

Hyperbolic paraboloid:

$\mathbf{r}(u,v)=(u,v,uv)^T$

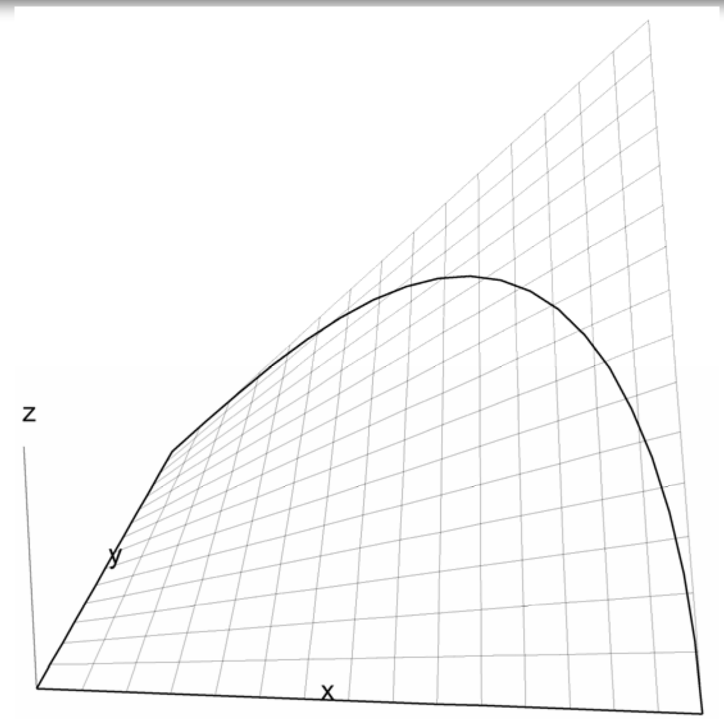
$0 \le u,v \le 1$

Bounded curves:

$u=0;\ v=0;$

$u^2+v^2=1$

Aim: Area of the surface patch bounded by the 3 curves?




Area bounded by positive $u$ and $v$ axes and a quarter circle

# Example: area of surface patch

First fundamental form coefficients

$$\mathbf{r}_u = (1, 0, v)^T, \quad \mathbf{r}_v = (0, 1, u)^T$$

$$E = \mathbf{r}_u \cdot \mathbf{r}_u = 1 + v^2$$

$$F = \mathbf{r}_u \cdot \mathbf{r}_v = uv$$

$$G = \mathbf{r}_v \cdot \mathbf{r}_v = 1 + u^2$$

$$\boxed{A = \int_D \sqrt{1 + u^2 + v^2}\, du dv}$$

# Example: area of surface patch

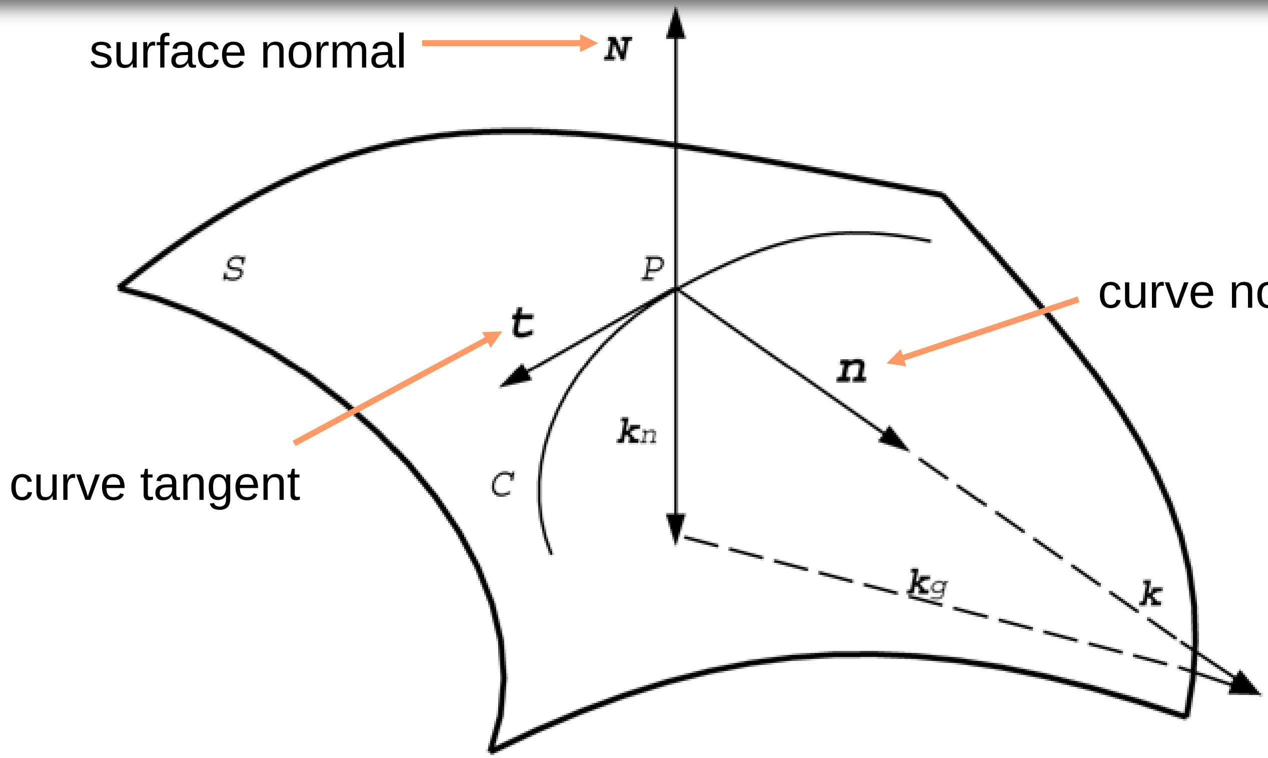
After reparametrization of the surface patch by setting $u=r\cos\theta$, $v=r\sin\theta$, we have

$$A = \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{1+r^2}\, r\, d\theta\, dr = \frac{\pi}{6}(\sqrt{8}-1)$$

# Second fundamental form II (curvature)

- The second fundamental form quantify the curvatures of a surface
- Consider a curve $C$ on surface $S$ which passes through point $P$
  - The differential geometry of curve

# Second fundamental form // (curvature)




Definition of normal curvature

# Second fundamental form // (curvature)

The relationship between unit tangent vector $\mathbf{t}$ and unit normal vector $\mathbf{n}$ of the curve $C$ at point $P$

$$\mathbf{k} = \frac{d\mathbf{t}}{ds} = \kappa\mathbf{n} = \mathbf{k}_n + \mathbf{k}_g$$

- Normal curvature vector $\mathbf{k}_n$ : component of $\mathbf{k}$ of curve $C$ in the surface normal direction
- Geodesic curvature vector $\mathbf{k}_g$ : component of $\mathbf{k}$ of curve $C$ in the direction perpendicular to $\mathbf{t}$ in the surface tangent plane

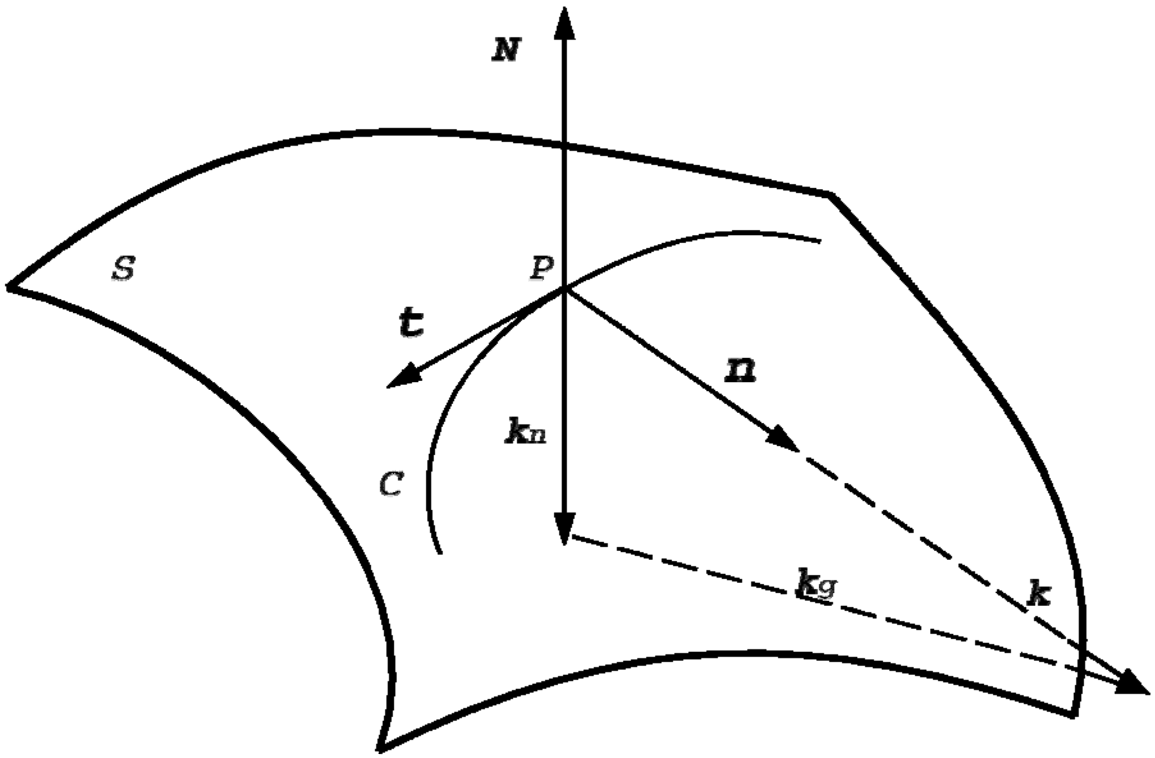
# Second fundamental form II (curvature)

The normal curvature vector can be expressed as

$$\mathbf{k}_n = \kappa_n \mathbf{N}$$

- $\kappa_n$ is called normal curvature of surface at point $P$ in the direction $\mathbf{t}$
- $\kappa_n$ is the magnitude of the projection of $\mathbf{k}$ onto the surface normal at $P$
- The sign of $\kappa_n$ is determined by the orientation of the surface normal at $P$.

# Second fundamental form // (curvature)




Definition of normal curvature

# Second fundamental form II (curvature)

Differentiating $\mathbf{N}\cdot\mathbf{t}=0$ along the curve respect to $s$:

$$\frac{d\mathbf{t}}{ds}\cdot\mathbf{N}+\mathbf{t}\cdot\frac{d\mathbf{N}}{ds}=0$$

Combined with $\mathbf{k}_n=\kappa_n\mathbf{N}$, Thus

$$\begin{aligned}\kappa_n &= \frac{d\mathbf{t}}{ds}\cdot\mathbf{N}=-\mathbf{t}\cdot\frac{d\mathbf{N}}{ds}=-\frac{d\mathbf{r}}{ds}\cdot\frac{d\mathbf{N}}{ds}=-\frac{d\mathbf{r}\cdot d\mathbf{N}}{d\mathbf{r}\cdot d\mathbf{r}}\\ &= \frac{Ldu^2+2Mdudv+Ndv^2}{Edu^2+2Fdudv+Gdv^2},\end{aligned}$$

# Second fundamental form // (curvature)

where

$$L = -\mathbf{r}_u \cdot \mathbf{N}_u$$

$$M = -\frac{1}{2}(\mathbf{r}_u \cdot \mathbf{N}_v + \mathbf{r}_v \cdot \mathbf{N}_u)$$

$$= -\mathbf{r}_u \cdot \mathbf{N}_v = -\mathbf{r}_v \cdot \mathbf{N}_u$$

$$N = -\mathbf{r}_v \cdot \mathbf{N}_v$$

# Second fundamental form // (curvature)

Since $\mathbf{r}_u \perp \mathbf{N}$ and $\mathbf{r}_v \perp \mathbf{N}$

$\mathbf{r}_u \cdot \mathbf{N} = 0$ and $\mathbf{r}_v \cdot \mathbf{N} = 0$

$$d(\mathbf{r}_u \cdot \mathbf{N})/du = \mathbf{r}_{uu} \cdot \mathbf{N} + \mathbf{r}_u \cdot \mathbf{N}_u = 0$$

$$d(\mathbf{r}_v \cdot \mathbf{N})/du = \mathbf{r}_{uv} \cdot \mathbf{N} + \mathbf{r}_v \cdot \mathbf{N}_u = 0$$

$$d(\mathbf{r}_u \cdot \mathbf{N})/dv = \mathbf{r}_{uv} \cdot \mathbf{N} + \mathbf{r}_u \cdot \mathbf{N}_v = 0$$

$$d(\mathbf{r}_v \cdot \mathbf{N})/dv = \mathbf{r}_{vv} \cdot \mathbf{N} + \mathbf{r}_v \cdot \mathbf{N}_v = 0$$

# Second fundamental form II (curvature)

Alternative expression of $L$, $M$ and $N$

$$L = \mathbf{r}_{uu} \cdot \mathbf{N} \quad M = \mathbf{r}_{uv} \cdot \mathbf{N} \quad N = \mathbf{r}_{vv} \cdot \mathbf{N}$$

The *second fundamental form* $II$

$$II = L du^2 + 2M du dv + N dv^2$$

$L$, $M$ and $N$ are called second fundamental form coefficients

# Second fundamental form // (curvature)

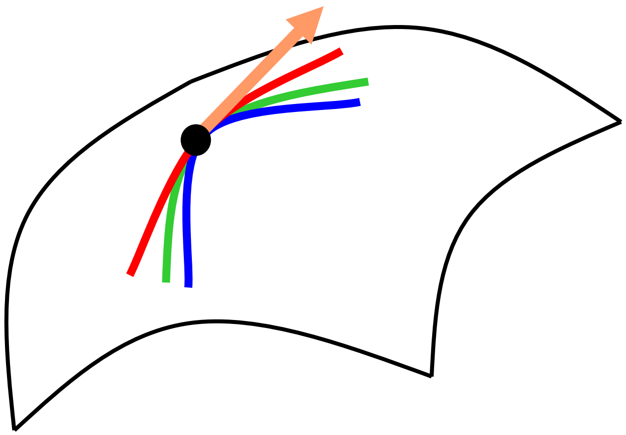
The normal curvature can be expressed as

$$\kappa_n = \frac{II}{I} = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2}$$

- $\lambda = dv/du$ is the direction of the tangent line to $C$ at $P$ (in the surface parametric domain)
- $\kappa_n$ at a given point $P$ on the surface depends only on $\lambda$

# Meusnier Theorem

*All curves lying on a surface $S$ passing through a given point $p \in S$ with the same tangent line have the same normal curvature at this point.*

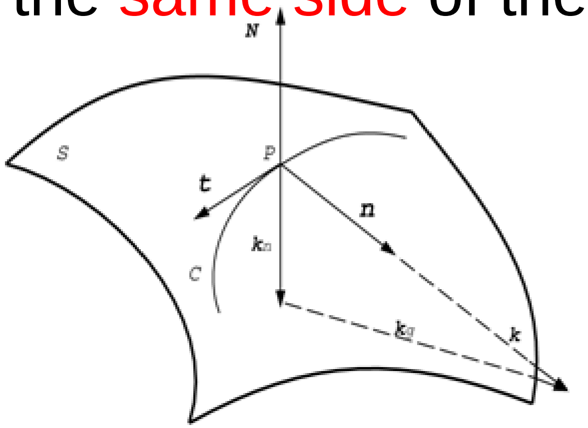


Meusnier Theorem

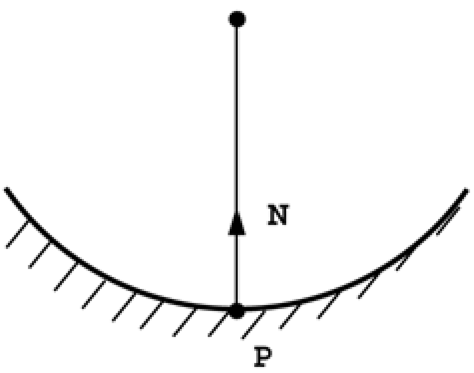
# About sign of normal curvature

- Convention (a): $\kappa\mathbf{n}\cdot\mathbf{N}=\kappa_n$

  The normal curvature is positive when the center of the curvature of the normal section curve, which is a curve through $P$ cut out by a plane that contains $\mathbf{t}$ and $\mathbf{N}$, is on the same side of the surface normal.




Definition of normal curvature (minus)

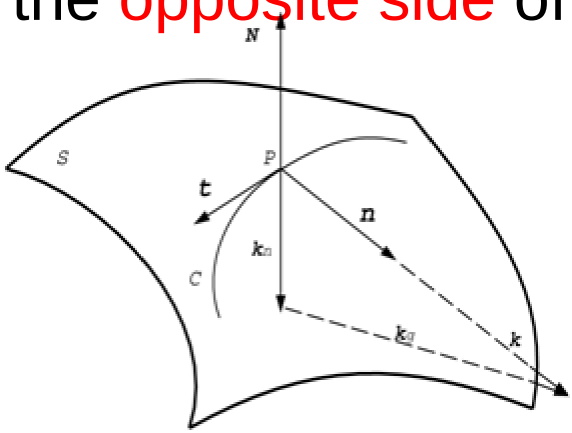



$\kappa\mathbf{n}\cdot\mathbf{N}=\kappa_n$

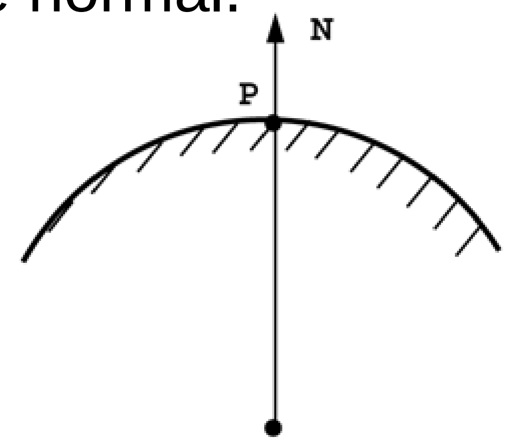
# About sign of normal curvature

- Convention (b): $\kappa\,\mathbf{n}\cdot\mathbf{N}=-\kappa_n$

  The normal curvature is positive when the center of the curvature of the normal section curve, which is a curve through $P$ cut out by a plane that contains $\mathbf{t}$ and $\mathbf{N}$, is on the opposite side of the surface normal.




Definition of normal curvature (positive)




$\kappa\mathbf{n}\cdot\mathbf{N}=-\kappa_n$

# About sign of normal curvature

- About convention (b)
  - The convention (b) is often used in the area of offset curves and surfaces in the context of NC machining

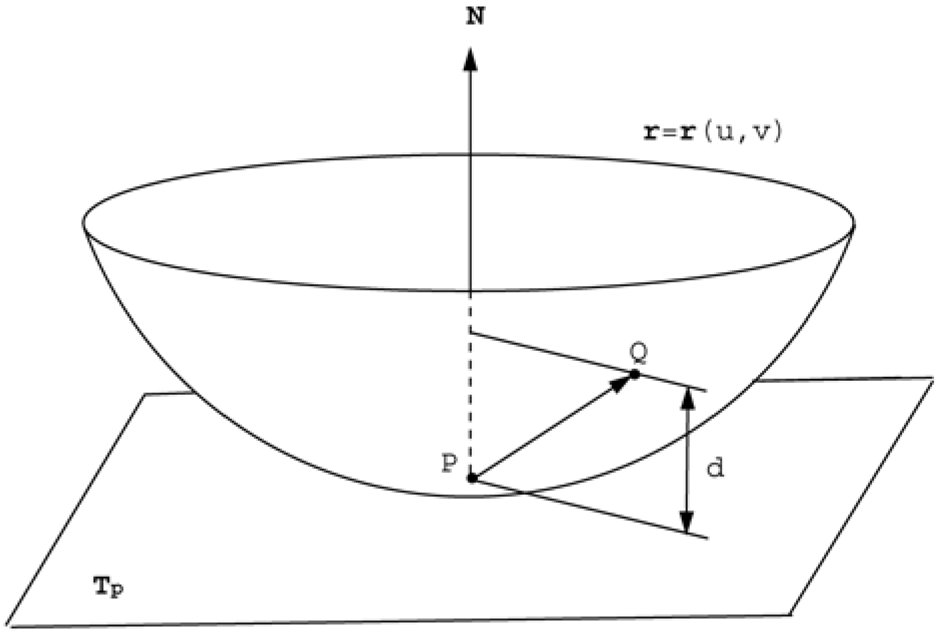
$$\mathbf{k}_n = -\kappa_n \mathbf{N}$$

$$\kappa_n = -\frac{II}{I} = -\frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2}$$

# Point classification by *II*

- Suppose $P$ and $Q$ on the surface $\mathbf{r}(u,v)$

$$P = \mathbf{r}(u,v),\quad Q = \mathbf{r}(u+du, v+dv)$$



# Point classification by *II*

$$\mathbf{r}(u+du, v+dv) = \mathbf{r}(u,v) + \mathbf{r}_u du + \mathbf{r}_v dv + \frac{1}{2}(\mathbf{r}_{uu} du^2 + 2\mathbf{r}_{uv} du dv + \mathbf{r}_{vv} dv^2) + \ldots$$

Thus $\mathbf{PQ} = \mathbf{r}(u+du, v+dv) - \mathbf{r}(u,v) = \mathbf{r}_u du + \mathbf{r}_v dv + \frac{1}{2}(\mathbf{r}_{uu} du^2 + 2\mathbf{r}_{uv} du dv + \mathbf{r}_{vv} dv^2) + \ldots\ .$

Projecting **PQ** onto **N**

$$d = \mathbf{PQ} \cdot \mathbf{N} = (\mathbf{r}_u du + \mathbf{r}_v dv) \cdot \mathbf{N} + \frac{1}{2} II$$

# Point classification by *II*

Finally

$$d = \frac{1}{2} II = \frac{1}{2}(L du^2 + 2M du dv + N dv^2)$$

Thus $|II|$ is equal to twice the distance from $Q$ to the tangent plane of the surface at $P$ within second order terms.

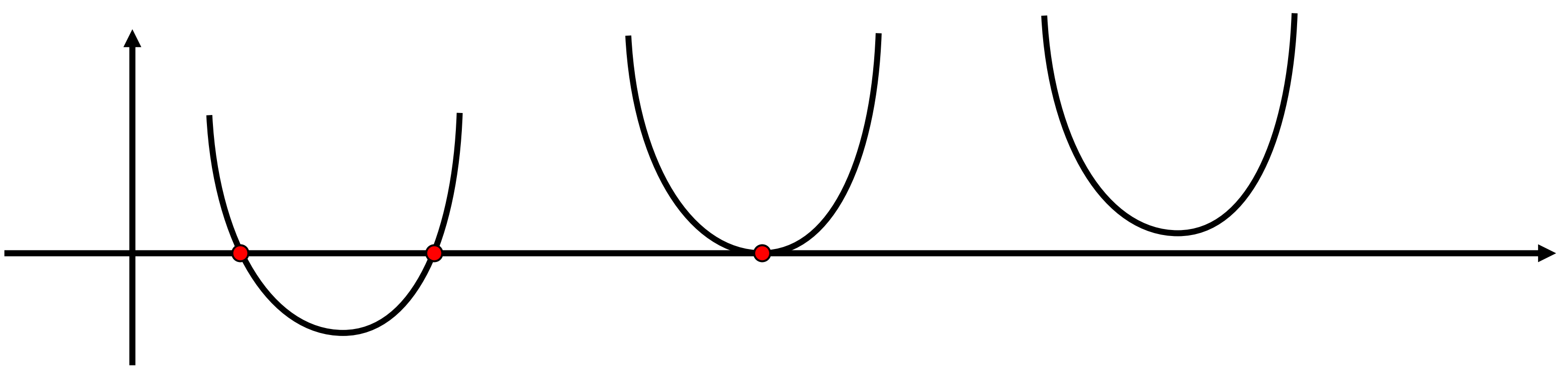
Next, we determine the sign of $II$, i.e. $Q$ lies in which side of tangent plane of $P$

# Point classification by *II*

$d=0$ : a quadratic equation interms of $du$ or $dv$

Assuming $L \neq 0$, we have

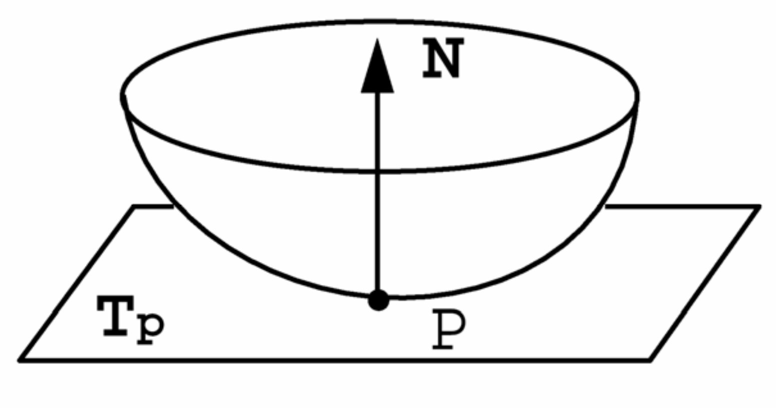
$$du = \frac{-M \pm \sqrt{M^2 - LN}}{L} dv$$



# Point classification: Elliptic point

$M^2$-$LN$<0 (Elliptic point) :

- There is no intersection between the surface and its tangent plane except at point $P$, e.g., ellipsoid

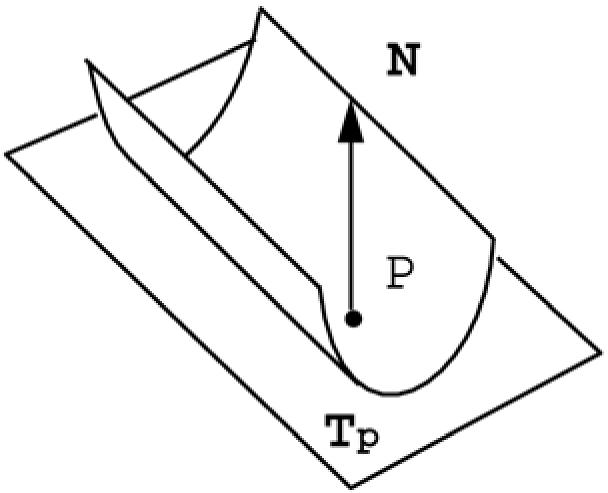



Elliptic point

# Point classification: Parabolic point

$M^2$-$LN$=0 (Parabolic point) :

- There are double roots. The surface intersects its tangent plane with one line $du = -\frac{M}{L}dv$ which passes through point *P*, e.g., a circular cylinder

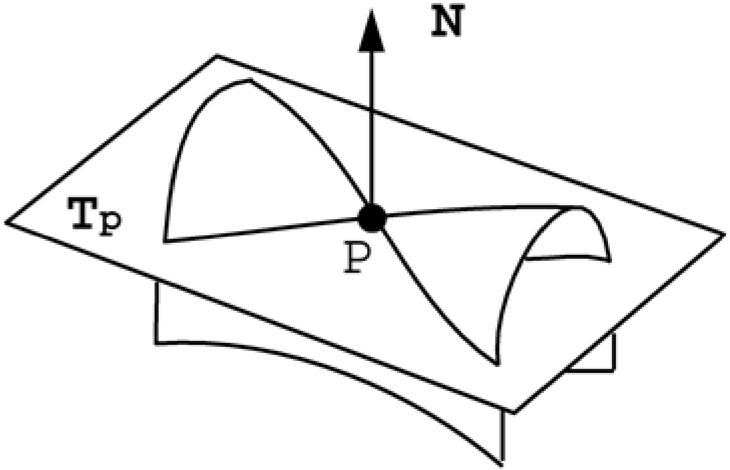



Parabolic point

# Point classification: Hyperbolic point

$M^2$-$LN$>0 (Hyperbolic point) :

- There are two roots. The surface intersects its tangent plane with two lines $du = \frac{-M \pm \sqrt{M^2 - LN}}{L} dv$ which intersect at point *P*, e.g., a hyperbolic of revolution




Hyperbolic point

# Point classification: flat/planar point

$L=M=N=0$ (flat or planar point)

- The surface and the tangent plane have a contact of higher order than in the preceding cases

# Point classification: other cases

- If $L=0$ and $N\neq 0$, we can solve for $dv$ instead of $du$
- If $L=N=0$ and $M\neq 0$, we have $2Mdudv=0$, thus the iso-parametric lines

$$u = const.$$

$$v = const.$$

will be the two intersection lines.

# Download the courses

http://www.cad.zju.edu.cn/home/zhx/GM/GM03.zip