Preliminary Mathematics of Geometric Modeling (3)

Hongxin Zhang and Jieqing Feng

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State Key Lab of CAD&CG, Zhejiang University
Differential Geometry of Surfaces

- Tangent plane and surface normal
- First fundamental form $I$ (metric)
- Second fundamental form $II$ (curvature)
- Principal curvatures
- Gaussian and mean curvatures
  - Explicit surfaces
  - Implicit surfaces
- Euler's theorem and Dupin's indicatrix
Tangent vector on the surface

A parametric surface $\mathbf{r} = \mathbf{r}(u, v)$

A curve $u = u(t)$, $v = v(t)$ in the parametric domain
Tangent vector on the surface

The tangent vector of the curve on the surface respect to the parameter $t$:

$$\mathbf{r}(t) = \mathbf{r}(u(t), v(t))$$

$$\dot{\mathbf{r}}(t) = \mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v}$$

where

$$\dot{u} = \dot{u}(t)$$

$$\dot{v} = \dot{v}(t)$$
The tangent plane at point $P$ can be considered as a union of the tangent vectors for all $r(t)$ through $P$. The tangent plane at a point on a surface.
Suppose: \( P = \mathbf{r}(u_p, v_p) \)

The equation of tangent plane at \( \mathbf{r}(u_p, v_p) \):

\[
\mathbf{T}_p(\mu, \nu) = \mathbf{r}(u_p, v_p) + \mu \mathbf{r}_u(u_p, v_p) + \nu \mathbf{r}_v(u_p, v_p)
\]

Where \( \mu, \nu \) are parameter
The *surface normal* vector is perpendicular to the tangent plane. The *unit normal* is

\[ \mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \]

The implicit form of tangent surface is

\[ (\mathbf{r} - \mathbf{r}(u_p, v_p)) \cdot \mathbf{N}(u_p, v_p) = 0 \]
Surface normal

The normal to the point on a surface
Definition: A regular (ordinary) point $P$ on a parametric surface is defined as a point where $\mathbf{r}_u \times \mathbf{r}_v \neq 0$. A point which is not a regular point is called a singular point.

Notes for

- Regular point:
  1. $\mathbf{r}_u \neq 0$ and $\mathbf{r}_v \neq 0$
  2. $\mathbf{r}_u$ is not parallel to $\mathbf{r}_v$

- Singular point:
  1. Normal may exist at the singular point
Examples of singular point

\[ r_u = 0 \text{ or } r_v = 0 \]

\[ r_u \parallel r_v \]
Types of singular points

- Essential singularities: specific features of the surface geometry
  - Apex of cone

- Artificial singularities
  - Parametrization
Regular surface

- Existence of a tangent plane everywhere on the surface
- Without self-intersection

Surface with intersection
Example: elliptic cone

Parametric form: \( \mathbf{r} = (at \cos \theta, b t \sin \theta, c t)^T \)

Where \( 0 \leq \theta \leq 2\pi, \; 0 \leq t \leq l, \; a, b, c \) are constants

\[
\begin{align*}
\mathbf{r}_\theta &= (-at \sin \theta, bt \cos \theta, 0)^T \\
\mathbf{r}_t &= (a \cos \theta, b \sin \theta, c)^T
\end{align*}
\]

\[
|\mathbf{r}_\theta \times \mathbf{r}_t| = |bct \cos \theta \mathbf{e}_x + act \sin \theta \mathbf{e}_y - abt \mathbf{e}_z|
\]

\[
= \sqrt{t^2(b^2c^2 \cos^2 \theta + a^2c^2 \sin^2 \theta + a^2b^2)}
\]

The apex of the cone \((t=0)\) is singular
Normal of implicit surface

Implicit surface: $f(x,y,z) = 0$

Considering the two parametric curves on the surfaces

$$\mathbf{r}_1 = (x_1(t_1), y_1(t_1), z_1(t_1))$$
$$\mathbf{r}_2 = (x_2(t_2), y_2(t_2), z_2(t_2))$$

The $\mathbf{r}_1$ and $\mathbf{r}_2$ intersect at point $P$

By substituting $\mathbf{r}_1$ and $\mathbf{r}_2$ into $f$, we have
Normal of implicit surface

\[ f(x_1(t_1), y_1(t_1), z(t_1)) = 0 \quad f(x_2(t_2), y_2(t_2), z(t_2)) = 0 \]

By differentiation with \( t_1 \) and \( t_2 \) respectively

\[
\begin{align*}
&f_x \frac{dx_1}{dt_1} + f_y \frac{dy_1}{dt_1} + f_z \frac{dz_1}{dt_1} = 0 \\
&f_x \frac{dx_2}{dt_2} + f_y \frac{dy_2}{dt_2} + f_z \frac{dz_2}{dt_2} = 0
\end{align*}
\]

After simplification, we can deduce:

\[
\begin{align*}
&f_x : f_y : f_z = \\
&\frac{dz_2}{dt_2} \frac{dy_1}{dt_1} - \frac{dz_1}{dt_1} \frac{dy_2}{dt_2} : \frac{dz_1}{dt_1} \frac{dx_2}{dt_2} - \frac{dz_2}{dt_2} \frac{dx_1}{dt_1} : \frac{dx_1}{dt_1} \frac{dy_2}{dt_2} - \frac{dx_2}{dt_2} \frac{dy_1}{dt_1}
\end{align*}
\]
Normal of implicit surface

As we know

\[
\frac{d\mathbf{r}_1(t_1)}{dt_1} \times \frac{d\mathbf{r}_2(t_2)}{dt_2}
\]

\[
= \left( \frac{dz_2}{dt_2} \frac{dy_1}{dt_1} - \frac{dz_1}{dt_1} \frac{dy_2}{dt_2}, \frac{dz_1}{dt_1} \frac{dx_2}{dt_2} - \frac{dz_2}{dt_2} \frac{dx_1}{dt_1}, \frac{dx_1}{dt_1} \frac{dy_2}{dt_2} - \frac{dx_2}{dt_2} \frac{dy_1}{dt_1} \right)^T
\]

Thus the normal is the gradient of \( f \), i.e.

\[
\nabla f = (f_x, f_y, f_z)^T \parallel \frac{d\mathbf{r}_1(t_1)}{dt_1} \times \frac{d\mathbf{r}_2(t_2)}{dt_2}
\]
Normal of implicit surface

Unit normal of the implicit surface

\[ \mathbf{N} = \frac{(f_x, f_y, f_z)^T}{\sqrt{f_x^2 + f_y^2 + f_z^2}} = \frac{\nabla f}{|\nabla f|} \]

provided that \(|\nabla f| \neq 0\)
Tangent plane of implicit surface

The tangent plane of point \( P(x_p, y_p, z_p) \) on the implicit surface \( f(x, y, z) = 0 \) is

\[ \nabla f \cdot (\mathbf{r} - P) = 0 \]

i.e.

\[ f_x(x - x_p) + f_y(y - y_p) + f_z(z - z_p) = 0 \]

\( \mathbf{r} = (x, y, z) \)
Example: elliptic cone

Elliptic cone in implicit form

\[ f(x, y, z) = \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 - \left( \frac{z}{c} \right)^2 = 0 \]

The gradient (normal) is

\[ \nabla f = \left( \frac{2x}{a^2}, \frac{2y}{b^2}, -\frac{2z}{c^2} \right)^T \]

subject to \( (x, y, z) \in f(x, y, z) = 0 \)
First fundamental form / (metric)

The differential arc length of parametric curve on the parametric surface

- Parametric surface \( \mathbf{r} = \mathbf{r}(u,v) \)
- Parametric curve defined the in parametric domain \( u = u(t), \ v = v(t) \)
- The differential arc length of parametric curve

\[
\begin{align*}
\frac{ds}{dt} &= \sqrt{\mathbf{r}(u,v) \cdot \mathbf{r}(u,v)}
\end{align*}
\]
First fundamental form / (metric)

$$ds = \left| \frac{dr}{dt} \right| dt = \left| r_u \frac{du}{dt} + r_v \frac{dv}{dt} \right| dt$$

$$= \sqrt{(r_u \dot{u} + r_v \dot{v}) \cdot (r_u \dot{u} + r_v \dot{v})} dt$$

$$= \sqrt{Edu^2 + 2F du dv + Gdv^2} ,$$

where \( E = r_u \cdot r_u, \quad F = r_u \cdot r_v, \quad G = r_v \cdot r_v \)
First fundamental form

\[ I = ds^2 = dr \cdot dr = Edu^2 + 2Fdu dv + Gdv^2 \]

- \(E, F, G\) : first fundamental form coefficients
- \(E, F, G\) are important for intrinsic properties
- Alternative representation

\[ I = \frac{1}{E} (E \ du + F \ dv)^2 + \frac{EG - F^2}{E} \ dv^2 \]
First fundamental form

\[(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)\]

\[(a \times b) \cdot (a \times b) = (a \cdot a)(b \cdot b) - (a \cdot b)^2\]

Thus

\[
(r_u \times r_v)^2 = (r_u \times r_v) \cdot (r_u \times r_v)
\]

\[
= (r_u \cdot r_u)(r_v \cdot r_v) - (r_u \cdot r_v)^2
\]

\[
= EG - F^2 > 0
\]
First fundamental form

- $I \geq 0$ for arbitrary surface
  - $I > 0$: positive definite provided that the surface is regular
  - $I = 0$ iff $du = 0$ and $dv = 0$
Example: First fundamental form

Hyperbolic paraboloid:
\[ \mathbf{r}(u,v) = (u,v,uv)^T \]
\[ 0 \leq u, v \leq 1 \]

Curve:
\[ u = t, \ v = t. \]
\[ 0 \leq t \leq 1 \]

Aim: arc length of the curve on the surface

Hyberbolic paraboloid
arc length along \( u = t, \ v = t \)
Example: First fundamental form

First fundamental form coefficients

\[ \mathbf{r}_u = (1, 0, v)^T, \quad \mathbf{r}_v = (0, 1, u)^T \]

\[
\begin{align*}
E &= \mathbf{r}_u \cdot \mathbf{r}_u = 1 + v^2 \\
F &= \mathbf{r}_u \cdot \mathbf{r}_v = uv \\
G &= \mathbf{r}_v \cdot \mathbf{r}_v = 1 + u^2
\end{align*}
\]
Example: First fundamental form

First fundamental form coefficients along the curve

\[ E = 1 + t^2, \quad F = t^2, \quad G = 1 + t^2 \]

The differential arc length of the curve

\[ ds = \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} dt = 2\sqrt{t^2 + \frac{1}{2}} dt \]
Example: First fundamental form

The arc length of the curve

\[ s = 2 \int_{0}^{1} \sqrt{t^2 + \frac{1}{2}} \, dt \]

\[ = \left[ t \sqrt{t^2 + \frac{1}{2}} + \frac{1}{2} \log \left( t + \sqrt{t^2 + \frac{1}{2}} \right) \right]_{0}^{1} \]

\[ = \sqrt{\frac{3}{2}} + \frac{1}{2} \log(\sqrt{2} + \sqrt{3}) \]
Application of first fundamental form: angle between curves on surface

- Two curves on a parametric surface
  \[ r_1 = r(u_1(t), v_1(t)) \quad r_2 = r(u_2(t), v_2(t)) \]

  - Angle between \( r_1 \) and \( r_2 \) is the angle between their tangent vectors

  - Angle between two vectors \( a \) and \( b \)

  \[ \cos \omega = \frac{a \cdot b}{|a||b|} \]

  where

  \[ a = r_u du_1 + r_v dv_1 \]

  \[ b = r_u du_2 + r_v dv_2 \]
Application of first fundamental form:
angle between curves on surface

- Angle between curves on the surface

\[
\cos \omega = \frac{E du_1 du_2 + F (du_1 dv_2 + dv_1 du_2) + G dv_1 dv_2}{\sqrt{Edu_1^2 + 2Fdu_1 dv_1 + Gdv_1^2} \sqrt{Edv_2^2 + 2Fdv_2 dv_2 + Gdv_2^2}}
\]

\[
= E \frac{du_1}{ds_1} \frac{du_2}{ds_2} + F \left( \frac{du_1}{ds_1} \frac{dv_2}{ds_2} + \frac{dv_1}{ds_1} \frac{du_2}{ds_2} \right) + G \frac{dv_1}{ds_1} \frac{dv_2}{ds_2}.
\]

- \( r_1 \) and \( r_2 \) is orthogonal \((\cos(\pi/2)=0)\) if

\[
Edu_1 du_2 + F (du_1 dv_2 + dv_1 du_2) + G dv_1 dv_2 = 0
\]
Application of first fundamental form: angle between curves on surface

- Special case: iso-parametric curves

\[ \mathbf{r}_1 : u_1(t) = t, \quad v_1(t) = 0 \]

\[ \mathbf{r}_2 : u_2(t) = 0, \quad v_2(t) = t \]

\[
\cos \omega = \frac{\mathbf{r}_u \cdot \mathbf{r}_v}{\|\mathbf{r}_u\| \|\mathbf{r}_v\|} = \frac{\mathbf{r}_u \cdot \mathbf{r}_v}{\sqrt{\mathbf{r}_u \cdot \mathbf{r}_u} \sqrt{\mathbf{r}_v \cdot \mathbf{r}_v}} = \frac{F}{\sqrt{EG}}
\]

The iso-parametric curves are orthogonal if \( F = 0 \)
Application of first fundamental form: area of the surface patch

The area bounded by four vertices \( r(u,v) \), \( r(u+\delta u,v) \), \( r(u+\delta u,v) \), \( r(u+\delta u,v+\delta v) \)

\[ \delta A \]

\[ r(u_0,v_0+\delta v) \]
\[ r(u_0,v_0) \]

\[ r(u_0,v_0+\delta v) - r(u_0,v_0) \]
\[ r(u_0,v_0) - r(u_0,v_0) \]

Area of small surface patch
Application of first fundamental form: area of the surface patch

The area bounded by four vertices $\mathbf{r}(u,v)$, $\mathbf{r}(u+\delta u,v)$, $\mathbf{r}(u+\delta u,v)$, $\mathbf{r}(u+\delta u,v+\delta v)$

\[
\delta A = |\mathbf{r}_u \delta u \times \mathbf{r}_v \delta v| = \sqrt{EG - F^2} \delta u \delta v
\]

In differential form

\[
dA = \sqrt{EG - F^2} \, du \, dv
\]
Example: area of surface patch

Hyperbolic paraboloid:
\[ \mathbf{r}(u,v) = (u, v, uv)^T \]
\[ 0 \leq u, v \leq 1 \]

Bounded curves:
\[ u = 0; \ v = 0; \]
\[ u^2 + v^2 = 1 \]

Aim: Area of the surface patch bounded by the 3 curves?

Area bounded by positive \( u \) and \( v \) axes and a quarter circle
Example: area of surface patch

First fundamental form coefficients

\[
\mathbf{r}_u = (1, 0, v)^T, \quad \mathbf{r}_v = (0, 1, u)^T
\]

\[
E = \mathbf{r}_u \cdot \mathbf{r}_u = 1 + v^2
\]

\[
F = \mathbf{r}_u \cdot \mathbf{r}_v = uv
\]

\[
G = \mathbf{r}_v \cdot \mathbf{r}_v = 1 + u^2
\]

\[
A = \int_D \sqrt{1 + u^2 + v^2} \, dudv
\]
Example: area of surface patch

After reparametrization of the surface patch by setting \( u = r \cos \theta \), \( v = r \sin \theta \), we have

\[
A = \int_{0}^{\frac{\pi}{2}} \int_{0}^{1} \sqrt{1 + r^2} \ r \ d\theta \ dr = \frac{\pi}{6} (\sqrt{8} - 1)
\]
Second fundamental form // (curvature)

• The second fundamental form quantify the curvatures of a surface

• Consider a curve $C$ on surface $S$ which passes through point $P$
  ◆ The differential geometry of curve
Second fundamental form
(curvature)

Definition of normal curvature
Second fundamental form // (curvature)

The relationship between unit tangent vector $\mathbf{t}$ and unit normal vector $\mathbf{n}$ of the curve $C$ at point $P$

$$\mathbf{k} = \frac{d\mathbf{t}}{ds} = \kappa \mathbf{n} = \mathbf{k}_n + \mathbf{k}_g$$

- Normal curvature vector $\mathbf{k}_n$: component of $\mathbf{k}$ of curve $C$ in the surface normal direction
- Geodesic curvature vector $\mathbf{k}_g$: component of $\mathbf{k}$ of curve $C$ in the direction perpendicular to $\mathbf{t}$ in the surface tangent plane
The normal curvature vector can be expressed as

\[ k_n = \kappa_n N \]

- \( \kappa_n \) is called normal curvature of surface at point \( P \) in the direction \( t \)
- \( \kappa_n \) is the magnitude of the projection of \( k \) onto the surface normal at \( P \)
- The sign of \( \kappa_n \) is determined by the orientation of the surface normal at \( P \)
Second fundamental form // (curvature)

Definition of normal curvature
Differentiating $\mathbf{N} \cdot \mathbf{t} = 0$ along the curve respect to $s$:

$$\frac{dt}{ds} \cdot \mathbf{N} + \mathbf{t} \cdot \frac{d\mathbf{N}}{ds} = 0$$

Combined with $k_n = \kappa_n \mathbf{N}$, Thus

$$\kappa_n = \frac{dt}{ds} \cdot \mathbf{N} = -\mathbf{t} \cdot \frac{d\mathbf{N}}{ds} = -\frac{d\mathbf{r}}{ds} \cdot \frac{d\mathbf{N}}{ds} = -\frac{d\mathbf{r} \cdot d\mathbf{N}}{d\mathbf{r} \cdot d\mathbf{r}}$$

$$= \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$$
Second fundamental form \( \parallel \) (curvature)

where

\[
L = -\mathbf{r}_u \cdot \mathbf{N}_u \\
M = -\frac{1}{2} (\mathbf{r}_u \cdot \mathbf{N}_v + \mathbf{r}_v \cdot \mathbf{N}_u) \\
= -\mathbf{r}_u \cdot \mathbf{N}_v = -\mathbf{r}_v \cdot \mathbf{N}_u \\
N = -\mathbf{r}_v \cdot \mathbf{N}_v
\]
Second fundamental form //
(curvature)

Since \( r_u \perp N \) and \( r_v \perp N \)

\[ r_u \cdot N = 0 \quad \text{and} \quad r_v \cdot N = 0 \]

\[
\begin{align*}
    d(r_u \cdot N)/du &= r_{uu} \cdot N + r_u \cdot N_u = 0 \\
    d(r_v \cdot N)/du &= r_{uv} \cdot N + r_v \cdot N_u = 0 \\
    d(r_u \cdot N)/dv &= r_{uv} \cdot N + r_u \cdot N_v = 0 \\
    d(r_v \cdot N)/dv &= r_{vv} \cdot N + r_v \cdot N_v = 0
\end{align*}
\]
Second fundamental form \( \text{II} \) (curvature)

Alternative expression of \( L, M \) and \( N \)

\[
L = r_{uu} \cdot N \quad M = r_{uv} \cdot N \quad N = r_{vv} \cdot N
\]

The second fundamental form \( \text{II} \)

\[
\text{II} = Ldu^2 + 2M du dv + N dv^2
\]

\( L, M \) and \( N \) are called second fundamental form coefficients
Second fundamental form \( II \) (curvature)

The normal curvature can be expressed as

\[
\kappa_n = \frac{II}{I} = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2}
\]

- \( \lambda = dv/du \) is the direction of the tangent line to \( C \) at \( P \) (in the surface parametric domain)
- \( \kappa_n \) at a given point \( P \) on the surface depends only on \( \lambda \)
Meusnier Theorem

All curves lying on a surface $S$ passing through a given point $p \in S$ with the same tangent line have the same normal curvature at this point.
About sign of normal curvature

- Convention (a): $\kappa n \cdot N = \kappa_n$

The normal curvature is positive when the center of the curvature of the normal section curve, which is a curve through $P$ cut out by a plane that contains $t$ and $N$, is on the same side of the surface normal.
About sign of normal curvature

- Convention (b): $\kappa n \cdot N = -\kappa_n$

The normal curvature is positive when the center of the curvature of the normal section curve, which is a curve through $P$ cut out by a plane that contains $t$ and $N$, is on the opposite side of the surface normal.

Definition of normal curvature (positive)
About sign of normal curvature

- About convention (b)
  - The convention (b) is often used in the area of offset curves and surfaces in the context of NC machining

\[
k_n = -\kappa_n N
\]

\[
k_n = -\frac{II}{I} = -\frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2}
\]
Point classification by $\parallel$

- Suppose $P$ and $Q$ on the surface $r(u,v)$
  
  $P = r(u,v), \quad Q = r(u+du,v+dv)$
Point classification by $\parallel$

$$\mathbf{r}(u + du, v + dv) = \mathbf{r}(u, v) + \mathbf{r}_u du + \mathbf{r}_v dv$$

$$+ \frac{1}{2}(\mathbf{r}_{uu} du^2 + 2\mathbf{r}_{uv} du dv + \mathbf{r}_{vv} dv^2) + \ldots$$

Thus $\mathbf{PQ} = \mathbf{r}(u + du, v + dv) - \mathbf{r}(u, v) = \mathbf{r}_u du + \mathbf{r}_v dv$

$$+ \frac{1}{2}(\mathbf{r}_{uu} du^2 + 2\mathbf{r}_{uv} du dv + \mathbf{r}_{vv} dv^2) + \ldots .$$

Projecting $\mathbf{PQ}$ onto $\mathbf{N}$

$$d = \mathbf{PQ} \cdot \mathbf{N} = (\mathbf{r}_u du + \mathbf{r}_v dv) \cdot \mathbf{N} + \frac{1}{2} II$$
Point classification by $\parallel$

Finally

$$d = \frac{1}{2} \parallel = \frac{1}{2}(Ldu^2 + 2Mdudv + Ndv^2)$$

Thus $|\parallel|$ is equal to twice the distance from $Q$ to the tangent plane of the surface at $P$ within second order terms.

Next, we determine the sign of $\parallel$, i.e. $Q$ lies in which side of tangent plane of $P$.
$d=0$: a quadratic equation in terms of $du$ or $dv$

Assuming $L \neq 0$, we have

$$
du = \frac{-M \pm \sqrt{M^2 - LN}}{L} dv
$$
Point classification: Elliptic point

$M^2 - LN < 0$ (Elliptic point):

- There is no intersection between the surface and its tangent plane except at point $P$, e.g., ellipsoid
Point classification: Parabolic point

\[ M^2 - LN = 0 \] (Parabolic point):

- There are double roots. The surface intersects its tangent plane with one line \( du = -\frac{M}{L} dv \) which passes through point \( P \), e.g., a circular cylinder.
Point classification: Hyperbolic point

$M^2-LN>0$ (Hyperbolic point):

- There are two roots. The surface intersects its tangent plane with two lines which intersect at point $P$, e.g., a hyperbolic of revolution.
Point classification: flat/planar point

$L=M=N=0$ (flat or planar point)

- The surface and the tangent plane have a contact of higher order than in the preceding cases
Point classification: other cases

- If $L=0$ and $N\neq 0$, we can solve for $dv$ instead of $du$

- If $L=N=0$ and $M\neq 0$, we have $2Mdudv=0$, thus the iso-parametric lines
  
  $u = \text{const.}$
  
  $v = \text{const.}$

  will be the two intersection lines.
Download the courses

http://www.cad.zju.edu.cn/home/zhx/GM/GM03.zip