Preliminary Mathematics of Geometric Modeling (2)

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Contents

Representation of Curves and Surfaces

- Planar Curves (2D)
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Representation of Planar Curves

- Parametric Form

$$x = x(t), \quad y = y(t)$$

where $$t_1 \leq t \leq t_2$$

$x(t)$ and $y(t)$ are assumed to be continuous with a sufficient number of continuous derivatives

Representation of Planar Curves

- Vector-Valued Parametric Form

$$\mathbf{r} = \mathbf{r}(t)$$

Representation of Planar Curves

- Implicit Form

$$f(x, y) = 0$$

  - Linear (line): $f(x,y) = ax+by+c = 0$
  - Quadric (Conic sections):

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + h = 0$$

Representation of Planar Curves

- Explicit Form

$$y = F(x) \quad or \quad x = G(y)$$

  - A special case of parametric and implicit forms

Representation of Planar Curves

- Special parametric form if $\frac{dx}{dt} \neq 0 \text{ or } \frac{dy}{dt} \neq 0$

  is satisfied at least locally.

- Special implicit form if $\frac{\partial f}{\partial y} \neq 0 \text{ or } \frac{\partial f}{\partial x} \neq 0$

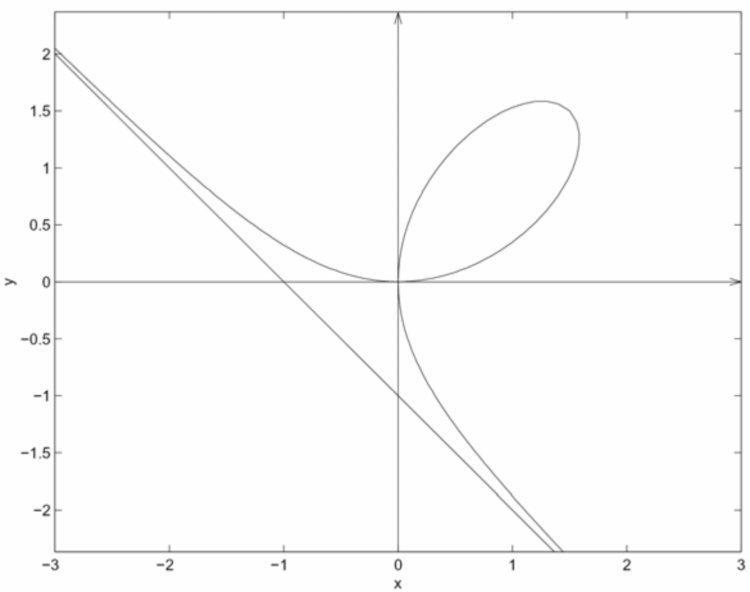
  is satisfied at least locally.

Representation of Planar Curves

- A planar curve can also be expressed as an intersection curve between a plane and a surface

Example of Planar Curves

- Folium of Descartes



Cubic curve with a single loop, a node, and two ends asymptotic to the same line

$$\mathbf{r}(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right)^T$$

$$-\infty < t < \infty \quad (t \neq -1)$$

Parametric form

$$f(x, y) = x^3 + y^3 - 3xy = 0$$

Implicit form

Representation of Space Curves

- The parametric representation of space curves

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

where

$$t_1 \leq t \leq t_2$$

Representation of Space Curves

- The implicit representation of space curves
  - intersection curve between two implicit surfaces

$$f(x, y, z) = 0 \ \cap \ g(x, y, z) = 0$$

  - intersection curve between parametric and implicit surfaces

$$\mathbf{r} = \mathbf{r}(u, v) \ \cap \ f(x, y, z) = 0$$

Representation of Space Curves

- The implicit representation of space curves
  - intersection curve between two parametric surfaces

$$\mathbf{r} = \mathbf{p}(\sigma, t) \ \cap \ \mathbf{r} = \mathbf{q}(u, v)$$

Representation of Space Curves

- The explicit representation of space curves

$$y = Y(x), \quad z = Z(x)$$

It is hold at least locally when

- Parametric form: $\frac{dx}{dt} \neq 0$
- Implicit form: $\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y} \neq 0$

Representation of Space Curves

- The explicit representation of space curves

$$y = Y(x), \quad z = Z(x)$$

The explicit equation for the space curve can be expressed as an intersection curve of two cylinders projecting the curve onto xy and xz planes.

Representation of Surfaces

- Parametric Form

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

where

$$u_1 \leq u \leq u_2, \quad v_1 \leq v \leq v_2$$

Note: the functions $x(u,v)$, $y(u,v)$ and $z(u,v)$ are continuous and possess a sufficient number of continuous partial derivatives

Representation of Surfaces

- Parametric form as vector-valued function

$$\mathbf{r} = \mathbf{r}(u, v)$$

Representation of Surfaces

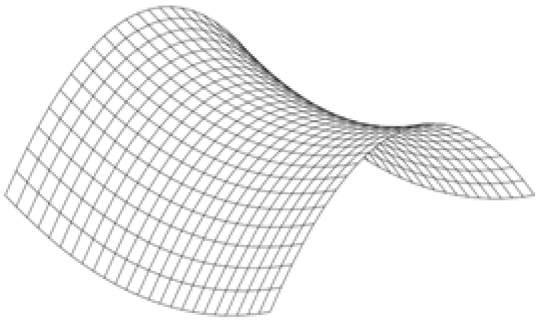
- Implicit form: an *implicit surface* is defined as the locus of points whose coordinates (x,y,z) satisfy an equation of the form
  - $f(x,y,z)$ is linear: $f(x,y,z)=0$
  - $f(x,y,z)$ is quadratic in the variables x,y,z: quadratic surface

$$ax^2+by^2+cz^2+dxy+eyz+hxz+kx+ly+mz+n=0$$

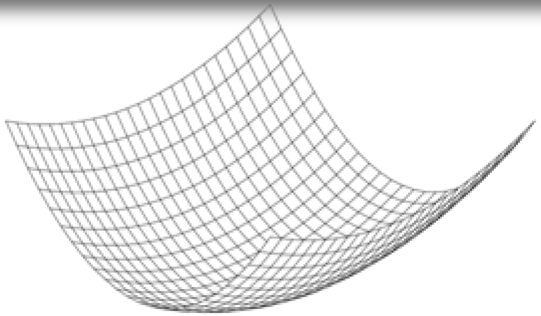
Representation of Surfaces

- Quadratic surfaces:
  - The natural quadrics, sphere, circular cone and circular cylinder: widely used in mechanical design and CAD/CAM systems
  - Result from standard manufacturing operations: rolling, turning(+), filleting(-),drilling and milling
  - 80-85% of mechanical parts were adequately represented by planes and cylinders, while 90-95% were modeled with the addition of cones. (Univ. of Rochester , in the mid 1970's)

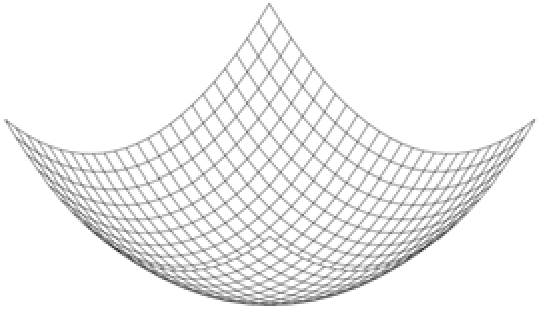
Representation of Surfaces



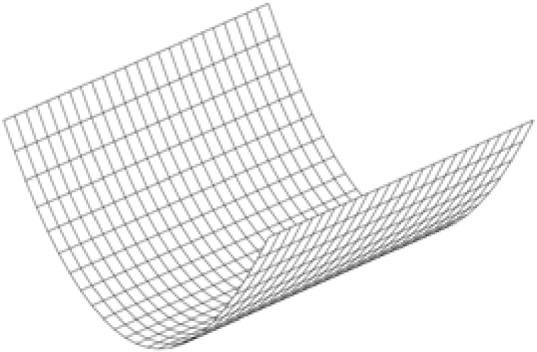
hyperbolic paraboloid



elliptic paraboloid



paraboloid of revolution



parabolic cylinder

Representation of Surfaces

- Explicit Form

$$z = F(x, y)$$

  - From implicit form: locally $\frac{\partial f}{\partial z} \neq 0$
  - From parametric form: locally $\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \neq 0$
  - $z=F(x,y)$: $x=u$, $y=v$, $z=f(u,v)$
    explicit form is special case of parametric form.

Representation of Surfaces

- Example: hyperbolic paraboloid surface patch

  parametric form:

  $$x = u + v, \quad y = u - v, \quad z = u^2 - v^2, \quad 0 \le u, v \le 1$$

  explicit form: $u=(x+y)/2$, $v=(x-y)/2$

  $$z = xy, \quad 0 \le x + y \le 2, \ 0 \le x - y \le 2$$

Summary

Representations of curves and surfaces

Geometry	Parametric	Implicit	Explicit
Plane curves	$x = x(t),\ y = y(t)$ $t_1 \leq t \leq t_2$	$f(x, y) = 0$ or $\mathbf{r} = \mathbf{r}(u, v) \cap$ plane	$y = F(x)$
Space curves	$x = x(t),\ y = y(t),$ $z = z(t),\ t_1 \leq t \leq t_2$	$f(x, y, z) = 0 \cap g(x, y, z) = 0$ or $\mathbf{r} = \mathbf{r}(u, v) \cap f(x, y, z) = 0$ or $\mathbf{r} = \mathbf{p}(\sigma, t) \cap \mathbf{r} = \mathbf{q}(u, v)$	$y = Y(x) \cap$ $z = Z(x)$
Surfaces	$x = x(u, v),$ $y = y(u, v),$ $z = z(u, v),$ $u_1 \leq u \leq u_2,$ $v_1 \leq v \leq v_2$	$f(x, y, z) = 0$	$z = F(x, y)$

Disadvantages		
Explicit	Implicit	Parametric
• Infinite slopes are impossible if $f(x)$ is a polynomial. • Axis dependent (difficult to transform). • Closed and multivalued curves are difficult to represent.	• Difficult to fit and manipulate free form shapes. • Axis dependent. • Complex to trace.	• High flexibility complicates intersections and point classification.
Advantages		
Explicit	Implicit	Parametric
• Easy to trace.	• Closed and multivalued curves and infinite slopes can be represented. • Point classification (solid modeling, interference check) is easy. • Intersections/offsets can be represented.	• Closed and multivalued curves and infinite slopes can be represented. • Axis independent (easy to transform). • Easy to generate composite curves. • Easy to trace. • Easy in fitting and manipulating free-form shapes.

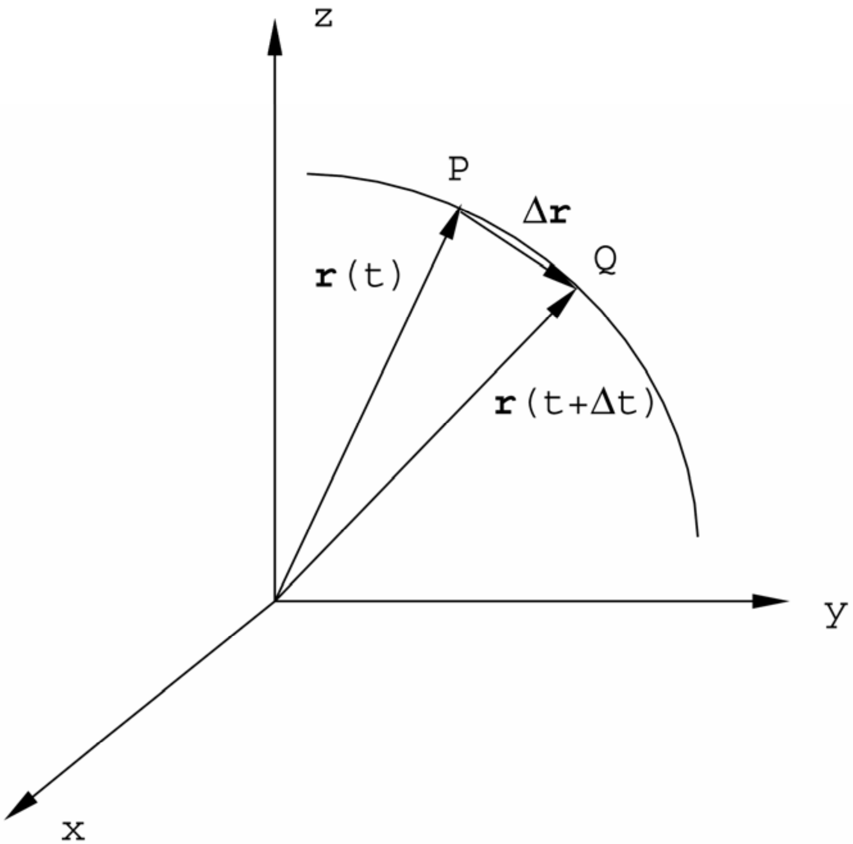
Summary

- The parametric form is the most versatile method among the three and the explicit is the least
- The explicit form can always be easily converted to parametric form
- Conversion between parametric form and implicit form

Differential Geometry of Curves

- Arc length and tangent vector
- Principal normal and curvature
- Binormal vector and torsion
- Frenet-Serret formulae

Arc length and tangent vector



Parametric curve $\mathbf{r}=\mathbf{r}(t)$

Segment between PQ

$P(\mathbf{r}(t))$ $Q(\mathbf{r}(t+\Delta t))$

Its arc length Δs

$$\Delta s \simeq |\Delta \mathbf{r}| = |\mathbf{r}(t+\Delta t) - \mathbf{r}(t)|$$

$$= \left| \frac{d\mathbf{r}}{dt}\Delta t + \frac{d^2\mathbf{r}}{dt^2}(\Delta t)^2 \right| \simeq \left| \frac{d\mathbf{r}}{dt} \right| \Delta t$$

$$\boxed{ds = \left| \frac{d\mathbf{r}}{dt} \right| dt = |\dot{\mathbf{r}}| dt = \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}\, dt}$$

Arc length and tangent vector

- Arc length between $\mathbf{r}(t_0)$ and $\mathbf{r}(t)$

$$\boxed{s(t) = \int_{t_o}^{t} ds = \int_{t_o}^{t} \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}dt = \int_{t_o}^{t} \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}dt}$$

- Tangent vector $d\mathbf{r}/dt$: $\dot{\mathbf{r}}$erivative about t

$$|\dot{\mathbf{r}}| = \frac{ds}{dt}$$

  - $|\dot{\mathbf{r}}|$ is called parametric speed

Arc length and tangent vector

- Useful formulae of the derivatives (s / t)

$$\dot{s} = \frac{ds}{dt} = |\dot{\mathbf{r}}| = \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}$$

$$\ddot{s} = \frac{d\dot{s}}{dt} = \frac{\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}}{\sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}} ,$$

$$\dddot{s} = \frac{d\ddot{s}}{dt} = \frac{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})(\dot{\mathbf{r}} \cdot \dddot{\mathbf{r}} + \ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}}) - (\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{\frac{3}{2}}}$$

$$t' = \frac{dt}{ds} = \frac{1}{|\dot{\mathbf{r}}|} = \frac{1}{\sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}} ,$$

$$t'' = \frac{dt'}{ds} = -\frac{\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}}}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^2} ,$$

$$t''' = \frac{dt''}{ds} = -\frac{(\ddot{\mathbf{r}} \cdot \ddot{\mathbf{r}} + \dot{\mathbf{r}} \cdot \dddot{\mathbf{r}})(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) - 4(\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}})^2}{(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})^{\frac{7}{2}}}$$

Arc length and tangent vector

- Derivative about arc length: $\mathbf{r}'$

$$\mathbf{t} = \frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} = \frac{\frac{d\mathbf{r}}{dt}}{\frac{ds}{dt}} = \frac{d\mathbf{r}}{ds} \equiv \mathbf{r}'$$

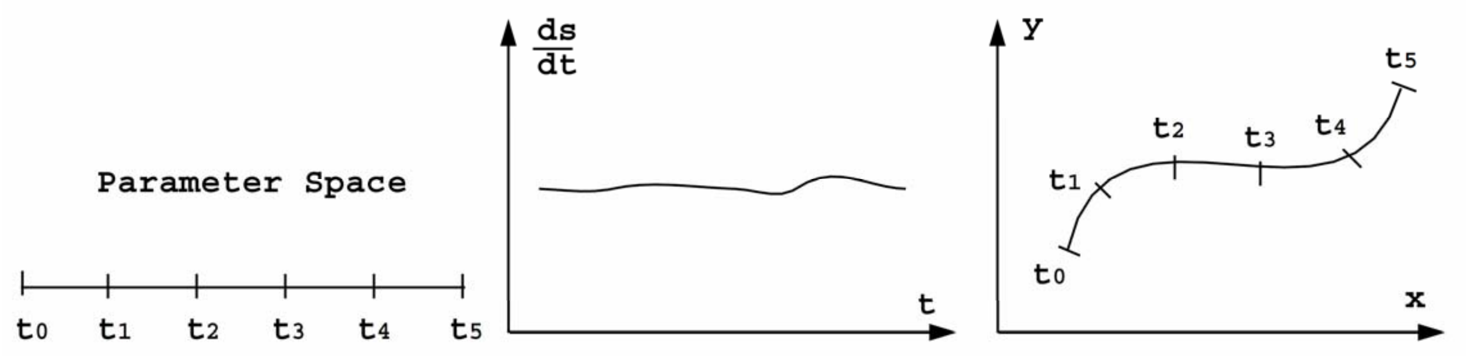
  - It is a unit vector

Arc length and tangent vector

- Definition: A regular (ordinary) point P on a parametric curve $\mathbf{r}=\mathbf{r}(t)=(x(t),y(t),z(t))^T$ is defined as a point where $\boxed{|\dot{\mathbf{r}}(t)| \neq 0}$. A point which is not a regular point is called a singular point.

Arc length and tangent vector

- Parameter t and Arc length parameter s
  - t: arbitrary speed
  - s: unit speed



When parametric speed does not vary significantly, points with uniformly spaced parameter values are nearly uniformly spaced along a parametric curve

Arc length and tangent vector

- Parameter t and Arc length parameter s
  - Every regular curve has an arc length parametrization
  - In practice it is very difficult to find it analytically, due to the fact that it is hard to integrate analytically
  - *Pythagorean hodograph* (*PH*) curves form a class of special planar polynomial curves whose parametric speed is a polynomial

Arc length and tangent vector

- Tangent vector of implicit planar curve
  - $f(x,y)=0$ total differentiation

$$df = f_x\,dx + f_y\,dy = 0$$

by assuming $f_y \neq 0$ $\quad \dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

  - Unit tangent vector :

$$\boxed{\mathbf{t} = \pm\frac{(f_y, -f_x)^T}{\sqrt{f_x^2 + f_y^2}}}$$

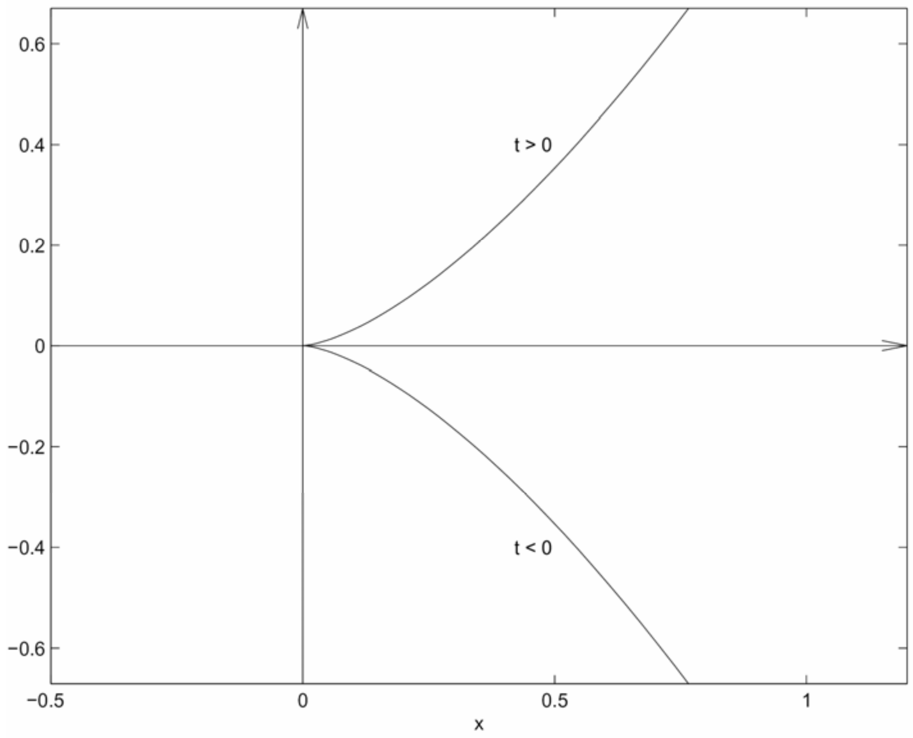
Arc length and tangent vector

- Tangent vector of implicit space curve
  - Curve: $f(x,y,z)=0 \cap g(x,y,z)=0$
  - Gradient vector operator: $$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T$$
  - Unit tangent vector : $$\boxed{\mathbf{t} = \pm\frac{\nabla f \times \nabla g}{|\nabla f \times \nabla g|}}$$

Arc length and tangent vector

- Example $\mathbf{r}(t) = (t^2, t^3)^T$
  - Parametric speed $|\dot{\mathbf{r}}(t)| = \sqrt{t^2(4+9t^2)}$
  - $\mathbf{r}(0)$ is a singular point
  - Implicit form: $f(x,y) = x^3 - y^2 = 0$
  - (0,0) is a singular point since $$f(0,0) = f_x(0,0) = f_y(0,0) = 0$$

Arc length and tangent vector



A singular point occurs on a semi-cubical parabola in the form of a cusp

Principal normal and curvature

- $\mathbf{r}(s)$: an arc length parametrized curve

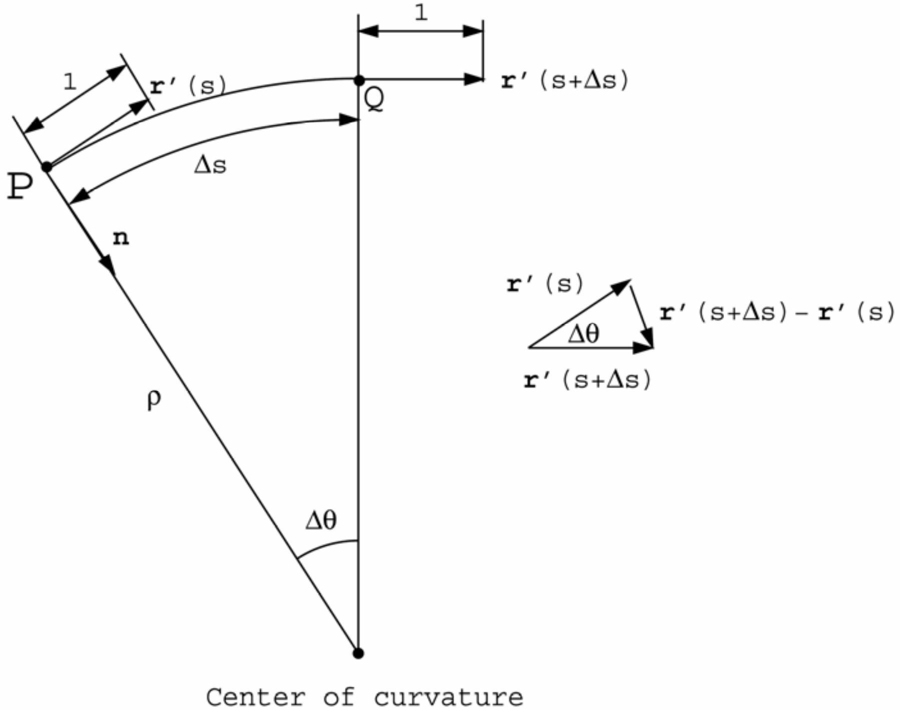
  $\mathbf{r}'(s)$ unit tangent vector

$$\mathbf{r}' \cdot \mathbf{r}' = 1 \quad \Longrightarrow \quad \mathbf{r}' \cdot \mathbf{r}'' = 0$$

$\mathbf{r}''(s)$ is orthogonal to $\mathbf{r}'(s)$

$$\mathbf{r}''(s) = \lim_{\Delta s \to 0} \frac{\mathbf{r}'(s + \Delta s) - \mathbf{r}'(s)}{\Delta s}$$

Principal normal and curvature



Derivation of the normal vector of a curve

$$\mathbf{r}''(s) = \lim_{\Delta s \to 0} \frac{\mathbf{r}'(s+\Delta s) - \mathbf{r}'(s)}{\Delta s}$$

$$\mathbf{n} = \frac{\mathbf{r}''(s)}{|\mathbf{r}''(s)|} = \frac{\mathbf{t}'(s)}{|\mathbf{t}'(s)|}$$

$\mathbf{n}$: unit principal normal vector

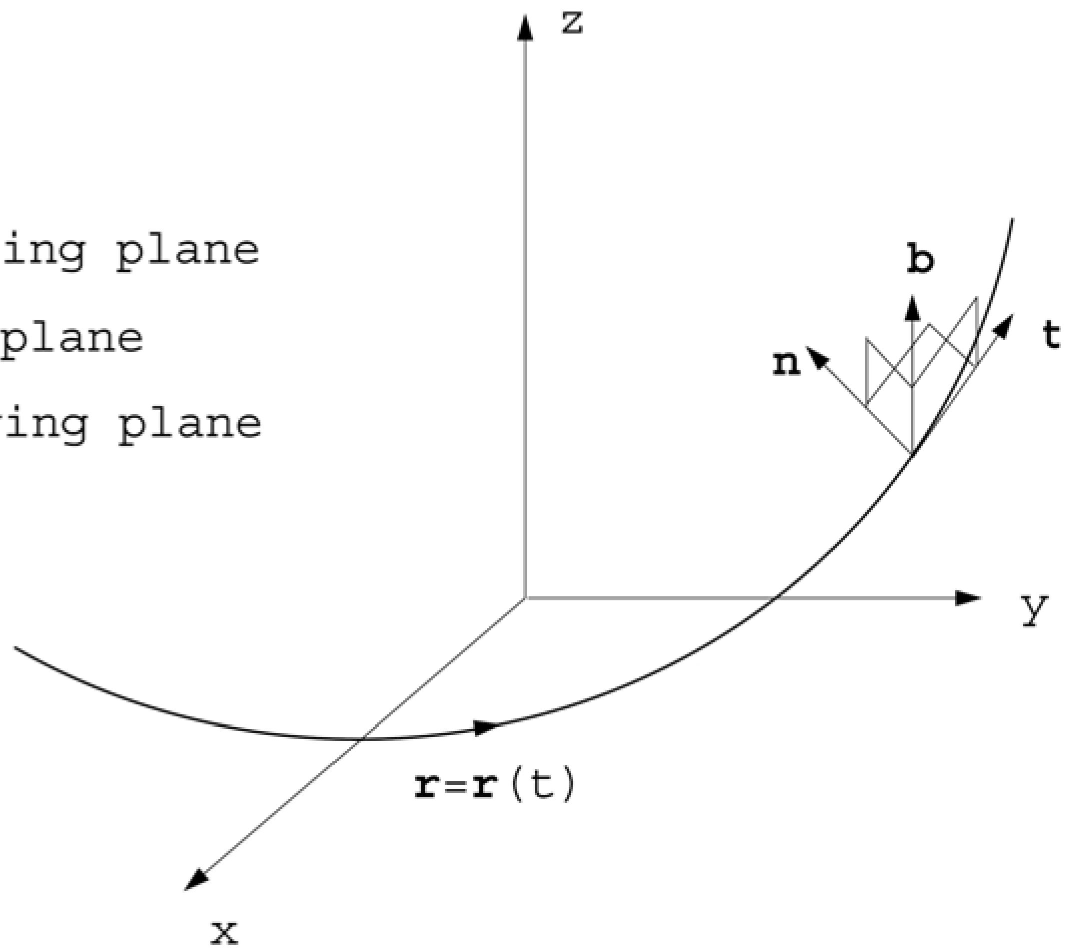
The plane determined by $\mathbf{t}(s)$ and $\mathbf{n}(s)$ is called osculating plane

Principal normal and curvature

(**t**,**n**): osculating plane

(**n**,**b**): normal plane

(**b**,**t**): rectifying plane



Principal normal and curvature

- Calculation of $|\mathbf{r}''(s)|$

$$|\mathbf{r}'(s+\Delta s)-\mathbf{r}'(s)| = \Delta\theta \cdot 1 = \Delta\theta$$

$$|\mathbf{r}''(s)| = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\varrho\Delta\theta} = \frac{1}{\varrho} \equiv \kappa$$

ϱ : *radius of curvature*

κ : *curvature*

Principal normal and curvature

- Curvature vector

$$\mathbf{k} = \mathbf{r}'' = \mathbf{t}' = \kappa \mathbf{n}$$

  - measures the rate of change of the tangent along the curve
  - It is the same as $\mathbf{r}''(s)$ provided defining κ is nonnegative

Principal normal and curvature

- Curvature for arbitrary speed curve (non-arc-length parametrized)

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{t}v \ ,$$

$$\ddot{\mathbf{r}} = \frac{d}{dt}[\mathbf{t}v] = \frac{d\mathbf{t}}{ds}v^2 + \mathbf{t}\frac{dv}{dt} = \kappa\mathbf{n}v^2 + \mathbf{t}\frac{dv}{dt}$$

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \kappa v^3 \mathbf{t} \times \mathbf{n}$$

$$\boxed{\kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}}$$

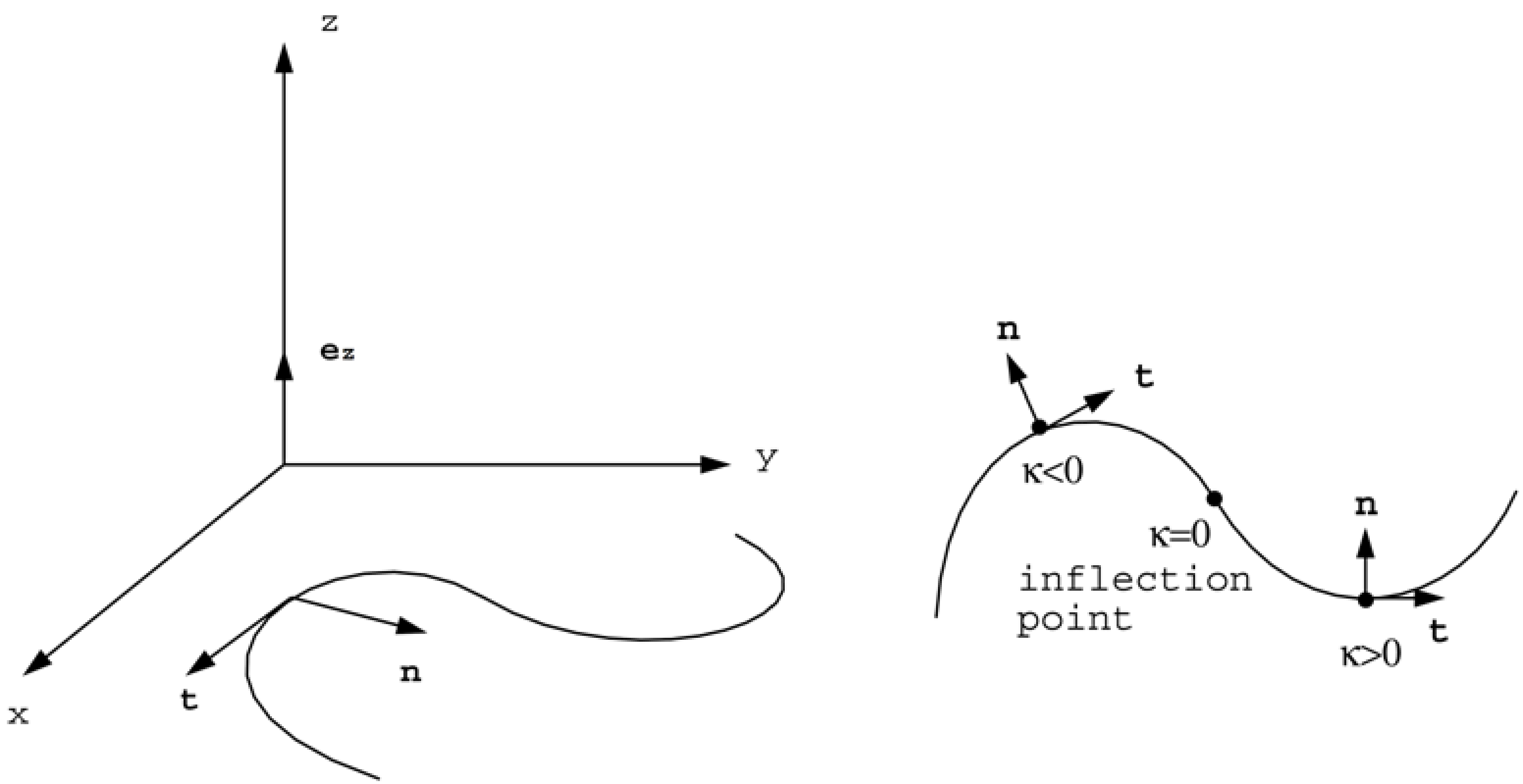
Principal normal and curvature

- Curvature for the planar curve
  - Give the curvature κ a sign by defining the normal vector such that $(\mathbf{t},\mathbf{n},\mathbf{e}_z)$ form a right-handed screw

$$\mathbf{e}_z=(0,0,1)^T$$

  - The point where the curvature changes sign is called an *inflection point*

Principal normal and curvature



Normal and tangent vectors along a 2D curve

Principal normal and curvature

- The unit normal vector of the plane curve

$$\mathbf{n} = \mathbf{e}_z \times \mathbf{t} = \frac{(-\dot{y}, \dot{x})^T}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

- The curvature of the plane curve

$$\kappa = \frac{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot \mathbf{e}_z}{v^3} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}$$

Principal normal and curvature

- The unit principal normal vector for planar implicit curves (2D)

Unit tangent vector

$$\mathbf{t} = \pm \frac{(f_y, -f_x)^T}{\sqrt{f_x^2 + f_y^2}}$$

Unit principal normal vector

$$\mathbf{n} = \mathbf{e}_z \times \mathbf{t} = \frac{(f_x, f_y)^T}{\sqrt{f_x^2 + f_y^2}} = \frac{\nabla f}{|\nabla f|}$$

Principal normal and curvature

- The curvature for planar implicit curves

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

$$\left(\frac{dx}{ds}, \frac{dy}{ds}\right)^T = \mathbf{t} = \pm\frac{(f_y, -f_x)^T}{\sqrt{f_x^2 + f_y^2}}$$

$$\Rightarrow \quad \frac{d}{ds} = \frac{1}{|\nabla f|}\left(f_y\frac{\partial}{\partial x} - f_x\frac{\partial}{\partial y}\right)$$

Principal normal and curvature

- The curvature for planar implicit curves

$$\frac{d}{ds} = \frac{1}{|\nabla f|}\left(f_y \frac{\partial}{\partial x} - f_x \frac{\partial}{\partial y}\right)$$

$$\left(\frac{dx}{ds}, \frac{dy}{ds}\right)^T = \mathbf{t} = \pm \frac{(f_y, -f_x)^T}{\sqrt{f_x^2 + f_y^2}}$$

$$\Rightarrow \quad \boxed{\kappa = -\frac{f_{xx} f_y^2 - 2 f_{xy} f_x f_y + f_x^2 f_{yy}}{(f_x^2 + f_y^2)^{\frac{3}{2}}}}$$

Principal normal and curvature

- The curvature for space implicit curves

$$\frac{d}{ds} = \frac{1}{|\boldsymbol{\alpha}|}\left(\alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial z}\right)$$

where

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = \nabla f \times \nabla g$$

$$\alpha_1 = \frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial z}$$

$$\alpha_2 = \frac{\partial g}{\partial x}\frac{\partial f}{\partial z} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial z}$$

$$\alpha_3 = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$$

Principal normal and curvature

$$\frac{d}{ds} = \frac{1}{|\boldsymbol{\alpha}|}\left(\alpha_1\frac{\partial}{\partial x} + \alpha_2\frac{\partial}{\partial y} + \alpha_3\frac{\partial}{\partial z}\right)$$

$$|\boldsymbol{\alpha}|\mathbf{t} = \boldsymbol{\alpha}$$

$$\Rightarrow \quad \frac{d|\boldsymbol{\alpha}|\mathbf{t}}{ds} = \frac{1}{|\boldsymbol{\alpha}|}\left(\alpha_1\frac{\partial \boldsymbol{\alpha}}{\partial x} + \alpha_2\frac{\partial \boldsymbol{\alpha}}{\partial y} + \alpha_3\frac{\partial \boldsymbol{\alpha}}{\partial z}\right)$$

$$\ddot{\mathbf{r}} = \frac{d}{dt}[\mathbf{t}v] = \frac{d\mathbf{t}}{ds}v^2 + \mathbf{t}\frac{dv}{dt} = \kappa\mathbf{n}v^2 + \mathbf{t}\frac{dv}{dt}$$

Principal normal and curvature

$$|\boldsymbol{\alpha}|^2\kappa\mathbf{n} + |\boldsymbol{\alpha}||\boldsymbol{\alpha}|'\mathbf{t} = \left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x} + \alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y} + \alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right)$$

Cross product with

$$|\boldsymbol{\alpha}|^3\kappa\mathbf{b} = \boldsymbol{\alpha} \times \left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x} + \alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y} + \alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right)$$

Thus

$$\boxed{\kappa = \frac{\left|\boldsymbol{\alpha} \times \left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x} + \alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y} + \alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right)\right|}{|\boldsymbol{\alpha}|^3}}$$

Binormal vector and torsion

- Define a unit binormal vector **b**

$$\boxed{\mathbf{b} = \mathbf{t} \times \mathbf{n}}$$

- (**t**,**n**,**b**) form a right-handed screw

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} \qquad \mathbf{t} = \mathbf{n} \times \mathbf{b} \qquad \mathbf{n} = \mathbf{b} \times \mathbf{t}$$

  - Normal plane: (**n**,**b**)
  - Rectifying plane: (**b**,**t**)
  - Osculating plane: (**t**,**n**)

Binomal for arbitrary speed parameter t

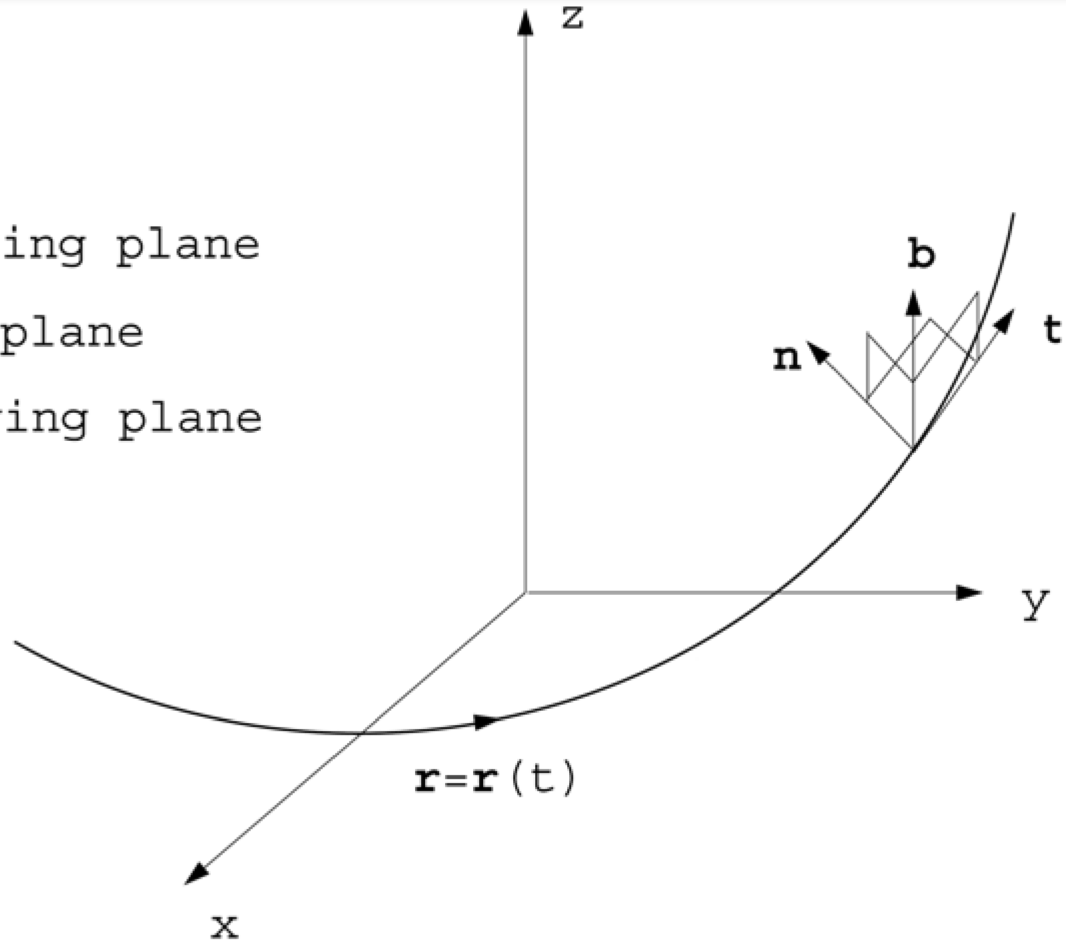
$$\boxed{\mathbf{b} = \frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}}$$

Binormal vector and torsion

(**t**,**n**): osculating plane

(**n**,**b**): normal plane

(**b**,**t**): rectifying plane



Binormal vector and torsion

- Torsion: measures how much the curve deviates from the osculating plane

$$\mathbf{b}' = \frac{d}{ds}(\mathbf{t} \times \mathbf{n}) = \frac{d\mathbf{t}}{ds} \times \mathbf{n} + \mathbf{t} \times \frac{d\mathbf{n}}{ds} = \mathbf{t} \times \mathbf{n}'$$

$$\mathbf{t}' = \kappa \mathbf{n}$$

$$\mathbf{n} \cdot \mathbf{n} = 1 \Rightarrow \mathbf{n} \cdot \mathbf{n}' = 0 \Rightarrow$$

$\mathbf{n}'$ is parallel to the rectifying plane $(\mathbf{b},\mathbf{t})$

$$\mathbf{n}' = \mu \mathbf{t} + \tau \mathbf{b}$$

$$\mathbf{b}' = \mathbf{t} \times (\mu \mathbf{t} + \tau \mathbf{b}) = \tau \mathbf{t} \times \mathbf{b} = -\tau \mathbf{b} \times \mathbf{t} = -\tau \mathbf{n}$$

Binormal vector and torsion

- Torsion for arc length parameter

$$\tau = -\mathbf{n} \cdot \mathbf{b}' = -\frac{\mathbf{r}''}{\kappa} \cdot \left(\mathbf{r}' \times \frac{\mathbf{r}''}{\kappa} \right)' = -\frac{\mathbf{r}''}{\kappa} \cdot \left(\mathbf{r}' \times \frac{\mathbf{r}'''}{\kappa} \right) = \frac{(\mathbf{r}'\mathbf{r}''\mathbf{r}''')}{\mathbf{r}'' \cdot \mathbf{r}''}$$

- Torsion for arbitrary speed parameter

$$\tau = \frac{(\dot{\mathbf{r}}\ddot{\mathbf{r}}\dddot{\mathbf{r}})}{(\dot{\mathbf{r}} \times \ddot{\mathbf{r}}) \cdot (\dot{\mathbf{r}} \times \ddot{\mathbf{r}})}$$

Binormal vector and torsion

- Geometric interpretation of torsion
  - $\tau > 0$: the rotation of the osculating plane is in the direction of a right-handed screw moving in the direction of $\mathbf{t}$ as s increases
  - $\tau < 0$: the rotation of the osculating plane is in the direction of a left-handed screw moving in the direction of $\mathbf{t}$ as s increases
  - $\tau \equiv 0$: planar curve

Binormal vector and torsion

- Binormal vector for space implicit curve

$$\mathbf{b} = \frac{\boldsymbol{\alpha} \times \left(\alpha_1 \frac{\partial \boldsymbol{\alpha}}{\partial x} + \alpha_2 \frac{\partial \boldsymbol{\alpha}}{\partial y} + \alpha_3 \frac{\partial \boldsymbol{\alpha}}{\partial z}\right)}{|\boldsymbol{\alpha} \times \left(\alpha_1 \frac{\partial \boldsymbol{\alpha}}{\partial x} + \alpha_2 \frac{\partial \boldsymbol{\alpha}}{\partial y} + \alpha_3 \frac{\partial \boldsymbol{\alpha}}{\partial z}\right)|}$$

Binormal vector and torsion

- Torsion of space implicit curve

$$\frac{d}{ds} = \frac{1}{|\boldsymbol{\alpha}|}\left(\alpha_1\frac{\partial}{\partial x} + \alpha_2\frac{\partial}{\partial y} + \alpha_3\frac{\partial}{\partial z}\right)$$

$$|\boldsymbol{\alpha}|^3\kappa\mathbf{b} = \boldsymbol{\alpha} \times \left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x} + \alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y} + \alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right)$$

$$\frac{d}{ds}(|\boldsymbol{\alpha}|^3\kappa\mathbf{b}) =$$

$$\frac{1}{|\boldsymbol{\alpha}|}\left(\alpha_1\frac{\partial}{\partial x} + \alpha_2\frac{\partial}{\partial y} + \alpha_3\frac{\partial}{\partial z}\right)\left(\boldsymbol{\alpha} \times \left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x} + \alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y} + \alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right)\right)$$

Binormal vector and torsion

$$|\boldsymbol{\alpha}|(|\boldsymbol{\alpha}|^3\kappa)'\mathbf{b} - |\boldsymbol{\alpha}|^4\kappa\tau\mathbf{n} =$$
$$\left(\alpha_1\frac{\partial}{\partial x} + \alpha_2\frac{\partial}{\partial y} + \alpha_3\frac{\partial}{\partial z}\right)\left(\boldsymbol{\alpha} \times \left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x} + \alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y} + \alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right)\right)$$

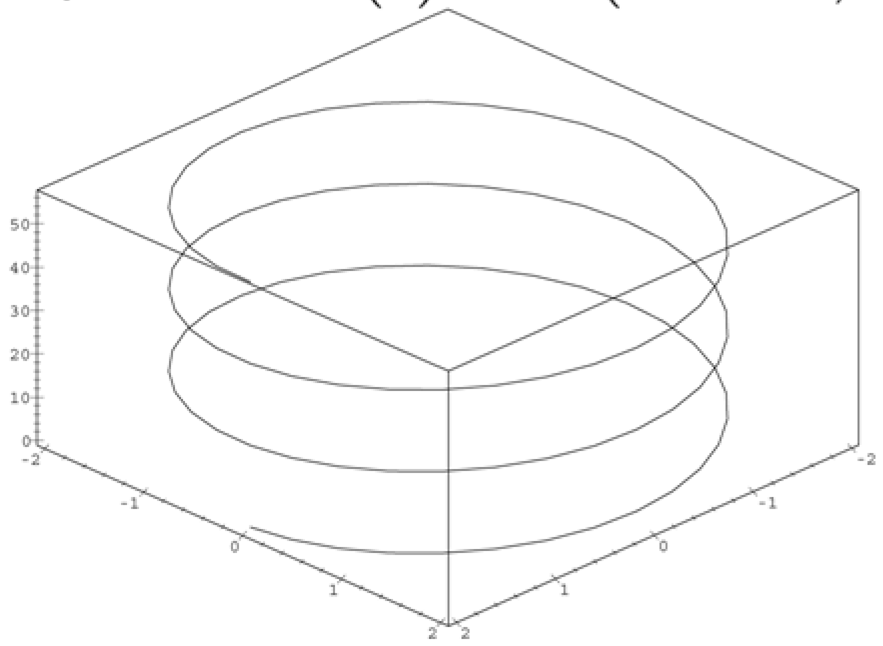
! dot product !

$$|\boldsymbol{\alpha}|^2\kappa\mathbf{n} + |\boldsymbol{\alpha}||\boldsymbol{\alpha}|'\mathbf{t} = \left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x} + \alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y} + \alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right)$$

$$\boxed{\begin{aligned}&-|\boldsymbol{\alpha}|^6\kappa^2\tau = \left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x} + \alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y} + \alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right)\\ &\cdot\left(\alpha_1\frac{\partial}{\partial x} + \alpha_2\frac{\partial}{\partial y} + \alpha_3\frac{\partial}{\partial z}\right)\left(\boldsymbol{\alpha} \times \left(\alpha_1\frac{\partial\boldsymbol{\alpha}}{\partial x} + \alpha_2\frac{\partial\boldsymbol{\alpha}}{\partial y} + \alpha_3\frac{\partial\boldsymbol{\alpha}}{\partial z}\right)\right)\end{aligned}}$$

Binormal vector and torsion

A circular helix $\mathbf{r}(t) = (a\cos t, a\sin t, bt)^T$



Circular helix with $a = 2, b = 3$ for $0 \leqslant t \leqslant 6\pi$

Binormal vector and torsion

The parametric speed $|\dot{\mathbf{r}}(t)| = \sqrt{a^2 + b^2} \equiv c$

Arc length $s(t) = \int_0^t |\dot{\mathbf{r}}| dt = \int_0^t \sqrt{a^2 + b^2} dt = ct$

Arc length parameterization

$$\mathbf{r} = (a \cos \tfrac{s}{c}, a \sin \tfrac{s}{c}, \tfrac{bs}{c})^T$$

Binormal vector and torsion

Derivatives to arc length

$$\mathbf{r}'(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right)^T$$

$$\mathbf{r}''(s) = \left(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right)^T$$

$$\mathbf{r}'''(s) = \left(\frac{a}{c^3}\sin\frac{s}{c}, -\frac{a}{c^3}\cos\frac{s}{c}, 0\right)^T$$

Binormal vector and torsion

Curvature and torsion

$$\kappa^2 = \mathbf{r}'' \cdot \mathbf{r}'' = \frac{a^2}{c^4}\left(\cos^2\frac{s}{c} + \sin^2\frac{s}{c}\right) = \frac{a^2}{c^4} = constant\ ,$$

$$\tau = \frac{(\mathbf{r}'\mathbf{r}''\mathbf{r}''')}{\mathbf{r}'' \cdot \mathbf{r}''} = \frac{(\mathbf{r}'\mathbf{r}''\mathbf{r}''')}{\kappa^2} = \frac{c^4}{a^2}\begin{vmatrix} -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{b}{c} \\ -\frac{a}{c^2}\cos\frac{s}{c} & -\frac{a}{c^2}\sin\frac{s}{c} & 0 \\ \frac{a}{c^3}\sin\frac{s}{c} & -\frac{a}{c^3}\cos\frac{s}{c} & 0 \end{vmatrix}$$

$$= \frac{c^4}{a^2}\frac{b}{c}\frac{a^2}{c^5}\left(\cos^2\frac{s}{c} + \sin^2\frac{s}{c}\right) = \frac{b}{c^2} = constant$$

Frenet-Serret formulae

We have known

$$\mathbf{t}' = \kappa \mathbf{n}$$

$$\mathbf{b}' = -\tau \mathbf{n}$$

$$\mathbf{n}' = (\mathbf{b} \times \mathbf{t})' = \mathbf{b}' \times \mathbf{t} + \mathbf{b} \times \mathbf{t}'$$

$$= -\tau \mathbf{n} \times \mathbf{t} + \mathbf{b} \times (\kappa \mathbf{n}) = -\kappa \mathbf{t} + \tau \mathbf{b}$$

Frenet-Serret formulae

For the arc length parameter

$$\boxed{\begin{pmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}}$$

Frenet-Serret formulae

Frenet-Serret formulae

Intrinsic equations of the curve

$$\kappa = \kappa(s), \ \tau = \tau(s)$$

They totally decide the shape of curve except for a rigid transformation!

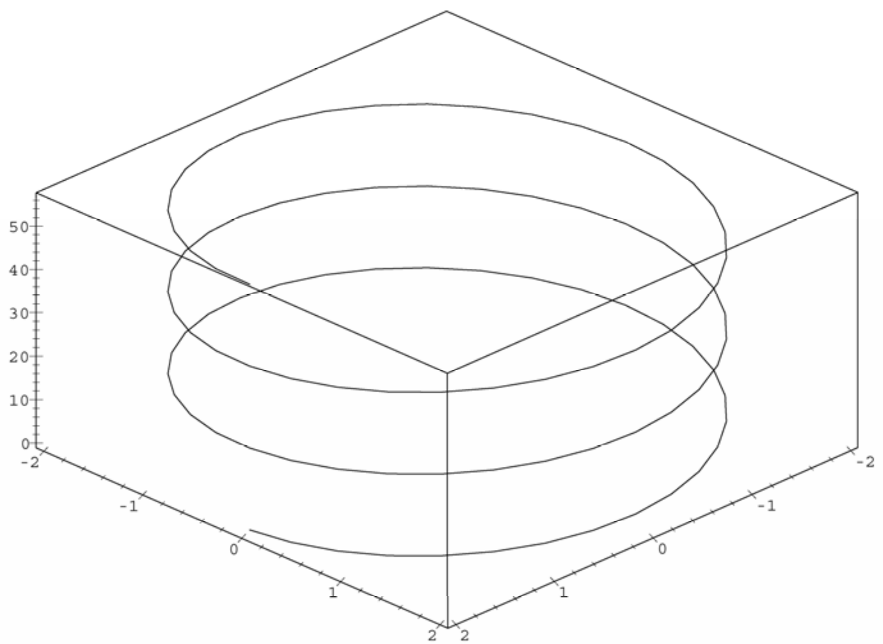
Frenet-Serret formulae

For the arbitrary speed parameter

$$\boxed{\begin{pmatrix} \dot{\mathbf{t}} \\ \dot{\mathbf{n}} \\ \dot{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} 0 & \upsilon\kappa & 0 \\ -\upsilon\kappa & 0 & \upsilon\tau \\ 0 & -\upsilon\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}}$$

Frenet-Serret formulae: example

A circular helix $\mathbf{r}(t) = (a\cos t, a\sin t, bt)^T$



Frenet-Serret formulae: example

A circular helix $\mathbf{r}(t) = (a\cos t, a\sin t, bt)^T$

Arc length parametrization

$$\mathbf{r} = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c})^T$$

Intrinsic equations

$$\kappa(s) = \frac{a}{c^2}, \tau(s) = \frac{b}{c^2}$$

$$c = \sqrt{a^2 + b^2}$$

Frenet-Serret formulae: example

Frenet-Serret equations

$$\frac{d\mathbf{t}}{ds} = \frac{a}{c^2}\mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = -\frac{a}{c^2}\mathbf{t} + \frac{b}{c^2}\mathbf{b}, \qquad \frac{d\mathbf{b}}{ds} = -\frac{b}{c^2}\mathbf{n}$$

Differentiate the first equation twice and the second equation once

$$\frac{d^2\mathbf{t}}{ds^2} = \frac{a}{c^2}\frac{d\mathbf{n}}{ds}, \qquad \frac{d^3\mathbf{t}}{ds^3} = \frac{a}{c^2}\frac{d^2\mathbf{n}}{ds^2}, \qquad \frac{d^2\mathbf{n}}{ds^2} = -\frac{a}{c^2}\frac{d\mathbf{t}}{ds} - \frac{b^2}{c^4}\mathbf{n}$$

Frenet-Serret formulae: example

Eliminating $\mathbf{n}, \frac{d\mathbf{n}}{ds}, \frac{d^2\mathbf{n}}{ds^2}$

$$\frac{d^4\mathbf{r}}{ds^4} + \frac{1}{c^2}\frac{d^2\mathbf{r}}{ds^2} = 0$$

The general solution is

$$\mathbf{r}(s) = \mathbf{C}_1 + \mathbf{C}_2 s + \mathbf{C}_3 \cos\frac{s}{c} + \mathbf{C}_4 \sin\frac{s}{c}$$

Frenet-Serret formulae: example

Given initial conditions

$$\mathbf{r}(0) = (a, 0, 0)^T$$

$$\mathbf{r}'(0) = \left(0, \frac{a}{c}, \frac{b}{c}\right)^T$$

$$\mathbf{r}''(0) = \left(-\frac{a}{c^2}, 0, 0\right)^T$$

$$\mathbf{r}'''(0) = \left(0, -\frac{a}{c^3}, 0\right)^T$$

$\Rightarrow$

$$\mathbf{C}_1 = (0, 0, 0)^T$$

$$\mathbf{C}_2 = \left(0, 0, \frac{b}{c}\right)^T$$

$$\mathbf{C}_3 = (a, 0, 0)^T$$

$$\mathbf{C}_4 = (0, a, 0)^T$$

Frenet-Serret formulae: example

Finally, A circular helix of arc length parametrization is

$$\mathbf{r} = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c}\right)^T$$

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