Preliminary Mathematics of Geometric Modeling (1)

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Contents

Coordinate Systems

- *Cartesian* coordinate system

Coordinate Systems

- Frame $\mathcal{F} = (\vec{u}, \vec{v}, \vec{w}, \mathbf{O})$
  - Origin **O**
  - Three Linear-Independent Vectors $(\vec{u}, \vec{v}, \vec{w})$

Vector Spaces

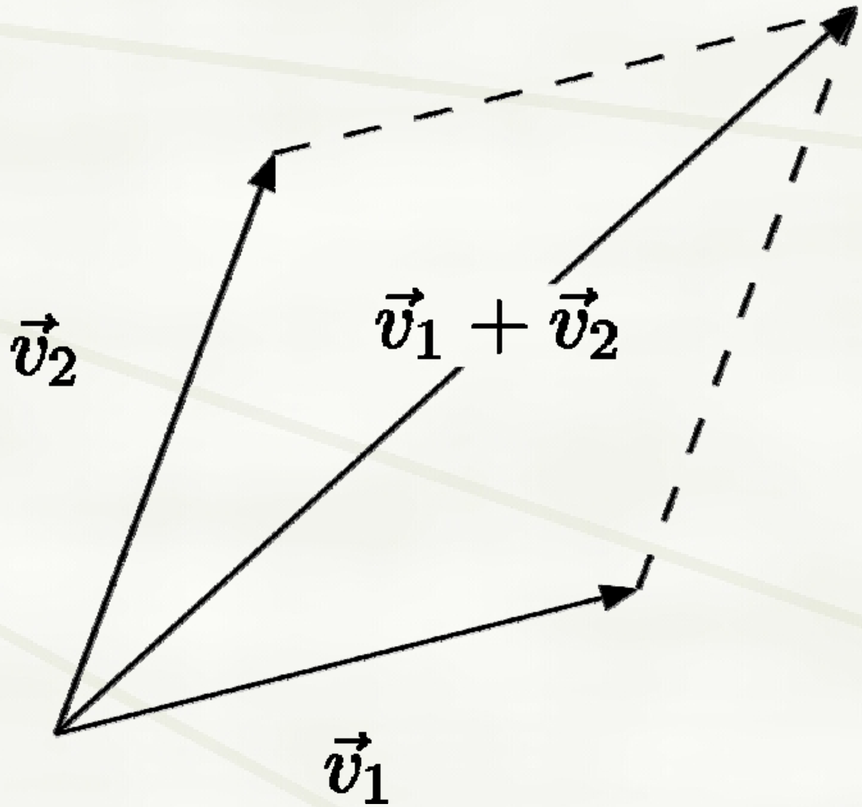
- Definition
  - A nonempty set ς of elements $\vec{v}, \vec{w}, \ldots$ is called a *vector space* if in ς there are two algebraic operations, namely *addition* and *scalar multiplication*
  - Examples of vector space
- Linear Independence and Bases

Vector Spaces Addition

- *Addition* associates with every pair of vectors $\vec{v}_1$ and $\vec{v}_2$ a unique vector $\vec{v} \in \mathcal{V}$ which is called the *sum* of $\vec{v}_1$ and $\vec{v}_2$ and is written $\vec{v}_1 + \vec{v}_2$.
- For 2D vectors, the summation is componentwise, i.e., if $\vec{v}_1 =< x_1, y_1 >$ and $\vec{v}_2 =< x_2, y_2 >$, then

$$\vec{v}_1 + \vec{v}_2 =< x_1 + x_2, y_1 + y_2 >$$

Vector Spaces Addition



parallelogram illustration

Addition Properties

- Commutativity
- Associativity
- Zero Vector
- Additive Inverse
- Vector Subtraction

Commutativity

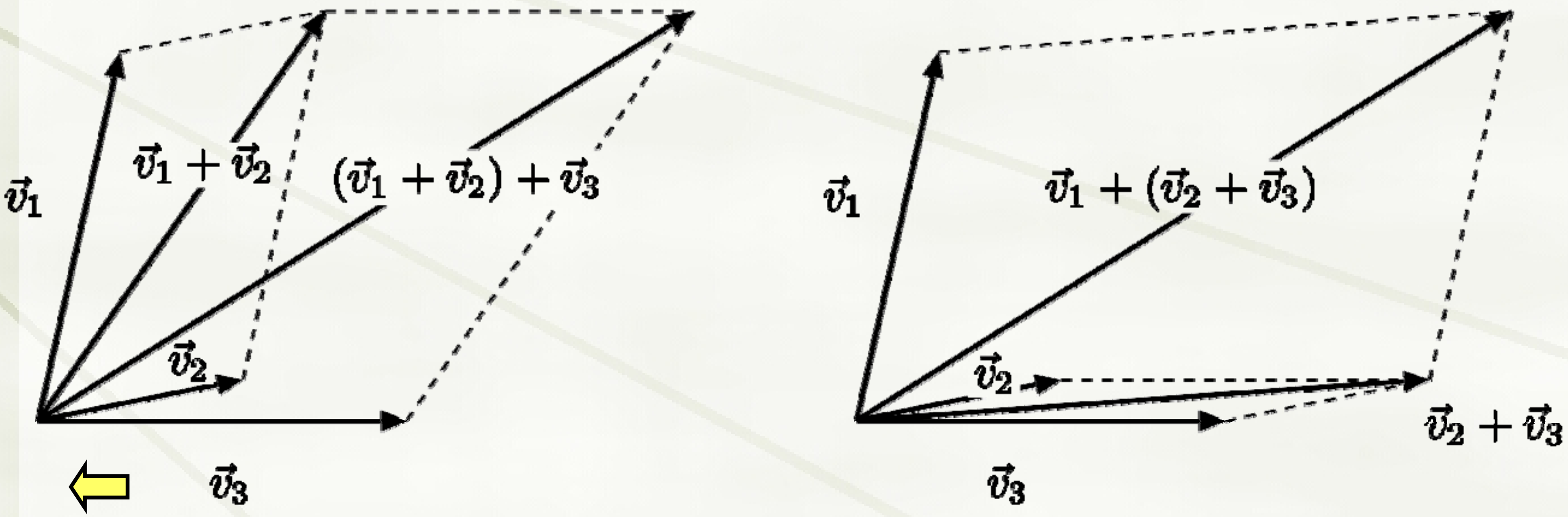
for any two vectors $\vec{v}_1$ and $\vec{v}_2$ in ς ,

$$\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$$

Associativity

for any three vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ in ς,

$$(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \vec{v}_1 + (\vec{v}_2 + \vec{v}_3)$$



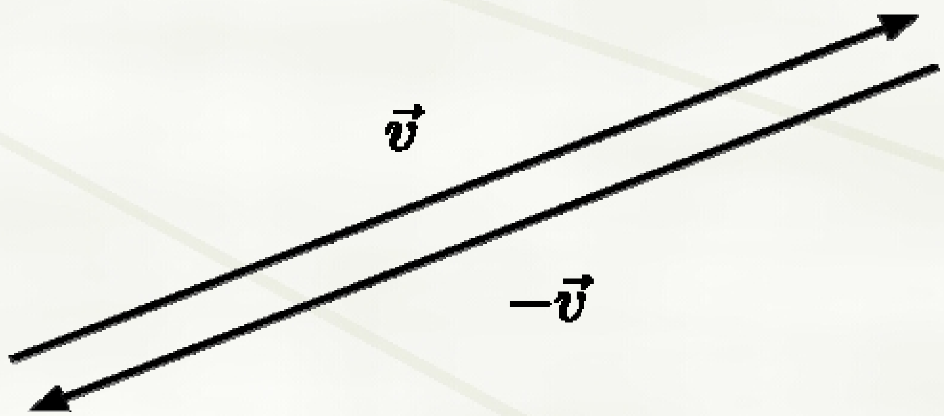
Zero Vector

There is a unique vector in ς called the *zero vector* and denoted $\vec{0}$ such that for every vector $\vec{v} \in \mathcal{V}$

$$\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$$

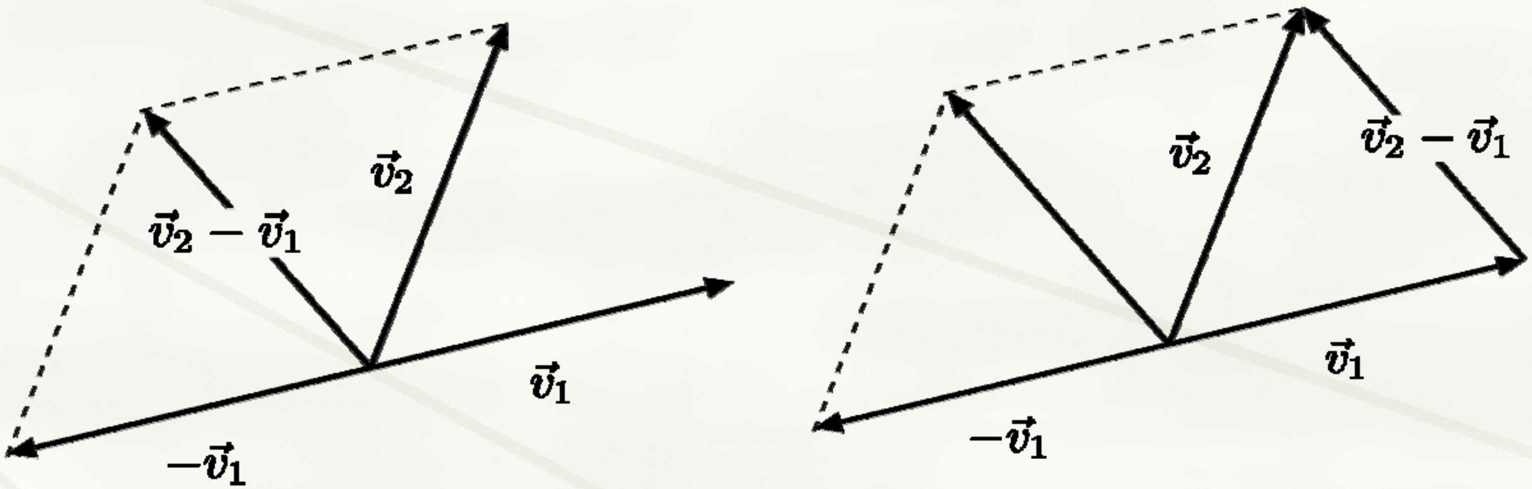
Additive Inverse

For each element $\vec{v} \in \mathcal{V}$, there is a unique element in ς , usually denoted $-\vec{v}$, so that $\vec{v} + (-\vec{v}) = \vec{0}$



Vector Subtraction

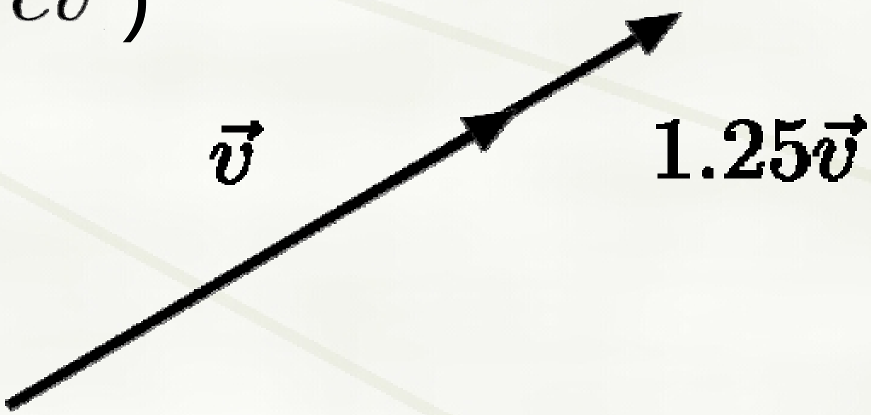
$$\vec{v}_2 - \vec{v}_1 = \vec{v}_2 + (-\vec{v}_1)$$



⇦ joining the ends of the two original vectors

Vector Spaces Scalar Multiplication

Scalar Multiplication associates with every vector $\vec{v} \in \mathcal{V}$ and every scalar c, another unique vector (usually written $c\vec{v}$)



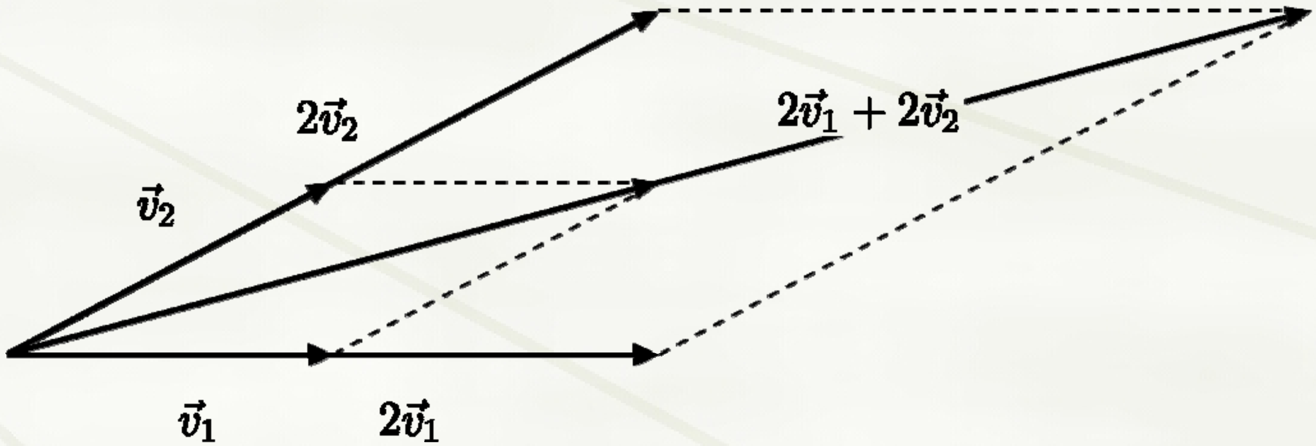
Scalar Multiplication Properties

- Distributivity
- Distributivity of Scalars
- Associativity
- Identity

Distributivity

For every scalar c and vectors $\vec{v}_1$ and $\vec{v}_2$ in ς

$$c(\vec{v}_1 + \vec{v}_2) = c\vec{v}_1 + c\vec{v}_2$$



Distributivity of Scalars

For every two scalars c_1 and c_2 and vector $\vec{v} \in \mathcal{V}$

$$(c_1 + c_2)\vec{v} = c_1\vec{v} + c_2\vec{v}$$

Associativity

For every two scalars c_1 and c_2 and vector $\vec{v} \in \mathcal{V}$

$$c_1(c_2\vec{v}) = (c_1 c_2)\vec{v}$$

Identity

For every vector $\vec{v} \in \mathcal{V}$

$$1\vec{v} = \vec{v}$$

Examples of Vector Spaces

- Vector Space of 3-Dimensional Vectors
- Vector Spaces of Polynomials
- Vector Spaces of Matrices

Vector Spaces of Polynomials

- The set of quadratic polynomials of the form $P(x)=ax^2+bx+c$

If $$P_1(x)=a_1x^2+b_1x+c_1$$
$$P_2(x)=a_2x^2+b_2x+c_2$$

Then

$$(P_1+P_2)(x)=(a_1+a_2)x^2+(b_1+b_2)x+(c_1+c_2)$$
$$sP(x)=(sa)x^2+(sb)x+(sc)$$

Linear Independence and Bases

- Linear Combinations
- Linear Independence
- A Basis for a Vector Space

Linear Combinations

- Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be any vectors in a vector space ς and let $c_1, c_2, \ldots, c_n$ be any set of scalars. Then an expression of the form

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

  is called a *linear combination* of the vectors
- This element is clearly a member of the vector space ς

Linear Combinations

- The set S that contains all possible linear combinations of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is called the *span* of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. We frequently say that S is *spanned* (or *generated*) by those n vectors
- The span of any set of vectors is again a vector space

Linear Independence

- Given a set of vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ from a vector space ς. This set is called *linearly independent* in ς if the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$

  implies that $c_i=0$ for all $i=1,2,\ldots n$.

Linearly Dependent

- *Linearly Dependent* implies that the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$ has a nonzero solution, *i.e.* there exist $c_1, c_2, \ldots, c_n$ which are not all zero
- This implies that at least one of the vectors $\vec{v}_i$ can be written in terms of the other n-1 vectors in the set. Assuming that c_1 is not zero, we can see that

$$\vec{v}_1 = \frac{c_2}{c_1}\vec{v}_2 + \cdots + \frac{c_n}{c_1}\vec{v}_n$$

A Basis for a Vector Space

- Let $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ be a set of vectors in a vector space ς and let Σ be the span of $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$. If $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ is linearly independent, then we say that these vectors form a *basis* for Σ and Σ has dimension n.

A Basis for a Vector Space

- If Σ is the entire vector space ς , we say that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ forms a basis for ς , and ς has dimension ***n*** .
- Any vector $\vec{v} \in \mathcal{V}$ can be written uniquely as

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

Points and Vectors

- The fundamental 3-dimensional space objects that form the basis for all operations in computer graphics are the *point* and the *vector* (sometimes called a *free vector*).

- Are points and vectors are ``essentially'' the same?

No!

Points and Vectors

- A *Point* has position in space. The only characteristic that distinguishes one point from another is its position. (bold letters such as **P** and **Q** in our course)
- A *Vector* has both magnitude and direction, but no fixed position in space. (lower case letters with an arrow above such as $\vec{v}$ and $\vec{w}$ in our course)

Affine Space

- An *affine space* is made up of a set of points Π and a vector space ς
- The relationship between points and vectors are described by the following axioms
  - Points: (x,y,z)
  - Vectors: $<u,v,w>$

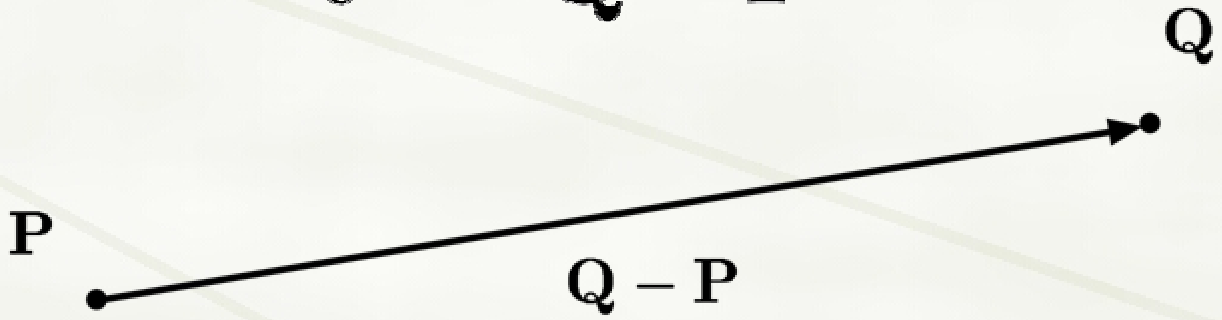
Relating Points and Vectors

- In general, the points are thought to play the primary role in the space, while the vectors are utilized to move about in the space from point to point.
- The General Axioms

The General Axioms (1)

- For each pair of points $\mathbf{P}$ and $\mathbf{Q}$, there exists a unique vector $\vec{v}$ such that

$$\vec{v} = \mathbf{Q} - \mathbf{P}$$



Geometric explanation: there is a direction and magnitude between any two points in the affine space

The General Axioms (2)

- For each point $\mathbf{P}$ and vector $\vec{v}$, there is a unique point $\mathbf{Q}$, such that $\mathbf{Q} = \mathbf{P} + \vec{v}$;

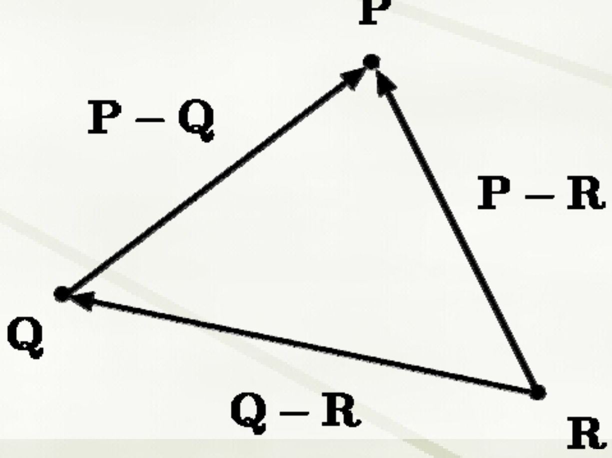
Geometric explanation: if we move point $\mathbf{P}$ a distance $|\vec{v}|$ in the direction of $\vec{v}$, we should find a point $\mathbf{Q} \in \Pi$ defined there

The General Axioms (3)

- Given three points **P**, **Q** and **R**, these points satisfy

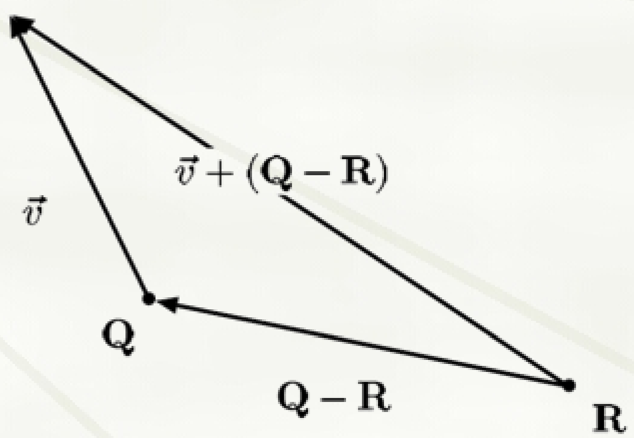
$$\mathbf{(P\text{-}Q)+(Q\text{-}R)=(P\text{-}R)}$$

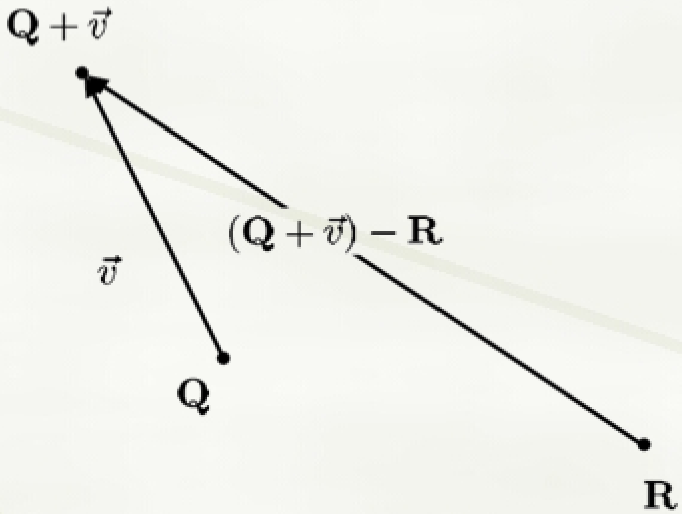
Geometric Explanation: head-to-tail axiom



Some corollaries

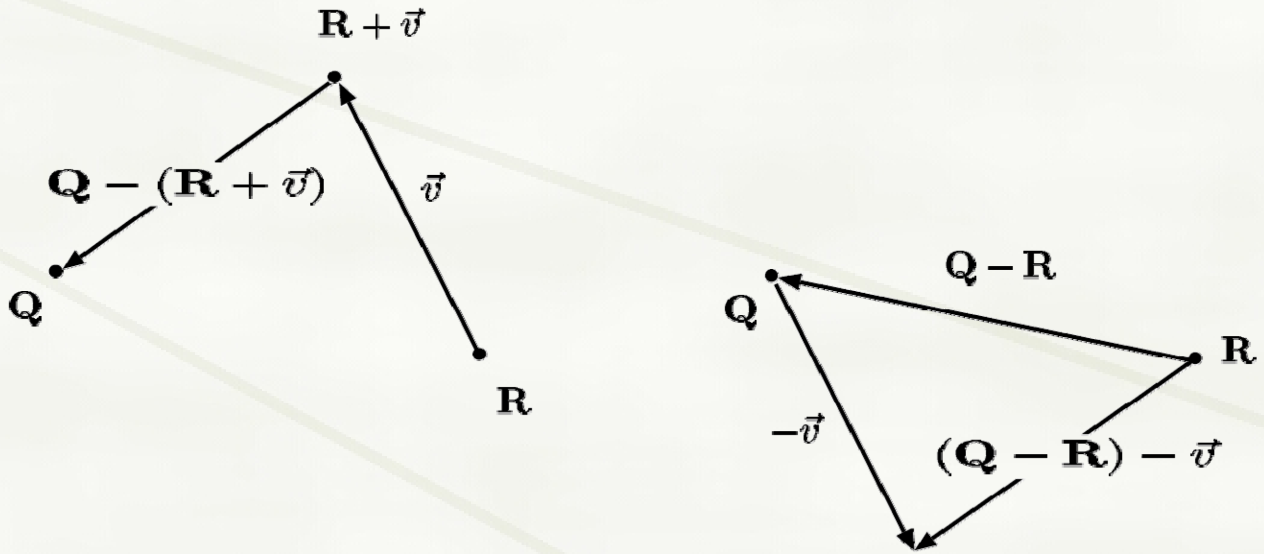
- $\mathbf{Q} - \mathbf{Q} = \vec{0}$
- $\vec{v} + (\mathbf{Q} - \mathbf{R}) = (\mathbf{Q} + \vec{v}) - \mathbf{R}$
- $\mathbf{R} - \mathbf{Q} = -(\mathbf{Q} - \mathbf{R})$





Some corollaries

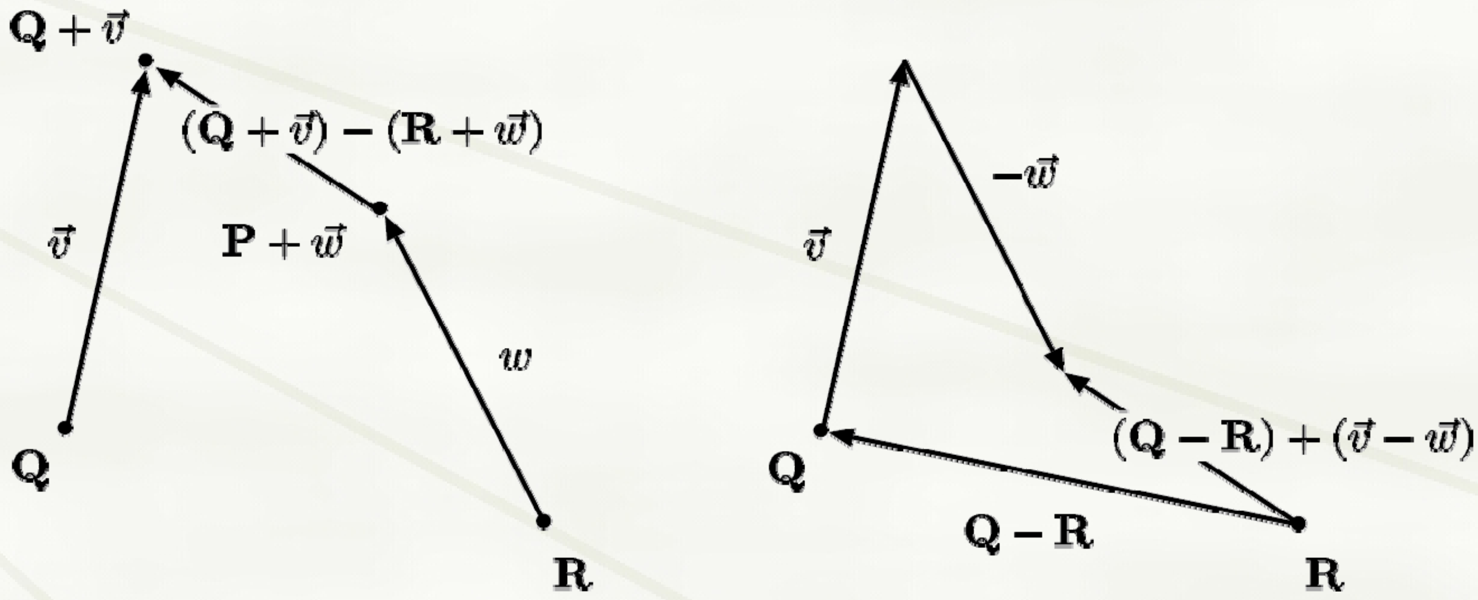
- $\mathbf{Q} - (\mathbf{R} + \vec{v}) = (\mathbf{Q} - \mathbf{R}) - \vec{v}$



- $\mathbf{P} = \mathbf{Q} + (\mathbf{P} - \mathbf{Q})$

Some corollaries

- $(\mathbf{Q}+\vec{v})-(\mathbf{R}+\vec{w})=(\mathbf{Q}-\mathbf{R})+(\vec{v}-\vec{w})$



Operations in Affine Space

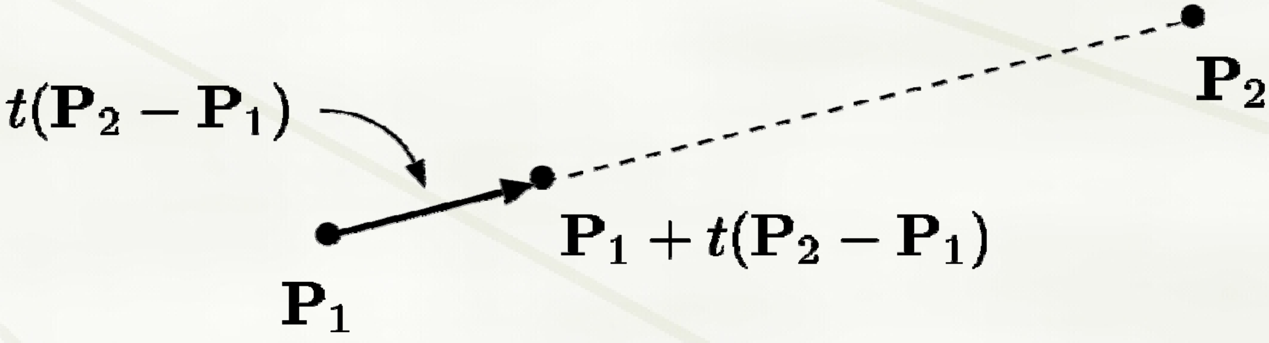
- Affine Combinations
- Barycentric Coordinates
- Convex Combinations

Affine Combinations of Points

Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be points in the affine space, the expression

$$\mathbf{P}=\mathbf{P}_1+t(\mathbf{P}_2-\mathbf{P}_1) \quad \text{or} \quad \mathbf{P}=(1-t)\mathbf{P}_1+t\mathbf{P}_2$$

represents a point $\mathbf{P}$ on the line that passes through $\mathbf{P}_1$ and $\mathbf{P}_2$.



Affine Combinations of Points

- The affine combination of two points $\mathbf{P}_1$ and $\mathbf{P}_2$ is

$$\mathbf{P} = \alpha_1\mathbf{P}_1 + \alpha_2\mathbf{P}_2$$

where $\alpha_1 + \alpha_2 = 1$

- The form $\mathbf{P}=(1-t)\mathbf{P}_1+t\mathbf{P}_2$ is affine transformation by setting $\alpha_2 = t$

Affine Combinations of Points

- An affine combination of an arbitrary number of points
  - $\mathbf{P}_1,\mathbf{P}_2,\ldots,\mathbf{P}_n$ are points
  - $\alpha_1,\alpha_2,\ldots\alpha_n$ are scalars such that $\alpha_1+\alpha_2+\ldots+\alpha_n=1$

$$\alpha_1\mathbf{P}_1+\alpha_2\mathbf{P}_2+\ldots+\alpha_n\mathbf{P}_n$$

Then $\mathbf{P}_1+\alpha_2(\mathbf{P}_2-\mathbf{P}_1)+\ldots+\alpha_n(\mathbf{P}_n-\mathbf{P}_1)$ is defined to be the point

Example of Affine Combination

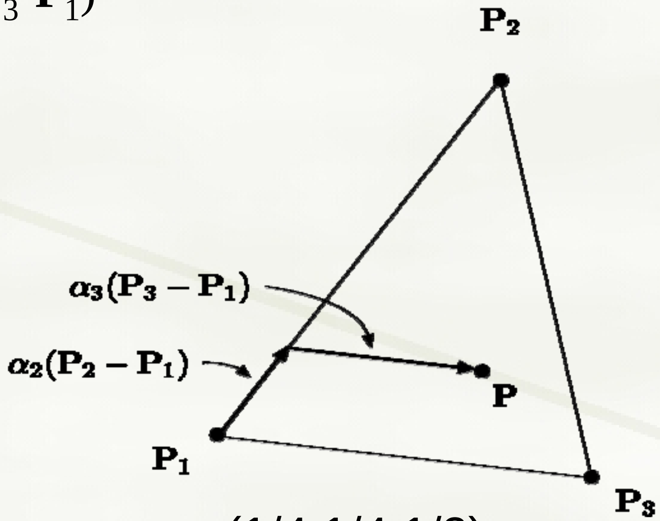
Consider three points $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$, a point $\mathbf{P}$ defined by

$$\mathbf{P}=\alpha_1\mathbf{P}_1+\alpha_2\mathbf{P}_2+\alpha_3\mathbf{P}_3$$

gives a point in the triangle. The definition of affine combination defines this point to be

$$\mathbf{P}=\mathbf{P}_1+\alpha_2(\mathbf{P}_2-\mathbf{P}_1)+\alpha_3(\mathbf{P}_3-\mathbf{P}_1)$$

- If $0\leqslant\alpha_1, \alpha_2, \alpha_3\leqslant 1$, the point $\mathbf{P}$ will be within (or on the boundary) of the triangle
- If any α_i is less than zero or greater than one, then the point will lie outside the triangle
- If any α_i is zero, then the point will lie on the boundary of the triangle.



(1/4,1/4,1/2)

Barycentric Coordinates

Given a frame $(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n, \mathbf{O})$ for an affine space A , we can write any point $\mathbf{P}$ uniquely as

$$\mathbf{P} = p_1\vec{v}_1 + p_2\vec{v}_2 + \cdots + p_n\vec{v}_n + \mathbf{O}$$

If we define $\mathbf{P}_i$ by

$$\mathbf{P}_0 = \mathbf{O}$$
$$\mathbf{P}_1 = \mathbf{O} + \vec{v}_1$$
$$\mathbf{P}_2 = \mathbf{O} + \vec{v}_2$$
$$\vdots$$
$$\mathbf{P}_n = \mathbf{O} + \vec{v}_n$$

And define p_0 to be

$$p_0=1-(p_1+p_2+\ldots+p_n)$$

Then we can see that $\mathbf{P}$ can be equivalently written as

$$\mathbf{P}=p_0\mathbf{P}_0+p_1\mathbf{P}_1+p_2\mathbf{P}_2+\ldots+p_n\mathbf{P}_n$$

where $p_0+p_1+p_2+\ldots+p_n=1$

In this form the value

$$(p_0, p_1, p_2, \ldots, p_n)$$

are called the *barycentric coordinates* of $\mathbf{P}$ relative to the points$(\mathbf{P}_0,\mathbf{P}_1,\mathbf{P}_2,\ldots,\mathbf{P}_n)$

Note: *barycentric coordinates* of point

Barycentric Coordinates

Vectors can also be expressed in barycentric form by letting

$$u_0 = -(u_1 + u_2 + \ldots + u_n)$$

Then we have

$$\vec{u} = u_0\mathbf{P}_0 + u_1\mathbf{P}_1 + u_2\mathbf{P}_2 + \cdots + u_n\mathbf{P}_n$$

where now we have that $u_0 + u_1 + u_2 + \ldots + u_n = 0$

Note: *barycentric coordinates* of vector sum=0

barycentric coordinates of point sum=1

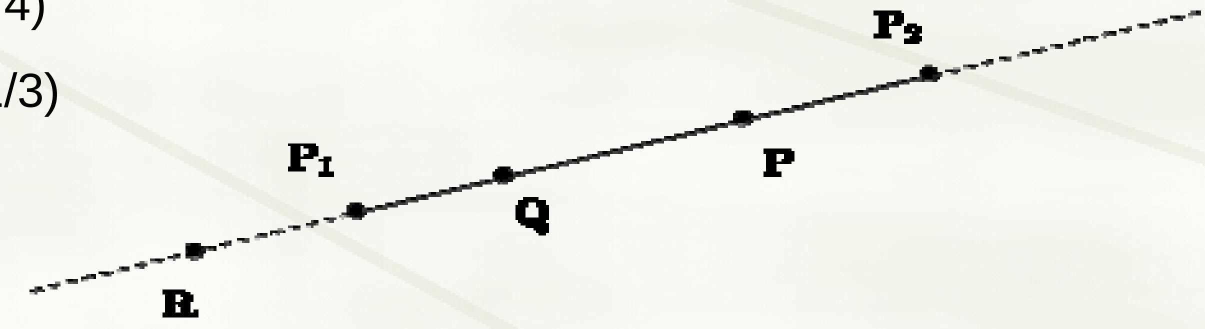
Example of Barycentric Coordinates

Consider two points $\mathbf{P}_1$ and $\mathbf{P}_2$ in the plane, if α_1 and α_2 are scalars such that $\alpha_1+\alpha_2=1$, then the point $\mathbf{P}$ defined by

$$\mathbf{P}=\alpha_1\mathbf{P}_1+\alpha_2\mathbf{P}_2$$

is a point on the line that passes through $\mathbf{P}_1$ and $\mathbf{P}_2$.

1. If $0\leqslant\alpha_1, \alpha_2\leqslant 1$, the point $\mathbf{P}$ on the line segment joining $\mathbf{P}_1$ and $\mathbf{P}_2$.
2. Some numerical examples
   - $\mathbf{P}$ (1/2, 2/3)
   - $\mathbf{Q}$ (3/4, 1/4)
   - $\mathbf{R}$ (4/3, -1/3)



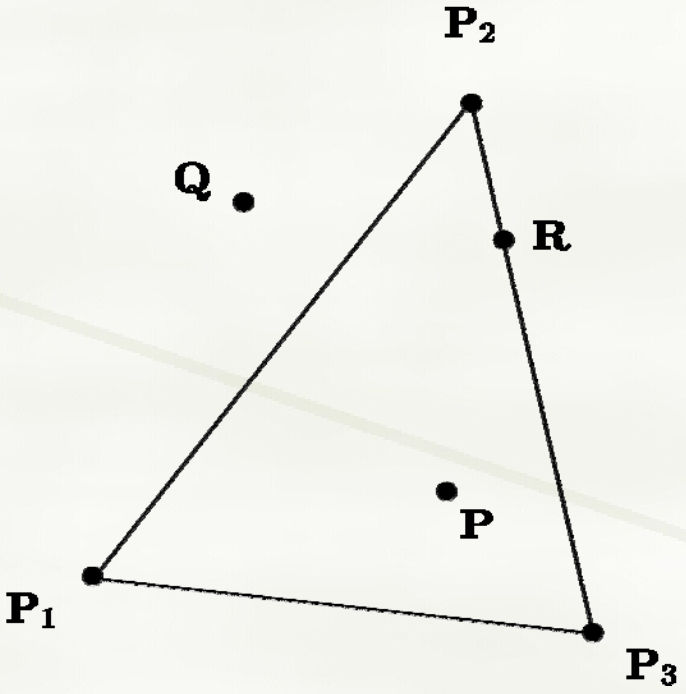
Example of Affine Combination

Consider three points $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ in the plane , if α_1, α_2, α_3 are scalars such that $\alpha_1+\alpha_2+\alpha_3=1$, then the point $\mathbf{P}$ defined by

$$\mathbf{P}=\alpha_1\mathbf{P}_1+\alpha_2\mathbf{P}_2+\alpha_3\mathbf{P}_3$$

is a point in the triangle $\mathbf{P}_1\mathbf{P}_2\mathbf{P}_3$.

1. If $0\leqslant\alpha_1, \alpha_2, \alpha_3\leqslant 1$, the point $\mathbf{P}$ will be within (or on the boundary) of the triangle
2. Some numerical examples
   - P (1/4, 1/4, 1/2)
   - Q (1/2, 3/4, -1/4)
   - R (0, 3/4, -1/4)



Convex Combinations

Given a set of points $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$, we can form affine combinations of these points by selecting $\alpha_0, \alpha_1, \ldots, \alpha_n$, with $\alpha_0+\alpha_1+\ldots+\alpha_n=1$ and form the point

$$\mathbf{P}=\alpha_0\mathbf{P}_0+\alpha_1\mathbf{P}_1+\ldots+\alpha_n\mathbf{P}_n$$

If each α_i is such that $0 \le \alpha_i \le 1$, then the points $\mathbf{P}$ is called a *convex combination* of the points $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$.

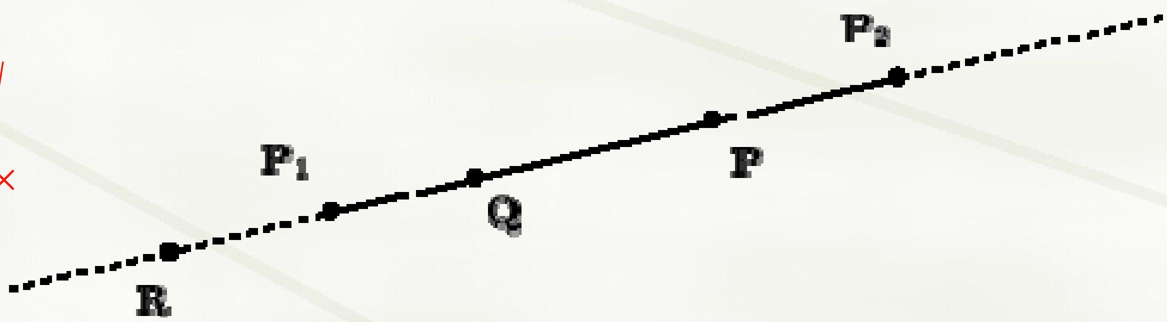
Example of Convex Combinations

Consider two points $\mathbf{P}_1$ and $\mathbf{P}_2$ in the plane, if α_1 and α_2 are scalars such that $\alpha_1+\alpha_2=1$, then the point $\mathbf{P}$ defined by

$$\mathbf{P}=\alpha_1\mathbf{P}_1+\alpha_2\mathbf{P}_2$$

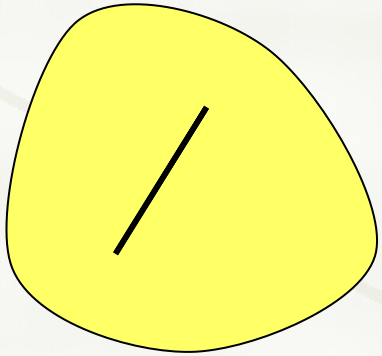
is a point on the line that passes through $\mathbf{P}_1$ and $\mathbf{P}_2$.

1. If $0\leqslant\alpha_1, \alpha_2\leqslant 1$, the point $\mathbf{P}$ on the line segment joining $\mathbf{P}_1$ and $\mathbf{P}_2$, and it is the convex combination of $\mathbf{P}_1$ and $\mathbf{P}_2$
2. Some numerical examples
   - **P** (1/2, 2/3) √
   - **Q** (3/4, 1/4) √
   - **R** (4/3, -1/3) ×

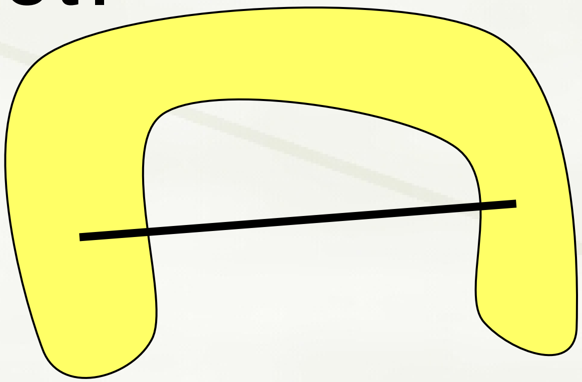


Convex Set

- Convex set : Given any set of points, if given any two points of the set, any convex combination of these two points is also in the set.



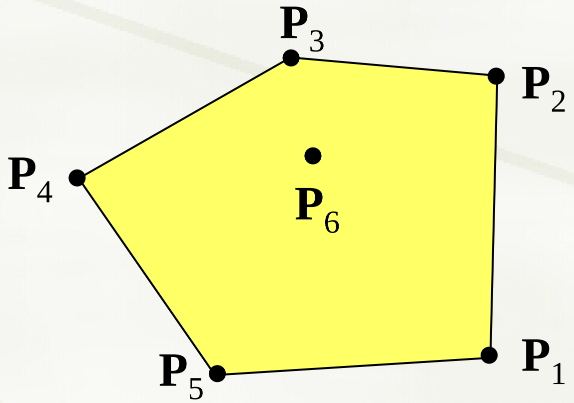
Convex set



Non-convex set

Convex Hull

- Convex hull of points $\mathbf{P}_0,\mathbf{P}_1,\ldots,\mathbf{P}_n$: The set of all points $\mathbf{P}$ that can be written as convex combinations of $\mathbf{P}_0,\mathbf{P}_1,\ldots,\mathbf{P}_n$
- The convex hull is the smallest convex set that contains the set of points $\mathbf{P}_0,\mathbf{P}_1,\ldots,\mathbf{P}_n$



Frames

- Definition of a Frame
- Matrix representation of Points and Vectors
- Converting Between Frames

Definition of a Frame

- Let $\mathbf{A}$ be an affine space of dimension n. Let $\mathbf{O}$ be a point in this space and let $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ be any basis for $\mathbf{A}$. We call the collection $\mathcal{F} = (\vec{v}_1, \vec{v}_2, ..., \vec{v}_n, \mathbf{O})$ a *frame* for $\mathbf{A}$.
- Frames form *coordinate systems* in our affine space $\mathbf{A}$.

Definition of a Frame

- The *coordinates* of point $\mathbf{P}$ relative to the frame Φ
  - Point $\mathbf{P}$ can be written as $\mathbf{O} + \vec{v}$.
  - $\vec{v}$ is a vector. The $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ forms a basis for A, then $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$
  - The point $\mathbf{P}$ can be written as

$$\mathbf{P} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n + \mathbf{O}$$

  - $(c_1, c_2, \ldots, c_n)$ are the *coordinates* of point $\mathbf{P}$ relative to the frame Φ

Example of Frames (1)

- The standard Cartesian frame $(\vec{u}, \vec{v}, \mathbf{O})$, where $\vec{u} =< 1,0 >, \vec{v} =< 0,1 >$ and $\mathbf{O} = (0,0)$ The coordinate (x,y) equals to the point

$$x\vec{u} + y\vec{v} + \mathbf{O}$$

The above statement can be extended to any dimension by setting origin (0,0,…,0), vectors

$<1,0,\ldots,0>, <0,1,\ldots,0>, \ldots , <0,0,\ldots,1>$.

Example of Frames (2)

- Consider the frame: the origin **O**=(2,2), the two vectors $\vec{u} = (1,0)$ and $\vec{v} = (1,1)$. The point **P** that has coordinates (5,3) can be written as

$$5<1,0> + 3<0,2> + <2,2>$$

which has the Cartesian coordinates (7,8)

Matrix representation of Points and Vectors

- Points and vectors can be uniquely identified by the coordinates relative to a specific frame.

- Given a frame
  $$(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n, \mathbf{O})$$
  in an affine space $\mathbf{A}$, we can write a point $\mathbf{P}$ uniquely as

$$\mathbf{P} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n + \mathbf{O}$$

This can also be written as

$$\mathbf{P} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n & 1 \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \\ \mathbf{O} \end{bmatrix}$$

Matrix representation of Points and Vectors

The vectors of affine space form a vector space, we can write a vector uniquely $\vec{v}$ as

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

This can be written as

$$\mathbf{P} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n & 0 \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \\ \mathbf{O} \end{bmatrix}$$

- Points are represented as row vectors whose last component is 1
- Vectors are represented as row vectors whose last component is 0

Converting Between Frames

- When given two different frames, to take a point that has a certain set of coordinates in one frame and find its coordinates in the second frame

  - if the second frame is the Cartesian frame √
  - if the second frame is not the Cartesian frame ?

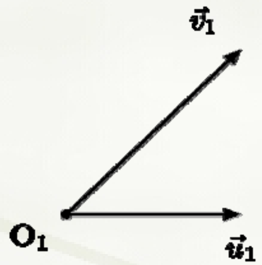
An Example of Converting Between Frames

Frame1 $(\vec{u}_1, \vec{v}_1, \mathbf{O}_1)$

$\vec{u}_1 =< 1,0 >$

$\vec{v}_1 =< 1,1 >$

$\mathbf{O}_1 =< 0,0 >$



Frame2 $(\vec{u}_2, \vec{v}_2, \mathbf{O}_2)$

$\vec{u}_2 =< 1,0 >$

$\vec{v}_2 =< 0,2 >$

$\mathbf{O}_2 =< 2,2 >$

$\mathbf{P}(3,2)$ in the Frame1, compute its coordinates in the Frame2?

$$3\vec{u}_1 + 2\vec{v}_1 + \mathbf{O}_1 = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} \vec{u}_1 \\ \vec{v}_1 \\ \mathbf{O}_1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} \vec{u}_2 \\ \vec{u}_2 + \frac{1}{2}\vec{v}_2 \\ -2\vec{u}_2 - \vec{v}_2 + \mathbf{O}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} \vec{u}_2 \\ \vec{v}_2 \\ \mathbf{O}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{u}_2 \\ \vec{v}_2 \\ \mathbf{O}_2 \end{bmatrix}$$

An Example of Converting Between Frames

- We wrote the vectors of the first frame in terms of the vectors of the second frame since the vectors of the second frame (any frame actually) form a basis for the space of vectors.
- We wrote the origin $\mathbf{O}_1$ in terms of the origin and vectors of the second frame
- The result is $\begin{bmatrix} u & v & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ -2 & -1 & 1 \end{bmatrix}$

Converting Between Frames

- Suppose a point **P** has coordinates $(c_1, c_2, \ldots, c_n, 1)$ relative to some frame $\mathcal{F} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n, \mathbf{O})$, Compute the coordinates of **P** relative to another frame

  $\mathcal{F}' = (\vec{v}'_1, \vec{v}'_2, \ldots, \vec{v}'_n, \mathbf{O}')$

a) Since $(\vec{v}'_1, \vec{v}'_2, \ldots, \vec{v}'_n)$ is a basis, we can write each of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ uniquely in terms of the $\vec{v}'_i$

$$\vec{v}_i = e_{i,1}\vec{v}'_1 + e_{i,2}\vec{v}'_2 + \cdots + e_{i,n}\vec{v}'_n \qquad (i=1,2,\ldots n)$$

b) Since **P**-**O** is a vector, we cam also write **O** uniquely in terms of $\vec{v}'_i$ and $\mathbf{O}'$

$$\mathbf{O} = e_{n+1,1}\vec{v}'_1 + e_{n+1,2}\vec{v}'_2 + \cdots + e_{n+1,n}\vec{v}'_n + \mathbf{O}'$$

Converting Between Frames

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n\mathbf{O} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n & 1 \end{bmatrix}\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \\ \mathbf{O} \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & c_2 & \cdots & c_n & 1 \end{bmatrix}\begin{bmatrix} e_{1,1}\vec{v}'_1 + e_{1,2}\vec{v}'_2 + \cdots + e_{1,n}\vec{v}'_n \\ e_{2,1}\vec{v}'_1 + e_{2,2}\vec{v}'_2 + \cdots + e_{2,n}\vec{v}'_n \\ \vdots \\ e_{n,1}\vec{v}'_1 + e_{n,2}\vec{v}'_2 + \cdots + e_{n,n}\vec{v}'_n \\ e_{n+1,1}\vec{v}'_1 + e_{n+1,2}\vec{v}'_2 + \cdots + e_{n+1,n}\vec{v}'_n + \mathbf{O}' \end{bmatrix}$$

$$= \begin{bmatrix} c_1 & c_2 & \cdots & c_n & 1 \end{bmatrix}\begin{bmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,n} & 0 \\ e_{2,1} & e_{2,2} & \cdots & e_{2,n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{n,1} & e_{n,2} & \cdots & e_{n,n} & 0 \\ e_{n+1,1} & e_{n+1,2} & \cdots & e_{n+1,n} & 1 \end{bmatrix}\begin{bmatrix} \vec{v}'_1 \\ \vec{v}'_2 \\ \vdots \\ \vec{v}'_n \\ \mathbf{O}' \end{bmatrix}$$

Converting Between Frames

The coordinates $(c_1, c_2, ..., c_n)$ of the point in the second frame is

$$\begin{bmatrix} c'_1 & c'_2 & \cdots & c'_n & 1 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n & 1 \end{bmatrix} \begin{bmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,n} & 0 \\ e_{2,1} & e_{2,2} & \cdots & e_{2,n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{n,1} & e_{n,2} & \cdots & e_{n,n} & 0 \\ e_{n+1,1} & e_{n+1,2} & \cdots & e_{n+1,n} & 1 \end{bmatrix}$$

1. The change of coordinates is accomplished via a matrix multiplication.
2. The rows of the matrix consist of the coordinates of the elements of the old frame Φ relative to the new frame Φ' .
3. The frames in n dimensional space, the matrix is $n \times n$.

Compute the matrix by utilizing Cramer's Rule (for 3D case)

Given two frame $(\vec{u}_1, \vec{v}_1, \vec{w}_1, \mathbf{O}_1)$ and $(\vec{u}_2, \vec{v}_2, \vec{w}_2, \mathbf{O}_2)$, compute the conversion the following matrix

$$\begin{bmatrix} e_{1,1} & e_{1,2} & e_{1,3} & 0 \\ e_{2,1} & e_{2,2} & e_{2,3} & 0 \\ e_{3,1} & e_{3,2} & e_{3,3} & 0 \\ e_{4,1} & e_{4,2} & e_{4,3} & 1 \end{bmatrix}$$

It can be accomplished by utilizing Cramer's Rule

Cramer's Rule

Given any frame $(\vec{u}, \vec{v}, \vec{w}, \mathbf{O})$ and a vector $\vec{t}$, it can be written as $\vec{t} = u\vec{u} + v\vec{v} + w\vec{w}$, for some u,v,w. The Cramer's rule is to compute the u,v,w. The formulae are:

$$D = \vec{u} \cdot (\vec{v} \times \vec{w}) \qquad u = \frac{D_1}{D}$$

$$D_1 = \vec{t} \cdot (\vec{v} \times \vec{w}) \qquad v = \frac{D_2}{D}$$

$$D_2 = \vec{u} \cdot (\vec{t} \times \vec{w}) \qquad w = \frac{D_3}{D}$$

$$D_3 = \vec{u} \cdot (\vec{v} \times \vec{t})$$

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