Parallel and Adaptive Visibility Sampling for Rendering Dynamic Scenes with Spatially-Varying Reflectance

Supplemental Document

Rui Wang[1], Minghao Pan, Xiang Han, Weifeng Chen, Hujun Bao

State Key Lab of CAD&CG, Zhejiang University

1. Proof of Spherical Distance Transform

In this section, we give the proof of our spherical distance transform. In [1], Paglieroni gave several conditions to apply dimensionality reduction Euclidean Distance Transform (EDT) algorithms on a 2D image as,

$$d(p_1, p_2) = f(|x_1 - x_2|, |y_1 - y_2|), \quad (1)$$

$$\forall y, |x_1| < |x_2| \Rightarrow f(|x_1|, |y|) \leq f(|x_2|, |y|), \quad (2)$$

$$\forall x, |y_1| < |y_2| \Rightarrow f(|x|, |y_1|) \leq f(|x|, |y_2|), \quad (3)$$

where $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$ are two points defined on a 2D image, $d(p_1, p_2)$ is the 2D distance between p_1 and p_2 computed by a distance function $f(\Delta x, \Delta y)$. If the distance function satisfies conditions (2) and (3), the EDT can be computed in a dimensionality reduction manner.

However, in the spherical distance transform, under the non-uniform parameterization, the spherical distance function does not satisfy Eq.(1). The parameterization of sphere is shown in Figure 1. The latitude is parameterized in θ and longitude is parameterized in ϕ. The hemisphere used in our paper to parameterize visibility map can be regarded as the right hemisphere under this parameterization. Given two points on the sphere, $p(\theta_1, \phi_1)$ and $q(\theta_2, \phi_2)$, the spherical distance $d_s(p, q)$ is computed by

$$d_s(p, q) = \arccos(\mathbf{p} \cdot \mathbf{q}) \quad (4)$$

where $\mathbf{p} = (\sin\theta_1 \cos\phi_1, \sin\theta_1 \sin\phi_1, \cos\theta_1)$ and $\mathbf{q} = (\sin\theta_2 \cos\phi_2, \sin\theta_2 \sin\phi_2, \cos\theta_2)$. Obviously, $d_s(p, q)$ is not the function of $\Delta\theta = |\theta_1 - \theta_2|$ and $\Delta\phi = |\phi_1 - \phi_2|$. Thus, the statement in [1] does not hold in the spherical distance transform.

In this document, we give two theorems to prove that under the non-uniform parameterization, if we reduce dimensions in a certain order, i.e. first compute along latitude and then scan the longitude, these two conditions (2) and (3) still hold. In this way, the spherical distance transform can be computed in a dimensionality reduction manner and be parallelized.

Before giving these two theorems, we first introduce some equations that will be used later. We define a function to compute the *longitude distance* between two points on the sphere, $p_1(\theta_0, \phi_1)$ and $p_2(\theta_0, \phi_2)$, as

$$\Psi(\phi_1, \phi_2) = min(|\phi_1 - \phi_2|, 2\pi - |\phi_1 - \phi_2|) \quad (5)$$

The *longitude distance* actually is the distance along the longitude between two points. Obviously, the range of $\Psi(\phi_1, \phi_2)$ is $[0, \pi]$.

Then, given a longitude, ϕ_0, and two points, p_1 and p_2, on sphere, we define a function to compute the *middle latitude*, θ_0, so that the point, $p_0(\phi_0, \theta_0)$ has the same distance to these two points, p_1 and p_2, (Figure. 2). θ_0 is computed by

$$\theta_0 = \Theta(\phi, p_1, p_2) = \quad (6)$$
$$\begin{cases} \frac{\pi}{2}, & A = B \\ \arctan\frac{\cos\theta_2 - \cos\theta_1}{A-B}, & \frac{\cos\theta_2 - \cos\theta_1}{A-B} \geq 0 \\ \arctan\frac{\cos\theta_2 - \cos\theta_1}{A-B} + \pi, & \frac{\cos\theta_2 - \cos\theta_1}{A-B} < 0 \end{cases}$$

where $A = \sin\theta_1 \cos\Delta\phi_1$, $B = \sin\theta_2 \cos\Delta\phi_2$, $\Delta\phi_1 = \phi_1 - \phi$ and $\Delta\phi_2 = \phi_2 - \phi$. Given any two points on sphere and a longitude, ϕ, utilizing Eq.(6), we are able to find one point on the longitude, ϕ, which has the same distance to these two points.

Given these two functions, we introduce two theorems.

Theorem 1. *Given two points on the same latitude, $p_1(\theta_0, \phi_1)$ and $p_2(\theta_0, \phi_2)$, for any longitude ϕ, if the longitude distance $\Psi(\phi, \phi_1) < \Psi(\phi, \phi_2)$, then for points at*

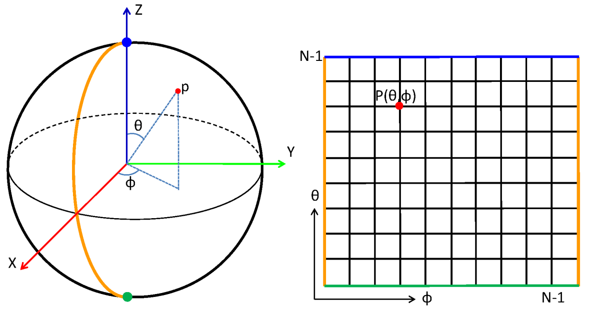

Figure 1: Spherical parameterization. The latitude is parameterized in θ and longitude is parameterized in ϕ.

the longitude ϕ, $p(\theta, \phi), \theta \in [0, \pi]$, we have

$$d_s(p, p_1) \leq d_s(p, p_2)$$

where equality holds iff $\sin\theta = 0$ or $\sin\theta_0 = 0$.

Proof
According to Eq.(4), we have

$$\begin{aligned} & \cos d_s(p, p_1) - \cos d_s(p, p_2) \qquad (7) \\ = & \sin\theta \sin\theta_0 (\cos(\phi - \phi_1) - \cos(\phi - \phi_2)) \\ = & \sin\theta \sin\theta_0 (\cos(\Psi(\phi - \phi_1)) - \cos(\Psi(\phi - \phi_2))) \end{aligned}$$

Since θ, θ_0 are both in $[0, \pi]$, $\sin\theta \sin\theta_0$ is always positive. Thus, according to Eq.(7), if $\Psi(\phi, \phi_1) < \Psi(\phi, \phi_2)$, we get $\cos d_s(p, p_1) > \cos d_s(p, p_2)$. Given the facts that the cosine function decreases monotonically at the range of $[0, \pi]$, and the range of spherical distance is $[0, \pi]$, we have $d_s(p, p_1) \leq d_s(p, p_2)$ and equality holds iff $\sin\theta = 0$ or $\sin\theta_0 = 0$. □

Theorem 2. *Given two points on the sphere, $p_1(\theta_1, \phi_1)$ and $p_2(\theta_2, \phi_2)$, where $\theta_1 < \theta_2$, and one longitude, ϕ_0, according to Eq.(6), we have the point, $p_0(\theta_0, \phi_0)$, with the same distance to p_0 and p_1. For any other point $p(\theta, \phi_0)$ on the longitude, ϕ_0, if $\theta < \theta_0$, then*

$$d_s(p, p_1) < d_s(p, p_2),$$

and if $\theta > \theta_0$, then

$$d_s(p, p_1) > d_s(p, p_2).$$

Proof
For simplicity, we assume $\phi_0 = 0$. If $\phi_0 \neq 0$, we can set a new coordinate with the x-axis at $\phi_0 = 0$ and rotate all points into the new coordinate. Then, we define a function $F(\theta)$ as,

$$F(\theta) = \cos d_s(p, p_1) - \cos d_s(p, p_2) \qquad (8)$$

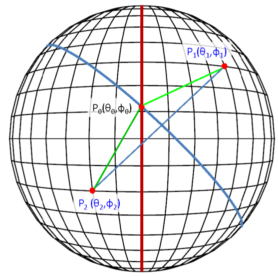

Figure 2: Illustration of *middle latitude*, θ_0, in Eq.(6). The point, p_0, is the intersection point of the longitude, ϕ, with two Voronoi regions defined by these two points, p_1 and p_2.

Thus,

$$\begin{aligned} F(\theta) = & \quad ((\sin\theta_1 \cos\phi_1, \sin\theta_1 \sin\phi_1, \cos\theta_1) \\ - & \quad (\sin\theta_2 \cos\phi_2, \sin\theta_2 \sin\phi_2, \cos\theta_2)) \\ \cdot & \quad (\sin\theta, 0, \cos\theta) \\ \Leftrightarrow & \begin{cases} F(\theta) = C \sin\theta + D \cos\theta \\ C = \sin\theta_1 \cos\phi_1 - \sin\theta_2 \cos\phi_2 \\ D = \cos\theta_1 - \cos\theta_2 \end{cases} \end{aligned}$$

Because $\theta_1 < \theta_2$, we have $D > 0$. We then formulate $F(\theta)$ as

$$F(\theta) = \sqrt{C^2 + D^2} \cos(\gamma - \theta) \qquad (9)$$

where $\gamma = \arccos(C/D)$. When we take θ_0 into the $F(\cdot)$ function, we have $F(\theta_0) = 0$. Thus, $\gamma - \theta_0 = \pm\pi/2$. Because $D > 0$, $\gamma \in (-\pi/2, \pi/2)$ and $\theta_0 \in (0, \pi)$, we have $\gamma - \theta_0 = -\pi/2$. By defining $\theta = \theta_0 + \Delta\theta$, we have

$$F(\theta) = \sqrt{A^2 + B^2} \cos(\frac{\pi}{2} + \Delta\theta) \qquad (10)$$

Since $\Delta\theta \in (-\pi, \pi)$ and the $\cos(\pi/2 + \Delta\theta)$ has the same sign with $\Delta\theta$, we have

$$\begin{cases} F(\theta) > 0 \Leftrightarrow d_s(p, p_1) < d_s(p, p_2), & \theta < \theta_0 \\ F(\theta) > 0 \Leftrightarrow d_s(p, p_1) > d_s(p, p_2), & \theta > \theta_0 \end{cases} \qquad (11)$$

□

Theorem 1 indicates that the condition (2) is true, when we take the LatitudeScans. **Theorem 2** indicates that the condition (3) is true, when we take the LongitudeScans. Actually, **Theorem 2** also proves that the point, p_0 computed from Eq.(6), is the intersection point of a longitude, ϕ, with two Voronoi regions defined by these two points, p_1 and p_2, Figure. 2. Using such intersection points, the scan on LongitudeScans can be conservatively taken out [2]. Combining these two theorems, we prove our spherical distance transform algorithm is a valid EDT algorithm on sphere.

2. Algorithm Details of Spherical Distance Transform

We list our spherical distance transform (SDT) algorithm in Algorithm 1 in the paper. It is extended from a 2D EDT algorithm [3]. Note that in our paper, the hemisphere distance transform is a special case of the spherical distance transform that only computes the distance filed at hemispheres. Our SDT algorithm takes two phases, the LatitudeScans and the LongitudeScans, to convert visibility maps into SSDFs. We represent SSDF in a square image with $N \times N$ resolutions, where one dimension represents ϕ_j, and the other dimension represents θ_i.

In LatitudeScans, the computation is taken in parallel for each θ_i. In each thread, we first scan ϕ_j from 0 to 3π and then backward. A temporary array, $g[i, j]$, is used to store the closest visible boundary point along latitude for each (θ_i, ϕ_j). The first scan will find the closest visible boundary point from forward direction and then update from backward direction. The scan range from 0 to 3π is to conservatively compute the distance on the sphere [4].

In LongitudeScans, for each ϕ_j, a forward scan and a backward scan are taken on θ_i to extend all closest visible boundary in the latitude dimension to the longitude dimension, and obtain final distance field. The algorithm is similar with that in [5], but we use Eq.(6) to compute the intersection point of regions in spherical Voronoi diagram, which is different from that in 2D EDT.

References

[1] D. Paglieroni, A unified distance transform algorithm and architecture, Machine Vision and Applications 5 (1) (1992) 47–55.

[2] J. Maurer, C.R., R. Qi, V. Raghavan, A linear time algorithm for computing exact euclidean distance transforms of binary images in arbitrary dimensions, Pattern Analysis and Machine Intelligence, IEEE Transactions on 25 (2) (2003) 265–270.

[3] J. B. T. M. R. A. Meijster, W. H. Hesselink, A general algorithm for computing distance transforms in linear time, Computational Imaging and Vision 18 (8) (2008) 331–340.

[4] T. Michikawa, H. Suzuki, Spherical distance transforms, in: Proceedings of the 2008 International Conference on Computational Sciences and Its Applications, 2008, pp. 405–412.

[5] A. Meijster, J. Roerdink, W. Hesselink, A general algorithm for computing distance transforms in linear time 18 (2002) 331–340.