1. Proof of Spherical Distance Transform

In this section, we give the proof of our spherical distance transform. In [1], Paglieroni gave several conditions to apply dimensionality reduction Euclidean Distance Transform (EDT) algorithms on a 2D image as,

\[ d(p_1, p_2) = f(|x_1 - x_2|, |y_1 - y_2|) \]

where \( d \) is computed by

\[ d(p, q) = \arccos(p \cdot q) \]  

where \( p = (\sin \theta_1 \cos \phi_1, \sin \theta_1 \sin \phi_1, \cos \theta_1) \) and \( q = (\sin \theta_2 \cos \phi_2, \sin \theta_2 \sin \phi_2, \cos \theta_2) \). Obviously, \( d(p, q) \) is not the function of \( \Delta \theta = |\theta_1 - \theta_2| \) and \( \Delta \phi = |\phi_1 - \phi_2| \). Thus, the statement in [1] does not hold in the spherical distance transform.

In this document, we give two theorems to prove that under the non-uniform parameterization, if we reduce dimensions in a certain order, i.e. first compute along latitude and then scan the longitude, these two conditions (2) and (3) still hold. In this way, the spherical distance transform can be computed in a dimensionality reduction manner and be parallelized.

Before giving these two theorems, we first introduce some equations that will be used later. We define a function to compute the longitude distance between two points on the sphere, \( p_1(\theta_1, \phi_1) \) and \( p_2(\theta_2, \phi_2) \), as

\[ \Psi(\phi_1, \phi_2) = \min(|\phi_1 - \phi_2|, 2\pi - |\phi_1 - \phi_2|) \]  

The longitude distance actually is the distance along the longitude between two points. Obviously, the range of \( \Psi(\phi_1, \phi_2) \) is \([0, \pi]\).

Then, given a longitude, \( \phi_0 \), and two points, \( p_1 \) and \( p_2 \), on sphere, we define a function to compute the middle latitude, \( \theta_0 \), so that the point, \( p_0(\phi_0, \theta_0) \) has the same distance to these two points, \( p_1 \) and \( p_2 \). (Figure. 2). \( \theta_0 \) is computed by

\[ \theta_0 = \Theta(\phi_0, p_1, p_2) = \begin{cases} \frac{\pi}{2}, & 0 \leq \frac{\arctan \frac{A}{B} - \arctan \frac{A}{B}}{\frac{A}{B} - \frac{A}{B}} \leq \pi, \\ \arctan \frac{A}{B} - \arctan \frac{A}{B}, & \frac{\arctan \frac{A}{B} - \arctan \frac{A}{B}}{\frac{A}{B} - \frac{A}{B}} < 0 \end{cases} \]  

where \( A = \sin \theta_1 \cos \Delta \phi_1, B = \sin \theta_2 \cos \Delta \phi_2, \Delta \phi_1 = |\phi_1 - \phi| \) and \( \Delta \phi_2 = |\phi_2 - \phi| \). Given any two points on sphere and a longitude, \( \phi \), utilizing Eq.(6), we are able to find one point on the longitude, \( \phi \), which has the same distance to these two points.

Given these two functions, we introduce two theorems.

**Theorem 1.** Given two points on the same latitude, \( p_1(\theta_0, \phi_1) \) and \( p_2(\theta_0, \phi_2) \), for any longitude \( \phi \), if the longitude distance \( \Psi(\phi, \phi_1) < \Psi(\phi, \phi_2) \), then for points at
the longitude $\phi$, $p(\theta, \phi), \theta \in [0, \pi]$, we have

$$d_i(p, p_1) \leq d_i(p, p_2)$$

where equality holds iff $\sin \theta = 0$ or $\sin \theta_0 = 0$.

**Proof**
According to Eq. (4), we have

$$\cos d_i(p, p_1) - \cos d_i(p, p_2)$$

$$= \sin \theta \sin \theta_0(\cos \phi - \cos \psi)$$

$$= \sin \theta \sin \theta_0(\cos(\Psi(\phi, \psi)) - \cos(\Psi(\theta, \phi)))$$

Since $\theta, \theta_0$ are both in $[0, \pi]$, $\sin \theta \sin \theta_0$ is always positive. Thus, according to Eq. (7), if $\Psi(\phi, \psi) < \Psi(\theta, \phi)$, we get $\cos d_i(p, p_1) > \cos d_i(p, p_2)$. Given the facts that the cosine function decreases monotonically in the range of $[0, \pi]$, and the range of spherical distance is $[0, \pi]$, we have $d_i(p, p_1) \leq d_i(p, p_2)$ and equality holds iff $\sin \theta = 0$ or $\sin \theta_0 = 0$. □

**Theorem 2.** Given two points on the sphere, $p_1(\theta_1, \phi_1)$ and $p_2(\theta_2, \phi_2)$, where $\theta_1 < \theta_2$, and one longitude, $\phi_0$, according to Eq. (6), we have the point, $p_0(\theta_0, \phi_0)$, with the same distance to $p_0$ and $p_1$. For any other point $p(\theta, \phi_0)$ on the longitude, $\phi_0$, if $\theta < \theta_0$, then

$$d_i(p, p_1) < d_i(p, p_2),$$

and if $\theta > \theta_0$, then

$$d_i(p, p_1) > d_i(p, p_2).$$

**Proof**
For simplicity, we assume $\phi_0 = 0$. If $\phi_0 \neq 0$, we can set a new coordinate with the x-axis at $\phi_0 = 0$ and rotate all points into the new coordinate. Then, we define a function $F(\theta)$ as,

$$F(\theta) = \cos d_i(p, p_1) - \cos d_i(p, p_2)$$

Thus,

$$F(\theta) = ((\sin \theta_1 \cos \phi_1, \sin \theta_1 \sin \phi_1, \cos \theta_1)$$

$$- (\sin \theta_2 \cos \phi_2, \sin \theta_2 \sin \phi_2, \cos \theta_2))$$

$$\cdot (\sin \theta, \cos \theta)$$

$$\Rightarrow \begin{cases} F(\theta) = C \sin \theta + D \cos \theta \\ C = \sin \theta_1 \cos \phi_1 - \sin \theta_2 \cos \phi_2 \\ D = \cos \theta_1 - \cos \theta_2 \end{cases}$$

Because $\theta_1 < \theta_2$, we have $D > 0$. We then formulate $F(\theta)$ as

$$F(\theta) = \sqrt{C^2 + D^2} \cos(\gamma - \theta)$$

Because $D > 0$, $\gamma \in (-\pi/2, \pi/2)$, and $\theta_0 \in (0, \pi)$, we have $\gamma - \theta_0 = -\pi/2$. By defining $\gamma = \theta_0 + \Delta \theta$, we have

$$F(\theta) = \sqrt{A^2 + B^2} \cos(\frac{\pi}{2} + \Delta \theta)$$

Since $\Delta \theta \in (-\pi, \pi)$ and the $\cos(\pi/2 + \Delta \theta)$ has the same sign with $\Delta \theta$, we have

$$\begin{cases} F(\theta) > 0 \Leftrightarrow d_i(p, p_1) < d_i(p, p_2), & \theta < \theta_0 \\ F(\theta) > 0 \Leftrightarrow d_i(p, p_1) > d_i(p, p_2), & \theta > \theta_0 \end{cases}$$

□

**Theorem 1** indicates that the condition (2) is true, when we take the LongitudeScans. **Theorem 2** indicates that the condition (3) is true, when we take the LongitudeScans. Actually, **Theorem 2** also proves that the point, $p_0$ computed from Eq. (6), is the intersection point of a longitude, $\phi$, with two Voronoi regions defined by these two points, $p_1$ and $p_2$. Figure [2] Using such intersection points, the scan on LongitudeScans can be conservatively taken out [2]. Combining these two theorems, we prove our spherical distance transform algorithm is a valid EDT algorithm on sphere.
2. Algorithm Details of Spherical Distance Transform

We list our spherical distance transform (SDT) algorithm in Algorithm 1 in the paper. It is extended from a 2D EDT algorithm [3]. Note that in our paper, the hemisphere distance transform is a special case of the spherical distance transform that only computes the distance filed at hemispheres. Our SDT algorithm takes two phases, the LatitudeScans and the LongitudeScans, to convert visibility maps into SSDFs. We represent SSDF in a square image with $N 	imes N$ resolutions, where one dimension represents $\phi_j$, and the other dimension represents $\theta_i$.

In LatitudeScans, the computation is taken in parallel for each $\theta_i$. In each thread, we first scan $\phi_j$ from 0 to $3\pi$ and then backward. A temporary array, $g[i, j]$, is used to store the closest visible boundary point along latitude for each $(\theta_i, \phi_j)$. The first scan will find the closest visible boundary point from forward direction and then update from backward direction. The scan range from 0 to $3\pi$ is to conservatively compute the distance on the sphere [4].

In LongitudeScans, for each $\phi_j$, a forward scan and a backward scan are taken on $\theta_i$ to extend all closest visible boundary in the latitude dimension to the longitude dimension, and obtain final distance field. The algorithm is similar with that in [5], but we use Eq. (6) to compute the intersection point of regions in spherical Voronoi diagram, which is different from that in 2D EDT.

References