Variational Progressive-iterative Approximation for Fairing Curve and Surface Generation

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***Abstract*—Fairing curve and surface generation is an important topic in geometric design. However, the conventional method for generating the fairing curve and surface, which fit the giving data points, is hard to control the fitting precision, because it is a minimization problem where the objective function is the weighted sum of a fitting term and a fairness term. In this paper, we develop the *variational progressive-iterative approximation* (abbr. variational PIA) method for fitting a data point sequence. While the variational PIA is easy to control the fitting precision, the generated fitting curve or surface is the *most* fairing one in some scope. Lots of comparisons show that the fairness results of the variational PIA are comparable to that of the conventional method.**

Keywords: fairness, progressive-iterative approximation, geometric design, variational method, energy minimization.

I. Introduction

Fairing curve and surface generation is an important topic in geometric design. Given a sequence of data points $\{\boldsymbol{P}_i, i = 0, 1, \cdots, n\}$, a fairing curve $\boldsymbol{P}(t), t \in [t_0, t_1]$ fitting the point sequence is often sought by solving the following minimization problem,

$$\min_{\boldsymbol{P}(t)} F_f(\boldsymbol{P}(t)) + \rho F_s(\boldsymbol{P}(t)), \tag{1}$$

where, $F_f(\boldsymbol{P}(t)) = \sum_{i=0}^{n} \|\boldsymbol{P}(t_i) - \boldsymbol{P}_i\|^2$ is the fitting term, $F_s(\boldsymbol{P}(t))$ is the fairness term, and ρ is the fairness weight.

The above method for seeking the fairing curve fitting the give point sequence has two deficiencies. On one hand, the fitting precision can not be controlled conveniently because it is a minimization problem; on the other hand, since the objective function in the minimization problem (1) is a sum of the fitting term and fairness term, the curve minimizing the object function is not ensured to be the *most* fairing one.

In this paper, a new technique, the *variational progressive-iterative approximation* (abbr. variational PIA), is developed to generate the fairing curve and surface which fit the given data points. Progressive-iterative approximation (abbr. PIA) is an efficient iterative approach to data fitting, which can generate a series of fitting curves or sufaces by adjusting the control points of a blending curve or surface iteratively. In the variational PIA, the adjusting step of every control point in each iteration is determined by a constrained minimization problem, guaranteeing that the resulting curve or surface is the *most* fairing in some scope. Moreover, since the fitting error after each iteration will decrease, it is easy to control the fitting precision in variational PIA. Last but not the least, for fitting the same given data points by the same degree of fitting curve or surface, the number of the unknowns in the variational PIA is one third of that in the conventional method, e.g. by solving the minimization problem (1).

This paper is organized as follows. In Section I-A, we briefly review the related work on progressive-iterative approximation, and fairness terms employed in the fairing curve and surface generation. Moreover, Section II-A introduces several energy functions as the fairness terms, Section II-B presents the iterative format for the variational PIA, and Section II-C addresses the method for solving the constrained minimization problem. In Section III the proposed variational PIA is discussed, and some results are presented. Finally, Section IV concludes the paper.

A. Related Work

The progressive-iterative approximation (PIA) property of the uniform cubic B-spline curve is discovered by Qi [1] and de Boor [2], respectively, and extended to non-uniform cubic B-spline curve and patch [3], the blending curve and patch with normalized total positive basis [4], and the non-uniform B-spline curve and surface [5]. Moreover, the convergence rate of the PIA format is analyzed in [6], and accelerated in [7].

On the other hand, the fairness term in Eq. (1) is usually taken as some energy of the fitting curve or surface, such as the strain energy [8], [9]. However, the stain energy is high nonlinear and the corresponding minimization problem is hard to be solved. Therefore, lots of energy models simplifying the strain energy are presented [9], [10], including the thin plate model [11], membrane model [12], jerk energy [13], etc. For more details on the fairness term, please refer to Refs. [9], [14].

II. Variational Progressive-iterative Approximation

The variational PIA method generates the fairing curve or surface by minimizing the energy of the curve or surface, which is introduced in Section II-A. In Section II-B, the iterative format of the variational PIA is presented, where the constrained minimization problem is solved by the method addressed in Section II-C.

A. Energy Function

In geometric design, the energy function is usually employed to measure the fairness of a piece of curve or surface.

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And the fairing curve and surface are generated by minimizing an energy function. Specifically, a commonly used energy function for generating a piece of fairing curve $\boldsymbol{P}(t), t \in I$ is the approximate strain energy [9],

$$E_c(\boldsymbol{P}(t)) = \int_I \|\boldsymbol{P}_{tt}(t)\|^2 \, dt. \tag{2}$$

Moreover, for generating the fairing surface $\boldsymbol{P}(u,v), (u,v) \in D$, we adopt the thin plate energy [9], that is,

$$E_s(\boldsymbol{P}(u,v)) = \iint_D (\|\boldsymbol{P}_{uu}\|^2 + 2\|\boldsymbol{P}_{uv}\|^2 + \|\boldsymbol{P}_{vv}\|^2) dudv. \tag{3}$$

B. The Iterative Format of the Variational PIA

In this section, we will present the iterative format of the variational PIA for the parametric curve and surface, respectively.

Given a data point sequence $\{\boldsymbol{P}_i, i = 0, 1, \cdots, n\}$ with non-decreasing parameters $t_i, i = 0, 1, \cdots, n$, the initial blending curve can be constructed as,

$$\boldsymbol{P}^0(t) = \sum_{i=0}^{n} \boldsymbol{P}_i^0 B_i(t), \ t \in I \tag{4}$$

where, $\boldsymbol{P}_i^0 = \boldsymbol{P}_i$, and $B_i(t), i = 0, 1, \cdots, n$ are the blending basis.

Suppose the k^{th} curve $\boldsymbol{P}^k(t)$ has been generated. To produce the $(k+1)^{th}$ curve $\boldsymbol{P}^{k+1}(t)$, we need to calculate the difference vectors,

$$\boldsymbol{\Delta}_i^k = \boldsymbol{P}_i - \boldsymbol{P}^k(t_i), i = 0, 1, \cdots, n. \tag{5}$$

Then, the variational PIA makes up the control points $\boldsymbol{P}_i^{k+1}, i = 0, 1, \cdots, n$ of the $(k+1)^{th}$ curve by,

$$\boldsymbol{P}_i^{k+1} = \boldsymbol{P}_i^k + \lambda_i^k \boldsymbol{\Delta}_i^k, i = 0, 1, \cdots, n, \tag{6}$$

where, the weights $\lambda_i^k, i = 0, 1, \cdots, n$ are determined by solving the following constrained minimization problem,

$$\begin{aligned} &\min_{\lambda_i^k} E_c(\boldsymbol{P}^{k+1}(t)) \\ &s.t. \quad 0 < \lambda_i^k < 2, \ i = 0, 1, \cdots, n. \end{aligned} \tag{7}$$

In the case of surface fitting by the variational PIA, the initial surface $\boldsymbol{P}^0(u,v), \ (u,v) \in D$ is constructed firstly, by taking the given data points $\boldsymbol{P}_{ij}, \ i = 0, 1, \cdots, m$, $j = 0, 1, \cdots, n$ as the control points, with the corresponding parameters (u_i, v_j), such that,

$$u_0 \le u_1 \le \cdots \le u_m, \ v_0 \le v_1 \le \cdots \le v_n.$$

Supposing the k^{th} surface $\boldsymbol{P}^k(u,v)$ has been obtained, we calculate the difference vectors,

$$\boldsymbol{\Delta}_{ij}^k = \boldsymbol{P}_{ij} - \boldsymbol{P}^k(u_i, v_j), \ i = 0, 1, \cdots, m, \ j = 0, 1, \cdots, n, \tag{8}$$

and the new control points $\boldsymbol{P}_{ij}^{k+1}$ for the $(k+1)^{th}$ surface $\boldsymbol{P}^{k+1}(u,v)$, that is,

$$\boldsymbol{P}_{ij}^{k+1} = \boldsymbol{P}_{ij}^k + \lambda_{ij}^k \boldsymbol{\Delta}_{ij}^k, \ i = 0, 1, \cdots, m, \ j = 0, 1, \cdots, n. \tag{9}$$

Similarly, the weights λ_{ij}^k in (9) are determined by the following constrained minimization problem,

$$\begin{aligned} &\min_{\lambda_{ij}^k} E_s(\boldsymbol{P}^{k+1}(u,v)) \\ &s.t. \quad 0 < \lambda_{ij}^k < 2, \ i = 0, 1, \cdots, m, \ j = 0, 1, \cdots, n, \end{aligned} \tag{10}$$

where, $\boldsymbol{P}^{k+1}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \boldsymbol{P}_{ij}^{k+1} B_i(u) B_j(v), \ (u,v) \in D$, and $B_i(u), B_j(v)$ are basis functions.

Based on the convergence analysis in Ref. [7], the curve sequence $\{\boldsymbol{P}^k(t), k = 0, 1, \cdots\}$ converges to the curve interpolating the given data points $\{\boldsymbol{P}_i, i = 0, 1, \cdots, n\}$, when $0 < \lambda_i^k < 2, \ i = 0, 1, \cdots, n$ (7), and the surface sequence $\{\boldsymbol{P}^k(u,v), k = 0, 1, \cdots,\}$ converges to the data points $\{\boldsymbol{P}_{ij}, \ i = 0, 1, \cdots, m, \ j = 0, 1, \cdots, n\}$ when $0 < \lambda_{ij}^k < 2, i = 0, 1, \cdots, m, j = 0, 1, \cdots, n$ (10).

C. Solving the Constrained Minimization Problem

The constrained minimization problems (7) and (10) are actually the box constrained quadratic programming problem,

$$\min_X \frac{1}{2} X^T H X + F^T X \tag{11}$$

$$s.t. \quad 0 < X < 2, \tag{12}$$

where, X is a column vector, and $0 < X < 2$ means that each element of X lies in $(0, 2)$. The matrices H and F will be deduced in the following sections.

1) The matrices H and F for Eq. (7): In the case of parametric curve, the $(k+1)^{th}$ curve is,

$$\begin{aligned} \boldsymbol{P}^{k+1}(t) &= (x^{k+1}(t), y^{k+1}(t), z^{k+1}(t)) \\ &= \sum_{i=0}^{n} (\boldsymbol{P}_i^k + \lambda_i^k \boldsymbol{\Delta}_i^k) B_i(t), \ t \in I, \end{aligned}$$

where, $\boldsymbol{P}_i^k = (x_i^k, y_i^k, z_i^k)$, and $\boldsymbol{\Delta}_i^k = (\Delta x_i^k, \Delta y_i^k, \Delta z_i^k)$ (5). By transforming the objective function $E_c(\boldsymbol{P}^{k+1}(t))$ in (7) into the form of (11), we have,

$$X = \{\lambda_0^k, \lambda_1^k, \cdots, \lambda_n^k\}^T,$$

$$H = D_x B D_x + D_y B D_y + D_z B D_z, \tag{13}$$

and,

$$F^T = L_x B D_x + L_y B D_y + L_z B D_z, \tag{14}$$

where,

$$L_w = [w_0^k, w_1^k, \cdots, w_n^k], \ D_w = diag(\Delta w_0^k, \Delta w_1^k, \cdots, \Delta w_n^k), \quad w = x, y, z,$$

and the matrix B is,

$$B = \left[\int_I B_i''(t) B_j''(t) dt \right]_{(n+1)\times(n+1)}, \quad i, j = 0, 1, \cdots, n.$$

Here, $diag(\cdot)$ denotes the diagonal matrix.

2) The matrices H and F for Eq. (10): Moreover, in the case of parametric surface, the $(k+1)^{th}$ surface is

$$\begin{aligned} \boldsymbol{P}^{k+1}(u,v) &= (x^{k+1}(u,v), y^{k+1}(u,v), z^{k+1}(u,v)) \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n}(\boldsymbol{P}_{ij}^k + \lambda_{ij}^k \boldsymbol{\Delta}_{ij}^k)B_i(u)B_j(v), \end{aligned} \tag{15}$$

where, $(u,v) \in D = [D_u, D_v]$, $\boldsymbol{P}_{ij}^k = (x_{ij}^k, y_{ij}^k, z_{ij}^k)$, and $\boldsymbol{\Delta}_{ij}^k = (\Delta x_{ij}^k, \Delta y_{ij}^k, \Delta z_{ij}^k)$.

To present the matrices H and F in (11), we arrange the control points and difference vectors in a one-dimensional sequence, where their subscripts are in the order below,

$$\{00, 01, \cdots, 0n, 10, 11, \cdots, 1n, \cdots, m0, m1, \cdots, mn\}.$$

According to the order above, we re-denotes the subscripts of the control points and difference vectors as, $\{0, 1, \cdots, (m+1)(n+1)\}$. Then,

$$X = \{\lambda_0^k, \lambda_1^k, \cdots, \lambda_{(m+1)(n+1)}^k\}^T.$$

Furthermore, substituting (15) into $E_s(\boldsymbol{P}^{k+1}(u,v))$ (10), the matrices H and F can also be represented as Eqs. (13) and (14). Here, $B = B_{uu} + 2B_{uv} + B_{vv}$,

$$B_{uu} = B_v^0 \otimes B_u^2, B_{uv} = B_v^1 \otimes B_u^1, B_{vv} = B_v^2 \otimes B_u^0.$$

where, $\otimes$ denotes the Kronecker product, and,

$$\begin{aligned} B_u^l &= \left[\int_{D_u} \frac{d^l B_i(u)}{du^l}\frac{d^l B_j(u)}{du^l}du\right]_{(m+1)\times(m+1)}, \\ & l = 0,1,2, i,j = 0,1,\cdots,m, \\ B_v^l &= \left[\int_{D_v} \frac{d^l B_i(v)}{dv^l}\frac{d^l B_j(v)}{dv^l}dv\right]_{(n+1)\times(n+1)}, \\ & l = 0,1,2, i,j = 0,1,\cdots,n. \end{aligned}$$

The box constrained quadratic programming problem (11) can be solved by the reflective Newton method [15].

III. Implementation and Results

Although the PIA iterative format converges for all of the blending curves and surfaces with normalized totally positive basis, we choose the cubic B-spline to demonstrate the variational PIA in this paper, since it is the most commonly used in practice.

As stated above, given a data point sequence $\{\boldsymbol{P}_i^0 = \boldsymbol{P}_i, i = 0, 1, \cdots, n\}$, with parameters $t_0 < t_1 < \cdots, t_n$, the knot of the cubic B-spline curve is constructed as,

$$k = \{t_0, t_0, t_0, t_1, t_2, \cdots, t_{n-1}, t_n, t_n, t_n\}.$$

Then, the initial cubic B-spline curve $\boldsymbol{P}^0(t)$ can be generated as,

$$\boldsymbol{P}^0(t) = \sum_{i=0}^{n} \boldsymbol{P}_i^0 B_i(t), \tag{16}$$

where, $B_i(t), i = 0, 1, \cdots, n$ are the cubic B-spline basis. The curve (16) is defined in the interval $[t_1, t_{n-1}]$, so in the k^{th} iteration, the point $\boldsymbol{P}^k(t_i)$ corresponds to the data point $\boldsymbol{P}_i, i = 1, 2, \cdots, n-1$. The difference vectors are $\boldsymbol{\Delta}_i^k = \boldsymbol{P}_i - \boldsymbol{P}^k(t_i), i = 1, 2, \cdots, n-1$, and we let $\boldsymbol{\Delta}_0^k = \boldsymbol{\Delta}_n^k = 0$.

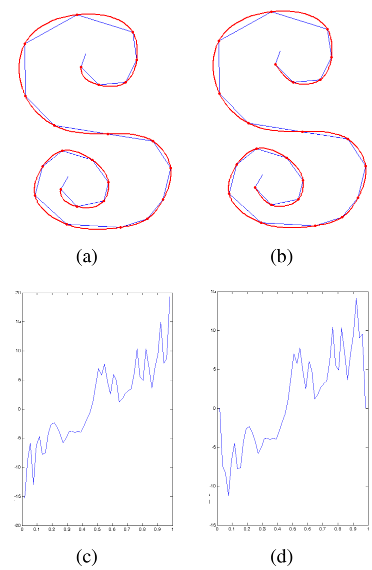

Fig. 1. Comparison of the fairness between the fitting curves generated by the variational PIA and the conventional method (1). (a.) Fitting curve generated by 64 variational PIA iterations with precision 2.1199×10^{-4}. (b.) Fitting curve by the conventional method (1) with precision 2.2273×10^{-4}. (c.) Curvature plot of Fig. 1(a). (d.) Curvature plot of Fig. 1(b).

In the limit, the curve interpolates the data points $\boldsymbol{P}_i, i = 1, 2, \cdots, n-1$.

On the other hand, given a data array $\{\boldsymbol{P}_{ij}^0 = \boldsymbol{P}_{ij}, i = 0, 1, \cdots, m, j = 0, 1, \cdots, n\}$ with parameters (u_i, v_j), where, $u_0 < u_1 < \cdots < u_m, v_0 < v_1 < \cdots < v_n$, the knots of the bi-cubic B-spline patch fitting the data array are constructed as,

$$\begin{aligned} k_u &= \{u_0, u_0, u_0, u_1, u_2, \cdots, u_{m-1}, u_m, u_m, u_m\}, \\ k_v &= \{v_0, v_0, v_0, v_1, v_2, \cdots, v_{n-1}, v_n, v_n, v_n\}. \end{aligned}$$

Then, the initial bi-cubic B-spline patch is,

$$\boldsymbol{P}^0(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} \boldsymbol{P}_{ij}^0 B_i(u)B_j(v), \tag{17}$$

defined on $[u_1, u_{m-1}] \times [v_1, v_{n-1}]$, where, $B_i(u)$ and $B_j(v)$ are the cubic B-spline basis.

In the k^{th} iteration, the point $\boldsymbol{P}^k(u_i, v_j)$ corresponds to the data point $\boldsymbol{P}_{ij}, i = 1, 2, \cdots, m-1, j = 1, 2, \cdots, n-1$. The difference vectors are, $\boldsymbol{\Delta}_{ij}^k = \boldsymbol{P}_{ij} - \boldsymbol{P}^k(u_i, v_j), i = 1, 2, \cdots, m-1, j = 1, 2, \cdots, n-1$, and the difference vectors corresponding to the boundary data points are set as 0 in each iteration. In the limit, the bi-cubic B-spline patch interpolates the data points $\boldsymbol{P}_{ij}, i = 1, 2, \cdots, m-1, j = 1, 2, \cdots, n-1$.

In our implementation, the fitting precision ε in the k^{th} iteration is taken as,

$$\varepsilon = \max\{\left\|\boldsymbol{\Delta}_i^k\right\|, i = 0, 1, \cdots, n\},$$

in the cases of cubic B-spline curve, and,

$$\varepsilon = \max\{\left\|\boldsymbol{\Delta}_{ij}^k\right\|, i = 0, 1, \cdots, m, j = 0, 1, \cdots, n\},$$

in the case of cubic B-spline surface.

As stated above, in the conventional methods (1) for generating the fairing curve or surface fitting the given data points, the objective function is the mixture of the fitting term and the fairness term. Integrating the two terms into one objective function makes the control of the fitting precision is inconvenient. To get the fitting curve or surface with desired fitting precision, we need to adjust the weight ρ in Eq. (1) by a trial and error procedure. In addition, since the fairness term is just one component of the objective function (1), the minimization of Eq. (1) does not mean the minimization of the fairness term.

On the contrary, the variational PIA presented in this paper is very easy in controlling the fitting precision, since the iterative format is convergent, and then the fitting precision will decrease after each iteration. Moreover, from the Eqs. (7) and (10), we can see that, the objective function in the variational PIA is just the energy function, so, the minimization of the objective function leads to the *most* fairing curve or surface in some scope.

Fig. 1 and Fig. 2 illustrate the comparison between the variational PIA and the conventional method (1). Fig. 1 are the cubic B-spline fitting curves and their curvature plots. Fig.2 is the bi-cubic B-spline fitting patches and the zebra on them. These experimental results show that, the fairness of the fitting curve and surface generated by the variational PIA is comparable to that generated by the conventional method (1).

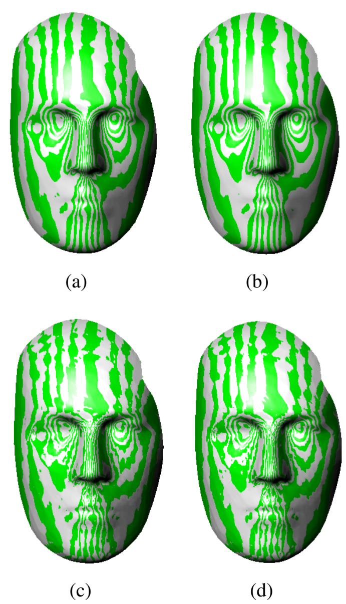

Fig. 2. The zebra strips on the surfaces fitting the data points sampled from a *face* model using the variational PIA and conventional method, respectively. (a.) Fitting surface by 1 variational PIA iteration with precision 4.0626×10^{-3}. (b.) Fitting surface by the conventional method with precision 3.6541×10^{-3}. (c.) Fitting surface by 39 variational PIA iterations with precision 1.2265×10^{-4}. (d.) Fitting surface by the conventional method with precision 1.2596×10^{-4}.

IV. Conclusion

In this paper, we develop the variational progressive-iterative approximation method, where each iteration step is determined by solving an energy minimization problem. Furthermore, we present the iterative format of variational PIA for the parametric curve and surface. The variational PIA is easy to control the fitting precision, while the generated fitting curve or surface is the most fairing one in some scope. Lots of examples illustrated that, the fairness of the fitting curve or surface generated by the variational PIA is comparable to the conventional method.

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