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Totally Positive Bases and Progressive Iteration Approximation

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Abstract—In this paper, we study the *progressive iteration approximation* property of a curve (tensor product surface) generated by blending a given data point set and a set of basis functions. The curve (tensor product surface) has the *progressive iteration approximation* property as long as the basis is totally positive and the corresponding collocation matrix is nonsingular. Thus, the B-spline and NURBS curve (surface) have the *progressive iteration approximation* property, and Bézier curve (surface) also has the property if the corresponding collocation matrix is nonsingular. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Progressive iteration approximation, Total positivity, NURBS, B-spline, Bézier.

1. INTRODUCTION

Given a sequence of points $\{\mathbf{P}_i\}_{i=0}^n$, the i^{th} point of which is assigned a parameter value t_i, $i = 0, 1, \ldots, n$, and a nonnegative basis $\{B_i(t) \geq 0 \mid t \in \mathrm{R},\ i = 0, 1, \ldots, n\}$ with $\sum_{i=0}^n B_i(t) = 1$, the initial curve can be generated as follows, i.e., $\mathbf{C}^0(t) = \sum_{i=0}^n \mathbf{P}_i^0 B_i(t)$, with $\{\mathbf{P}_i^0 = \mathbf{P}_i\}_{i=0}^n$. By calculating the adjusting vector for each control point $\mathbf{\Delta}_i^0 = \mathbf{P}_i - \mathbf{C}^0(t_i)$, $i = 0, 1, \ldots, n$, and letting $\{\mathbf{P}_i^1 = \mathbf{P}_i^0 + \mathbf{\Delta}_i^0\}_{i=0}^n$, we can get the next curve $\mathbf{C}^1(t) = \sum_{i=0}^n \mathbf{P}_i^1 B_i(t), \ldots$, and so on. Thus, at last, we get a sequence of curves $\{\mathbf{C}^k(t) \mid k = 0, 1, \ldots\}$ (see Figure 1).

Qi and de Boor have shown that, if the given nonnegative basis is a uniform cubic B-spline basis, and the parameter value t_i assigned to each data point happens to be at the knot of the knot vector on which the uniform cubic B-spline basis is defined, the curve sequence converges to a curve interpolating the given point sequence, i.e., $\lim_{k \to \infty} \mathbf{C}^k(t_i) = \mathbf{P}_i^0$, $i = 0, 1, \ldots, n$ [1,2]. We say that the initial curve has the *progressive iteration approximation* property.

Furthermore, in [3], the authors have shown that not only the nonuniform cubic B-spline curve, but the nonuniform cubic B-spline tensor product surface also has the *progressive iteration approximation* property.

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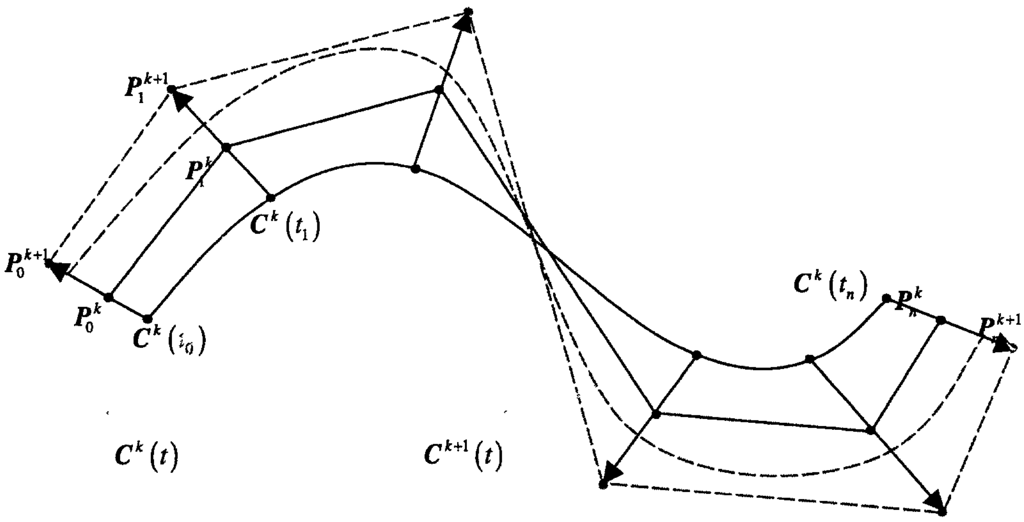

Figure 1. Progressive iteration approximation, from $\mathbf{C}^k(t)$ to $\mathbf{C}^{k+1}(t)$.

In this paper, we will show that, as long as the given basis is totally positive, and its corresponding collocation matrix is nonsingular, the curve and tensor product surface generated by the basis have the *progressive iteration approximation* property. So, the B-spline, and NURBS curve and surface all have the *progressive iteration approximation* property, and Bézier curves and surfaces also have the *progressive iteration approximation* property, if the corresponding collocation matrix is nonsingular.

The layout of this paper is as follows. In Section 2, we establish the *progressive iteration approximation* property of the curve and tensor product surface generated by a totally positive blending basis with nonsingular collocation matrix. In Section 3, we point out that the NURBS curve and surface have the *progressive iteration approximation* property. In Section 4, some results illustrating the *progressive iteration approximation* property of the Bézier (B-spline, NURBS) curve (surface) are given, and the fitting errors are also listed. At last, we conclude the paper in Section 5.

2. PROGRESSIVE ITERATION APPROXIMATION OF CURVES AND SURFACES

A nonnegative basis $\{B_i \geq 0 \mid i = 0, 1, \dots, n\}$ with $\sum_{i=0}^{n} B_i = 1$ is called a *blending basis*. Based on the blending basis and a given data point set $\{\mathbf{P}_i \in \mathrm{R}^3\}_{i=0}^{n}$ ($\{\mathbf{P}_{ij} \in \mathrm{R}^3\}_{i=0,j=0}^{m\ \ n}$), we can generate a blending curve,

$$\mathbf{C}(t) = \sum_{i=0}^{n} \mathbf{P}_i B_i(t), \tag{2.1}$$

or a tensor product blending surface,

$$\mathbf{S}(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{P}_{ij} B_i(u) B_j(v), \tag{2.2}$$

where $\mathbf{P}_i$ and $\mathbf{P}_{ij}$ are called *control points*.

In the following, we first present the definition of a totally positive basis.

DEFINITION 2.1. *Given a basis $\{B_i(t) \geq 0 \mid i = 0, 1, \dots, n\}$ defined on $\Xi \subseteq \mathrm{R}$ and an increasing sequence $\tau_0 < \tau_1 < \cdots < \tau_m$ in Ξ, the collocation matrix of $B_0, \dots, B_n$ at $\tau_0 < \tau_1 < \cdots < \tau_m$ is the matrix,*

$$M\begin{pmatrix} B_0, \dots, B_n \\ \tau_0, \dots, \tau_m \end{pmatrix} := \left(B_j(\tau_i)\right)_{i=0,\dots,m,\, j=0,\dots,n}. \tag{2.3}$$

The basis $\{B_0, B_1, \ldots, B_n\}$ is called *totally positive basis* if its collocation matrix at any increasing sequence is a totally positive matrix, that is, all of its minors are nonnegative [4,5].

2.1. Progressive Iteration Approximation of Blending Curves

Given a point sequence $\{\mathbf{P}_i \in \mathrm{R}^3 \mid i = 0, 1, \ldots, n\}$, we first parameterize the points with the real increasing sequence,

$$t_0 < t_1 < \cdots < t_n. \tag{2.4}$$

Namely, the parameter t_i is assigned to the i^{th} point $\mathbf{P}_i$ $(i = 0, 1, \ldots, n)$. Then, we can generate the first curve by blending the point sequence $\{\mathbf{P}_i^0 = \mathbf{P}_i \mid i = 0, 1, \ldots, n\}$ and the totally positive blending basis $\{B_i \geq 0 \mid i = 0, 1, \ldots, n\}$, that is,

$$\mathbf{C}^0(t) = \sum_{i=0}^{n} \mathbf{P}_i^0 B_i(t). \tag{2.5}$$

By computing the adjusting vectors of the control points,

$$\mathbf{\Delta}_i^0 = \mathbf{P}_i - \mathbf{C}^0(t_i), \qquad i = 0, 1, \ldots, n, \tag{2.6}$$

and letting

$$\mathbf{P}_i^1 = \mathbf{P}_i^0 + \mathbf{\Delta}_i^0, \qquad i = 0, 1, \ldots, n, \tag{2.7}$$

we can get the second curve,

$$\mathbf{C}^1(t) = \sum_{i=0}^{n} \mathbf{P}_i^1 B_i(t). \tag{2.8}$$

Similarly, if we get the $(k+1)^{\text{st}}$ curve $C^k(t)$ after the k^{th} iteration, and let

$$\mathbf{\Delta}_i^k = \mathbf{P}_i - \mathbf{C}^k(t_i), \quad i = 0, 1, \ldots, n, \qquad \text{and} \qquad \mathbf{P}_i^{k+1} = \mathbf{P}_i^k + \mathbf{\Delta}_i^k, \quad i = 0, 1, \ldots, n, \tag{2.9}$$

we can get the $(k+2)^{\text{nd}}$ curve after the $(k+1)^{\text{st}}$ iteration,

$$\mathbf{C}^{k+1}(t) = \sum\nolimits_{i=0}^{n} \mathbf{P}_i^{k+1} B_i(t). \tag{2.10}$$

Thus, we get a curve sequence $\{\mathbf{C}^k(t) \mid k = 0, 1, \ldots\}$. If $\lim_{k\to\infty} \mathbf{C}^k(t_i) = \mathbf{P}_i^0$, $i = 0, 1, \ldots, n$, the initial curve (2.5) has the *progressive iteration approximation* property.

Due to

$$\begin{aligned}
\mathbf{\Delta}_j^{k+1} &= \mathbf{P}_j - \mathbf{C}^{k+1}(t_j) = \mathbf{P}_j - \sum_{i=0}^{n} (\mathbf{P}_i^k + \mathbf{\Delta}_i^k) B_i(t_j) \\
&= (\mathbf{P}_j - \mathbf{C}^k(t_j)) - \sum_{i=0}^{n} \mathbf{\Delta}_i^k B_i(t_j) \\
&= -\sum_{i=0}^{j-1} \mathbf{\Delta}_i^k B_i(t_j) \\
&\quad + (1 - B_j(t_j)) \mathbf{\Delta}_j^k - \sum_{i=j+1}^{n} \mathbf{\Delta}_i^k B_i(t_j), \qquad (j = 0, 1, \ldots, n;\ k = 0, 1, \ldots),
\end{aligned} \tag{2.11}$$

we can get the iterative format in matrix form of the adjusting vectors of the control points,

$$\left[\mathbf{\Delta}_0^{k+1}, \mathbf{\Delta}_1^{k+1}, \ldots, \mathbf{\Delta}_n^{k+1}\right]^\top = \mathbf{D}\left[\mathbf{\Delta}_0^k, \mathbf{\Delta}_1^k, \ldots, \mathbf{\Delta}_n^k\right]^\top, \qquad \mathbf{D} = \mathbf{I} - \mathbf{B}; \qquad k = 0, 1, \ldots, \tag{2.12}$$

where $\mathbf{I}$ is the $n+1$ rank identity matrix, and $\mathbf{B}$ is the collocation matrix of the blending basis $\{B_i \geq 0 \mid i = 0,1,\ldots,n\}$ at $\{t_0, t_1, \ldots, t_n\}$, namely,

$$\mathbf{B} = \begin{bmatrix} B_0(t_0) & B_1(t_0) & \ldots & B_n(t_0) \\ B_0(t_1) & B_1(t_1) & \ldots & B_n(t_1) \\ \ddots & \ddots & & \ddots \\ B_0(t_n) & B_1(t_n) & \ldots & B_n(t_n) \end{bmatrix}. \tag{2.13}$$

In Theorem 2.1, we give a sufficient condition for the iterative format (2.12) to converge to zero, and then the curve (2.5) has *progressive iteration approximation* property. In the following, we denote by $\lambda_i(M)$, $i = 1,2,\ldots,m$, the eigenvalues of the $m \times m$ matrix M, and by $\rho(M)$, the spectrum radius of M.

THEOREM 2.1. *A piece of blending curve (2.5) has the progressive iteration approximation property, if the basis is totally positive and its collocation matrix* $\mathbf{B}$ *at* $\{t_0, t_1, \ldots, t_n\}$ *is nonsingular.*

PROOF. Since the blending basis $\{B_i \geq 0 \mid i = 0,1,\ldots,n\}$ is totally positive, its collocation matrix $\mathbf{B}$ has $n+1$ nonnegative eigenvalues $\lambda_i(B)$, $i = 0,1,\ldots,n$ [6,7]. Together with the fact that the collocation matrix $\mathbf{B}$ is nonsingular, its $n+1$ eigenvalues are all positive. Note that the basis $\{B_i \geq 0 \mid i = 0,1,\ldots,n\}$ is a blending basis, namely, $\sum_{i=0}^{n} B_i = 1$, so $\|\mathbf{B}\|_\infty = 1$. Therefore, $0 < \lambda_i(\mathbf{B}) \leq 1$, $i = 0,1,\ldots,n$, so, $0 \leq \lambda_i(\mathbf{D}) = 1 - \lambda_i(\mathbf{B}) < 1$, $i = 0,1,\ldots,n$. This result implies $\rho(\mathbf{D}) < 1$, so the iterative format (2.12) converges to zero vector; hence, $\lim_{k\to\infty} \mathbf{C}^k(t_i) = \mathbf{P}_i^0$, $i = 0,1,\ldots,n$.

2.2. Progressive Iteration Approximation of Blending Surfaces

In this section, we study the *progressive iteration approximation* property of tensor product blending surfaces. Given an ordered point set $\{P_{ij} \in \mathrm{R}^3\}_{i=0}^{m}{}_{j=0}^{n}$, we first assign the following parameter values $\{(u_i, v_j)\}_{i=0}^{m}{}_{j=0}^{n}$ to the points $\{P_{ij} \in \mathrm{R}^3\}_{i=0}^{m}{}_{j=0}^{n}$,

$$\begin{aligned} &u_0 < u_1 < \cdots < u_m, \\ &v_0 < v_1 < \cdots < v_n. \end{aligned} \tag{2.14}$$

Similar to section 2.1, we can generate an initial surface,

$$\mathbf{S}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{P}_{ij} B_i(u) B_j(v), \tag{2.15}$$

and a surface sequence,

$$\left\{ \mathbf{S}^k(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{P}_{ij}^k B_i(u) B_j(v) \mid k = 0,1,\ldots \right\} \text{ with } \{\mathbf{P}_{ij}^0 = \mathbf{P}_{ij}\}_{i=0}^{m}{}_{j=0}^{n}. \tag{2.16}$$

If

$$\lim_{k\to\infty} S^k(u_i, v_j) = \mathbf{P}_{ij}^0, \qquad i = 0,1,\ldots,m, \quad j = 0,1,\ldots,n,$$

the tensor product surface generated by blending the blending basis $\{B_i \geq 0 \mid i = 0,1,\ldots,n\}$ and the control points $\{\mathbf{P}_{ij} \in \mathrm{R}^3\}_{i=0}^{m}{}_{j=0}^{n}$ has the *progressive iteration approximation* property.

Suppose that $\mathbf{S}^k(u,v)$ is the surface after the k^{th} iteration. The adjusting vector of the $(h,l)^{\text{th}}$ control point in the $(k+1)^{\text{st}}$ iteration is

$$\begin{aligned} \mathbf{\Delta}_{hl}^{k+1} &= \mathbf{P}_{hl} - \sum_{i=0}^{m} \sum_{j=0}^{n} (\mathbf{P}_{ij}^k + \mathbf{\Delta}_{ij}^k) B_i(u_h) B_j(v_l) \\ &= \mathbf{P}_{hl} - \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{P}_{ij}^k B_i(u_h) B_j(v_l) - \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{\Delta}_{ij}^k B_i(u_h) B_j(v_l) \\ &= \mathbf{\Delta}_{hl}^k - \sum_{i=0}^{m} \sum_{j=0}^{n} \mathbf{\Delta}_{ij}^k B_i(u_h) B_j(v_l), \\ &\qquad (h = 0,1,\ldots,m, \ l = 0,1,\ldots,n, \ k = 0,1,\ldots). \end{aligned} \tag{2.17}$$

Thus, we can get the iterative format in matrix form of the adjusting vectors of the control points,

$$\mathbf{\Delta}^{k+1} = \mathbf{D}\Delta^k, \quad \mathbf{D} = \mathbf{I} - \mathbf{B}, \qquad k = 0, 1, \dots. \tag{2.18}$$

Here,

$$\mathbf{\Delta}^j = \left[\mathbf{\Delta}^j_{00}, \mathbf{\Delta}^j_{01}, \dots, \mathbf{\Delta}^j_{0,n}, \mathbf{\Delta}^j_{10}, \dots, \mathbf{\Delta}^j_{1n}, \dots, \mathbf{\Delta}^j_{m1}, \dots, \mathbf{\Delta}^j_{mn}\right]^{\top}, \qquad j = k, k+1, \tag{2.19}$$

$\mathbf{I}$ is the identity matrix, and matrix $\mathbf{B}$ is the Kronecker product of the matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ [8], that is, $\mathbf{B} = \mathbf{B}_1 \otimes \mathbf{B}_2$, where

$$\mathbf{B}_1 = \begin{bmatrix} B_0(u_0) & B_1(u_0) & \dots & B_m(u_0) \\ B_0(u_1) & B_1(u_1) & \dots & B_m(u_1) \\ \ddots & \ddots & & \ddots \\ B_0(u_m) & B_1(u_m) & \dots & B_m(u_m) \end{bmatrix} \quad \text{and}$$

$$\mathbf{B}_2 = \begin{bmatrix} B_0(v_0) & B_1(v_0) & \dots & B_n(v_0) \\ B_0(v_1) & B_1(v_1) & \dots & B_n(v_1) \\ \ddots & \ddots & & \ddots \\ B_0(v_n) & B_1(v_n) & \dots & B_n(v_n) \end{bmatrix}. \tag{2.20}$$

LEMMA 2.1. *Consider the matrices $\mathbf{A} \in \mathrm{R}^{m\times m}$, and $\mathbf{B} \in \mathrm{R}^{n\times n}$. Every eigenvalue of their Kronecker product $\mathbf{A} \otimes \mathbf{B}$ can be expressed as the product of the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$. Namely, if $\lambda(\mathbf{A}) = \{\lambda_1, \dots, \lambda_m\}$, and $\lambda(\mathbf{B}) = \{\mu_1, \mu_2, \dots, \mu_n\}$,*

$$\lambda(\mathbf{A} \otimes \mathbf{B}) = \{\lambda_i \mu_j \mid i = 1, 2, \dots, m;\ j = 1, 2, \dots, n\}.$$

Here, the eigenvalues are counted with their algebraic multiplicity [8].

THEOREM 2.2. *A piece of a tensor product blending surface (2.15) has the progressive iteration approximation property, if the bases $\{B_i\}_{i=0}^m$ and $\{B_j\}_{j=0}^n$ are totally positive and their collocation matrices at $\{u_0, \dots, u_m\}$ and $\{v_0, \dots, v_n\}$ are nonsingular.*

PROOF. As to the iterative format (2.18), we can know first from the proof of Theorem 2.1 that $0 < \lambda_i(\mathbf{B}_1) \le 1$, $i = 0, 1, \dots, m$, and $0 < \lambda_i(\mathbf{B}_2) \le 1$, $i = 0, 1, \dots, n$; second, from Lemma 2.1, we can get $0 < \lambda_{ij}(\mathbf{B}_1 \otimes \mathbf{B}_2) = \lambda_i(\mathbf{B}_1) \cdot \lambda_j(\mathbf{B}_2) \le 1$, $i = 0, 1, \dots, m$, $j = 0, 1, \dots, n$. So,

$$0 \le \lambda_i(\mathbf{D}) = 1 - \lambda_i(\mathbf{B}_1 \otimes \mathbf{B}_2) < 1, \qquad i = 0, 1, \dots, mn.$$

That is, $\rho(\mathbf{D}) < 1$. Therefore, the iterative format (2.18) converges to the zero vector, so, $\lim_{k\to\infty} \mathbf{S}^k(u_i, v_j) = \mathbf{P}^0_{ij}$, $i = 0, 1, \dots, m$, $j = 0, 1, \dots, n$. ∎

3. PROGRESSIVE ITERATION APPROXIMATION OF NURBS CURVES AND SURFACES

In this section, we will study the *progressive iteration approximation* property of Bézier, B-spline, and NURBS curves and surfaces.

Because the Bernstein basis is totally positive [4], Bézier curves and surfaces have the property of *progressive iteration approximation* if the corresponding collocation matrix is nonsingular.

Lin *et al.* have shown that nonuniform cubic B-spline curves and surfaces have the property of *progressive iteration approximation* [3]. Specifically, given a sequence of points $\{\mathbf{P}_i \in \mathrm{R}^3\}_{i=0}^n$, we first define a knot vector,

$$\mathbf{T} = \left\{ \underbrace{0, \dots, 0}_{p+1}, t_{p+1}, \dots, t_{p+n-1}, \underbrace{t_{p+n}, \dots, t_{p+n}}_{p+1} \right\}, \tag{3.1}$$

where

$$0 = t_p < t_{p+1} < \cdots < t_{p+n},$$

and assign the parameter value t_{p+i} to the i^{th} point $\mathbf{P}_i$, $i = 0, 1, \ldots, n$. Similar to section 2.1, we can get a sequence of curves

$$\left\{ \mathbf{C}^l(t) = \sum_{i=0}^{p+n-1} \mathbf{P}_i^l N_i^p(t) \mid t \in [t_p, t_{p+n}],\ l = 0, 1, \ldots \right\},$$

where $\{N_i^p(t)\}_{i=0}^{p+n-1}$ are the p^{th}-degree nonuniform B-spline basis functions defined on the above knot vector (3.1),

$$\begin{aligned} &\left\{ \mathbf{P}_0^l = \mathbf{P}_1^l = \cdots = \mathbf{P}_{\lceil p-1 \rceil - 1}^l = \mathbf{P}_0 \mid l = 0, 1, \ldots \right\}, \\ &\left\{ \mathbf{P}_{p+n-1}^l = \mathbf{P}_{p+n-2}^l = \cdots = \mathbf{P}_{p+n-\lfloor p-1 \rfloor}^l = \mathbf{P}_n \mid l = 0, 1, \ldots \right\}, \quad \text{and} \\ &\left\{ \mathbf{P}_{\lceil p-1 \rceil + i}^0 = \mathbf{P}_i \mid i = 0, 1, \ldots, n \right\}. \end{aligned} \tag{3.2}$$

Due to the fact that the nonuniform B-spline basis, $\{N_i^p(t)\}_{i=0}^{p+n-1}$, is totally positive [9,10] and its collocation matrix at the knot vector $\{t_p, t_{p+1}, \ldots, t_{p+n}\}$ is obviously nonsingular from the local support property of the B-spline basis, the curve sequence converges to the B-spline curve interpolating the given points, that is, the initial curve has the *progressive iteration approximation* property.

For surfaces, given an ordered point set

$$\{\mathbf{P}_{ij} \in \mathrm{R}^3\}_{i=0}^{m}{}_{j=0}^{n},$$

we first define two knot vectors along the u-direction and the v-direction, i.e.,

$$0 = u_0 = \cdots = u_p < u_{p+1} < \cdots < u_{p+m} = u_{p+m+1} = \cdots = u_{p+2m}, \tag{3.3}$$

$$0 = v_0 = \cdots = v_q < v_{q+1} < \cdots < v_{q+n} = v_{q+n+1} = \cdots = v_{2q+n}. \tag{3.4}$$

Then, we assign the parameter vector (u_{p+i}, v_{q+j}) to the $(i,j)^{\text{th}}$ point $\mathbf{P}_{ij}$ ($i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, n$). Similar to Section 2.2, a sequence of surfaces can be generated as follows,

$$\left\{ \mathbf{S}^l(u,v) = \sum_{i=0}^{m+p-1} \sum_{j=0}^{n+q-1} \mathbf{P}_{ij}^l N_i^p(u) N_j^q(v) \,|\, u \in [u_p, u_{p+m}],\ v \in [v_q, v_{q+n}],\ l = 0, 1, \ldots \right\}, \tag{3.5}$$

where

$$\{N_i^p(u)\}_{i=0}^{m+p-1}$$

and

$$\{N_j^q(v)\}_{j=0}^{n+q-1}$$

are the p^{th}-degree and the q^{th}-degree nonuniform B-spline basis functions defined on the above two knot vectors (3.3) and (3.4), and

$$\begin{aligned} &\left\{ \mathbf{P}_{ij}^l = \mathbf{P}_{\lceil p-1 \rceil, \lceil q-1 \rceil} \right\}_{i=0,\ j=0}^{\lceil p-1 \rceil\ \lceil q-1 \rceil}, \\ &\left\{ \mathbf{P}_{ij}^l = \mathbf{P}_{\lceil p-1 \rceil + m, \lceil q-1 \rceil} \right\}_{i=\lceil p-1 \rceil + m,\ j=0}^{p+m-1\ \ \lceil q-1 \rceil}, \\ &\left\{ \mathbf{P}_{ij}^l = \mathbf{P}_{\lceil p-1 \rceil, \lceil q-1 \rceil + n} \right\}_{i=0,\ j=\lceil q-1 \rceil + n}^{\lceil p-1 \rceil\ q+n-1}, \end{aligned} \tag{3.6}$$

$$\begin{aligned}
\left\{\mathbf{P}^l_{ij} = \mathbf{P}_{\lceil p-1\rceil+m,\lceil q-1\rceil+n}\right\}_{i=\lceil p-1\rceil+m,\ j=\lceil q-1\rceil+n}^{p+m-1 \qquad q+n-1};\\
\left\{\mathbf{P}^l_{i,\lceil q-1\rceil+j} = \mathbf{P}_{0j}\right\}_{i=0,\quad j=0}^{\lceil p-1\rceil \ n},\\
\left\{\mathbf{P}^l_{i,j} = \mathbf{P}_{i0}\right\}_{i=\lceil p+1\rceil+m,\, j=0}^{p+m-1 \qquad \lceil q-1\rceil},\\
\left\{\mathbf{P}^l_{i,\lceil q-1\rceil+j} = \mathbf{P}_{m,j}\right\}_{i=\lceil p-1\rceil+m,\, j=0}^{p+m-1 \qquad n},\\
\left\{\mathbf{P}^l_{\lceil p-1\rceil+i,j} = \mathbf{P}_{i,n}\right\}_{i=0,\ j=\lceil q-1\rceil+n}^{m \quad q+n-1};\\
\left\{\mathbf{P}^0_{\lceil p+1\rceil+i,\lceil q+1\rceil+j} = \mathbf{P}_{ij}\right\}_{i=0,\, j=0}^{m \quad n}.
\end{aligned} \tag{3.6 (cont.)}$$

Here, $\lceil p\rceil$ denotes the least integer not less than the integer p, and $\lfloor p\rfloor$ the biggest integer not greater than p. Again, the initial surface $\mathbf{S}^0(u,v)$ has the *progressive iteration approximation* property.

On the other hand, since the NURBS basis is also totally positive, and its collocation matrix at a knot vector is obviously nonsingular from the local support of the NURBS basis, similar to the B-spline case, the first curve of the curve sequence,

$$\left\{ \mathbf{C}^l(t) = \sum_{i=0}^{p+n-1} \frac{\mathbf{P}^l_i w_i N^p_i(t)}{\sum\limits_{j=0}^{n} w_j N^p_j(t)} \;\middle|\; t \in [t_0, t_n],\ l = 0, 1, \dots \right\}, \tag{3.7}$$

where

$$\left\{ \frac{w_i N^p_i(t)}{\sum\limits_{j=0}^{n} w_j N^p_j(t)} \right\}_{i=0}^{p+n-1}$$

is the p^{th}-degree NURBS basis defined on the knot vector (3.1), and

$$\left\{\mathbf{P}^l_i\right\}_{i=0}^{p+n-1}$$

is defined as (3.2), has the *progressive iteration approximation* property. Furthermore, the first surface of the surface sequence,

$$\left\{ \mathbf{S}^l(u,v) = \sum_{i=0}^{m+p-1} \sum_{j=0}^{n+q-1} \mathbf{P}^l_{ij} \frac{w_i N^p_i(u)}{\sum\limits_{k=0}^{m+p-1} w_k N^p_k(u)} \frac{w_j N^q_j(v)}{\sum\limits_{k=0}^{n+q-1} w_k N^q_k(v)} \;\middle|\; u \in [u_p, u_{p+m}],\ v \in [v_q, v_{q+n}],\ l = 0, 1, \dots \right\}, \tag{3.8}$$

where

$$\left\{ \frac{w_i N^p_i(u)}{\sum\limits_{l=0}^{m+p-1} w_l N^p_l(u)} \right\}_{i=0}^{m+p-1}$$

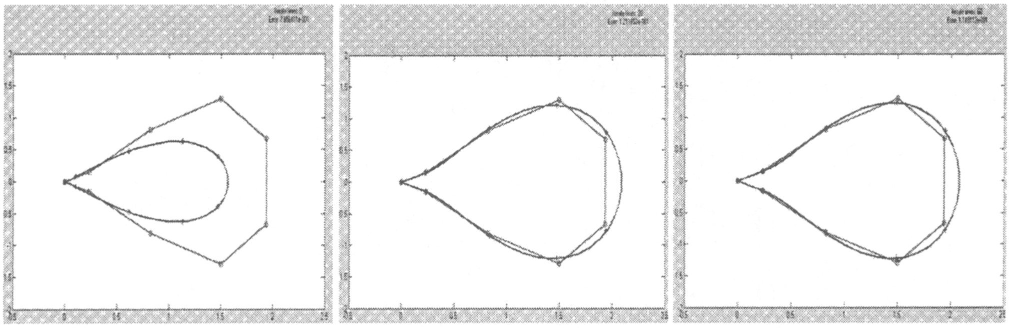

Figure 2. Fitting a Piriform curve with a Bézier curve. Left: the iteration level is 0, the error is 7.6564e − 001, middle: the iteration level is 20, the error is 1.2117e − 001, right: the iteration level is 60, the error is 1 1491e − 001

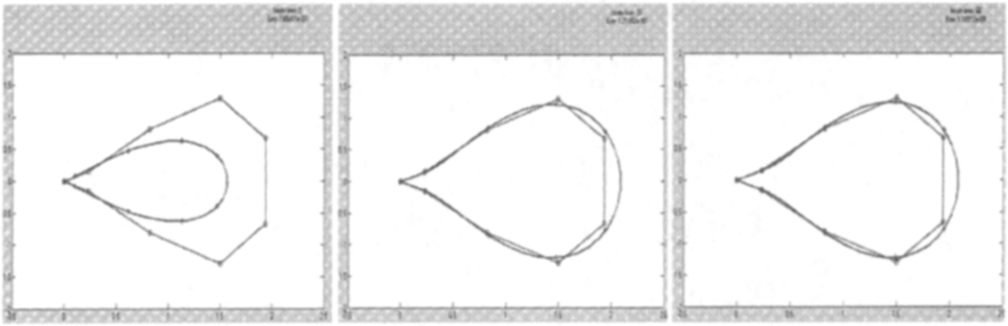

Figure 3. Fitting a Piriform curve with a 3rd-degree B-spline curve. Left: the iteration level is 0, the error is 2.6908e − 001; middle: the iteration level is 20, the error is 8.3898e − 005; right: the iteration level is 60, the error is 1.1618e − 011

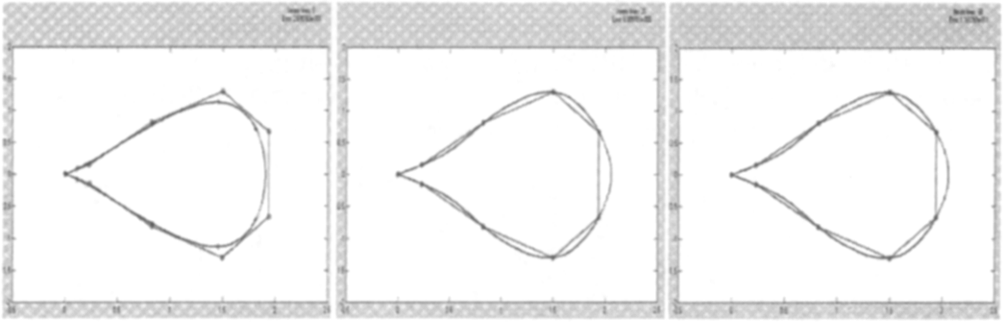

Figure 4. Fitting a Piriform curve with a 3rd-degree NURBS curve. Left: the iteration level is 0, the error is 1.7006e − 001; middle: the iteration level is 20, the error is 1 8778e − 005, right: the iteration level is 60, the error is 8 1035e − 013

and

$$\left\{ \frac{w_j N_j^k(v)}{\sum\limits_{l=0}^{n+q-1} w_l N_l^k(v)} \right\}_{j=0}^{n+q-1}$$

are NURBS bases defined on the knot vectors (3.3) and (3.4), respectively, and

$$\left\{ \mathbf{P}_{ij}^{l} \right\}_{i=0,\ \ j=0}^{m+p-1\ \ n+q-1}$$

is defined as (3.6), also has the *progressive iteration approximation* property.

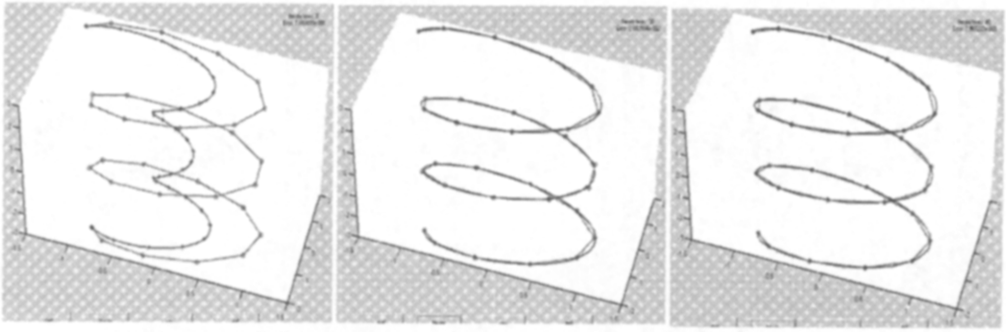

Figure 5 Fitting a helix with a Bézier curve. Left: the iteration level is 0, the error is 7.8931e − 001, middle the iteration level is 20, the error is 2.8426e − 002; right: the iteration level is 40, the error is 7 9672e − 003

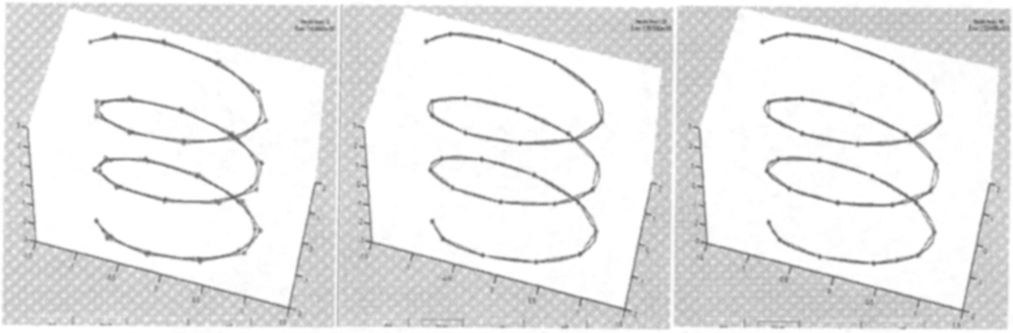

Figure 6. Fitting a helix with a 3rd-degree B-spline curve. Left: the iteration level is 0, the error is 1 0245e−001, middle the iteration level is 20, the error is 1.5577e−006; right the iteration level is 40, the error is 2 5245e − 010.

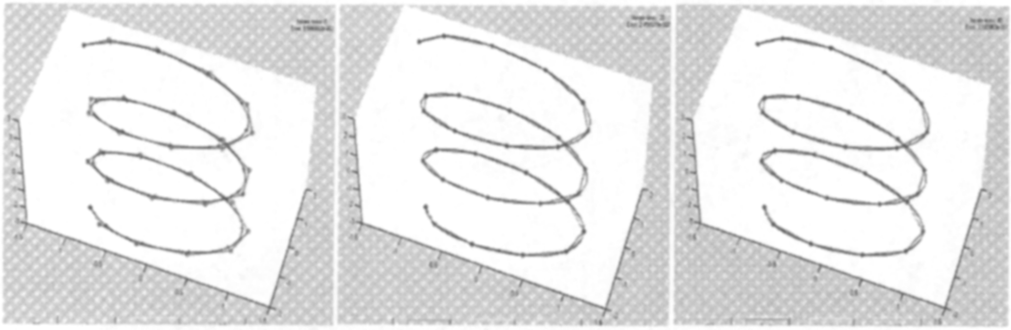

Figure 7 Fitting a helix with a 3rd-degree NURBS curve. Left the iteration level is 0, the error is 8 5859e−002, middle. the iteration level is 20, the error is 2 4585e−007; right the iteration level is 40, the error is 3 1921e − 011

4. RESULTS

In this section, we will illustrate the *progressive iteration approximation* property of Bézier, B-spline, and NURBS curves and surfaces, namely, the convergence of the corresponding curve and surface sequences. Specifically, in Figures 2–4, are curve sequences fitting the Piriform curve, i.e.,

$$x = a(1 + \cos\theta),$$
$$y = a\sin\theta(1 + \cos\theta),$$

generated by *progressive iteration approximation* of a Bézier curve, a 3rd-degree B-spline curve,

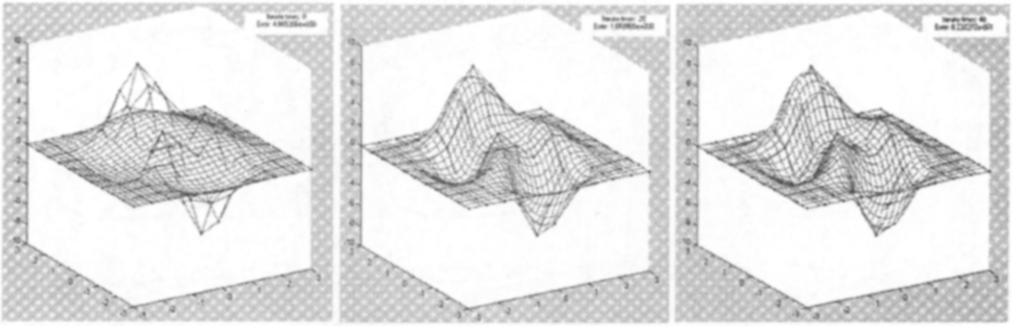

Figure 8. Fitting a Peaks function in MATLAB with a Bézier surface Left: the iteration level is 0, the error is 4.8653e + 000, middle: the iteration level is 20, the error is 1.6600e + 000, right: the iteration level is 40, the error is 8.2323e − 001.

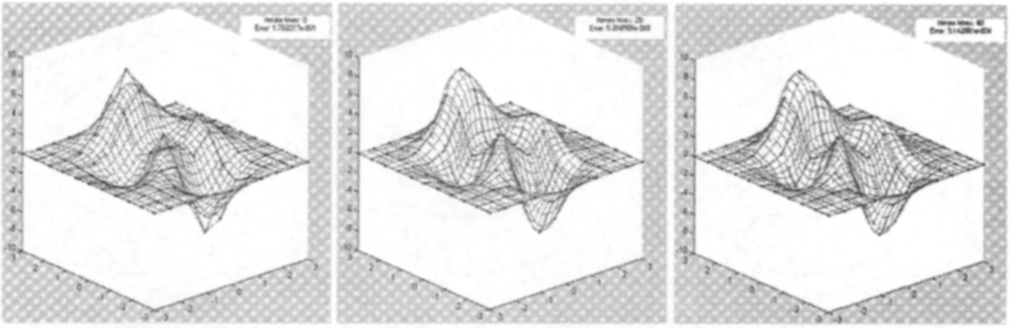

Figure 9. Fitting a Peaks function in MATLAB with a 3rd-degree B-spline surface. Left. the iteration level is 0, the error is 1.7822e − 001, middle the iteration level is 20, the error is 5.2065e−003, right the iteration level is 40, the error is 3.1430e−004

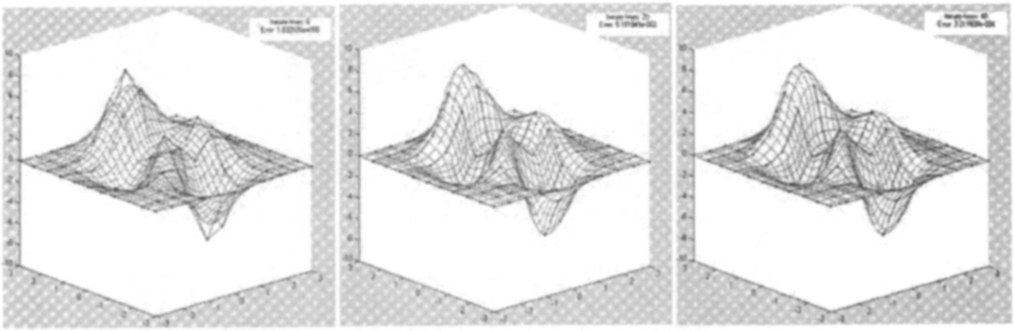

Figure 10 Fitting a Peaks function in MATLAB with a 3rd-degree NURBS surface. Left: the iteration level is 0, the error is 1.8328e + 000, middle: the iteration level is 20, the error is 5.1819e−003, right: the iteration level is 40, the error is 3.2116e−004.

and a 3rd-degree NURBS curve. In Figures 5–7, are curve sequences fitting a helix generated by *progressive iteration approximation* of a Bézier curve, a 3rd-degree B-spline curve, and a 3rd-degree NURBS curve. Finally, in Figures 8–10, are surface sequences fitting the Peaks function in MATLAB generated by *progressive iteration approximation* of a Bézier surface, a 3rd-degree B-spline surface, and a 3rd-degree NURBS surface. The weights $\{w_i\}_{i=0}^{n}$ of the NURBS curves in Figure 4 and Figure 7 are taken as $\{1, 2, \dots, (n+1)/2, (n+1)/2, \dots, 2, 1\}$ if $n+1$ is even, or $\{1, 2, \dots, (n+2)/2, \dots, 2, 1\}$ if $n+1$ is odd. Similarly, the weights, $\{w_{ij}\}_{i=0,}^{m}{}_{j=0}^{n}$, of the NURBS surface in Figure 10 are taken as

$$\{w_{i0}, w_{i1}, \dots, w_{in}\} = \left\{1, 2, \dots, \frac{n+1}{2}, \frac{n+1}{2}, \dots, 2, 1\right\},$$

if $n+1$ is even, or

$$\{w_{i0}, w_{i1}, \ldots, w_{in}\} = \left\{1, 2, \ldots, \frac{n+2}{2}, \ldots, 2, 1\right\},$$

if $n+1$ is odd, where $i = 0, 1, \ldots, m$. All of illustrations are programmed with MATLAB, and run on a PC with 2.8 GHz CPU and 512 MB Memory. In Table 1, we list the fitting errors of the curve (surface) sequences after specific iteration levels. The fitting error is taken as the maximum norm of the adjusting vectors defined in (2.11) and (2.17), that is,

$$\max\{\|\mathbf{\Delta}_{ij}\| \mid i = 0, 1, \ldots, m;\ j = 0, 1, \ldots, n\}$$

and

$$\max\{\|\mathbf{\Delta}_i\| \mid i = 0, 1, \ldots, n\}.$$

Table 1 Fitting errors of the curve (surface) sequences in Figures 2–10

Figures	0^{th} Level	10^{th} Level	20^{th} Level	30^{th} Level
Figure 2	7 6564e − 001	2 1043e − 001	1.2117e − 001	1.2002e − 001
Figure 3	2.6908e − 001	4.2569e − 003	8.3898e − 005	1 6284e − 006
Figure 4	1.7006e − 001	1.3073e − 003	1.8778e − 005	2 7050e − 007
Figure 5	7 8931e − 001	8 9408e − 002	2 8426e − 002	1 3637e − 002
Figure 6	1 0245e − 001	1 6375e − 004	1 5577e − 006	1.8920e − 008
Figure 7	8.5859e − 002	3.3348e − 005	2.4585e − 007	2.6284e − 009
Figure 8	4 8653e − 000	2 5202e − 000	1 6600e − 000	1 1453e − 000
Figure 9	1.7822e − 001	4 1888e − 002	5.2065e − 003	1.2696e − 003
Figure 10	1.8328e − 000	3 5677e − 002	5 1819e − 003	1 2834e − 003

Figures	40^{th} Level	50^{th} Level	60^{th} Level
Figure 2	1 1943e − 001	1 1844e − 001	1.1491e − 001
Figure 3	3.1410e − 008	6 0438e − 010	1.1618e − 011
Figure 4	3 8990e − 009	5 6206e − 011	8 1035e − 013
Figure 5	7 9672e − 003	5 2059e − 003	3 6700e − 003
Figure 6	2 5245e − 010	3 5481e − 012	5 2134e − 014
Figure 7	3.1921e − 011	4.1400e − 013	5 5511e − 015
Figure 8	8.2323e − 001	7.2588e − 001	7 1093e − 001
Figure 9	3 1430e − 004	8.2850e − 005	2 3009e − 005
Figure 10	3.2116e − 004	8.4641e − 005	2 3278e − 005

5. CONCLUSION

Given a blending basis and a set of ordered data points, a curve (tensor product surface) generated by blending the data points and the basis functions has the *progressive iteration approximation* property, as long as the given basis is totally positive and its collocation matrix at the corresponding parameter set is nonsingular That is, the curve (tensor product surface) sequence generated by adjusting the control points iteratively converges to the curve (tensor product surface) interpolating the given data points. Specifically, Bézier, B-spline, and NURBS curves (surfaces) have the *progressive iteration approximation* property. However, because different bases have different convergence rates, it needs to be studied in the future which bases have the fastest convergence rates

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