The PIA Property of Low Degree Non-uniform Triangular B-B Patches

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***Abstract*—Progressive-iterative approximation presents an intuitive way to generate a sequence of curves or patches, whose limit interpolates the given data points. It has been shown that the blending curves and tensor product blending patches with normalized totally positive basis have the progressive-iterative approximation property. In this paper, we prove that, the quadratic, cubic, and quartic non-uniform triangular Bernstein-Bézier patches also have the progressive-iterative approximation property. Since the most often empolyed in geometric design are the low degree curves or patches, especially the cubic curves and patches, the result shown in this paper has practical significance for geometric design.**

Keywords: Progressive-iterative approximation, convergence, data fitting, triangular Bernstein-Bézier patch, geometric design

I. Introduction

The *progressive-iterative approximation* (abbr. PIA) property of the uniform cubic B-spline curve, first discovered by Qi et. al. [1] and de Boor [2], respectively, generates a sequence of curves by adjusting the control points iteratively, and the limit curve interpolates the control points of the initial curve. In Ref. [3], the authors show that the non-uniform cubic B-spline curves and surfaces also hold the property. Furthermore, the result is extended to the blending curve and surface with normalized totally positive basis [4]. That is, any blending curve or surface with normalized totally positive basis has the progressive-iterative approximation property. In Ref. [5], the convergence rates of different bases are compared, and the basis with the fastest convergence rate is found. Moreover, it is proved that the rational B-spline curve and surface (NURBS) have the property, too [6]. Recently, Martin et al. [7] devise an iterative format for fitting, which is actually the progressive-iterative approximation (PIA) format for the uniform periodic cubic B-spline. Lu [8] devises a weighted PIA format to speed up the convergence of the PIA.

Moreover, the *local* progressive-iterative approximation property for the blending curve and surface with normalized totally positive basis is proved in Ref. [9]. By the local PIA property, we can adjust only a subset of the control points progressively, and the corresponding points on the limit curve still interpolate the corresponding subset of the initial data points. The local progressive-iterative approximation format brings more flexibility to data fitting.

Till now, it has been shown that the PIA property is held by the blending curve and tensor product blending surface with normalized totally positive basis, by Loop, Doo-Sabin, and Catmull-Clark subdivision surfaces, respectively. Moreover, besides these types of patches, *triangular Bernstein-Bézier* patch (abbr. B-B patch) is also widely employed in geometric design, especially in computer graphics [10], [11]. In this paper, we show that the PIA format for the quadratic, cubic, and quartic non-uniform triangular Bernstein-Bézier patch is convergent. Since the most often employed in geometric design are the low degree curve and patches, this result has practical significance for geometric design.

This paper is organized as follows. In Section II, we develop the progressive-iterative approximation (PIA) format for a non-uniform triangular Bernstein-Bézier patch, and show its convergence for quadratic, cubic non-uniform B-B patch. Its convergence for the quartic non-uniform B-B patch is presented in Section III. In Section IV, the local PIA format is developed, and its convergence is proven. Finally, Section V concludes the paper.

II. The PIA format for the non-uniform B-B patch and its convergence

Suppose we are given a data point set $\{\boldsymbol{T}_{ijk}, i+j+k = n, i, j, k \in \bar{\mathbb{Z}}\}$, where $\bar{\mathbb{Z}}$ is the nonnegative integer set. Taking them as the initial control points, an initial triangular Bernstein-Bézier patch (abbr. B-B patch) $\boldsymbol{T}^0(u,v,w)$ of degree n is generated, that is,

$$\begin{aligned}\boldsymbol{T}^0(u,v,w) = &\sum_{i+j+k=n} B^n_{ijk}(u,v,w)\boldsymbol{T}^0_{ijk},\\ &u,v,w \geq 0, u+v+w=1,\end{aligned} \tag{1}$$

where,

$$\begin{aligned}B^n_{ijk}(u,v,w) =&\frac{n!}{i!j!k!}u^i v^j w^k, i+j+k=n,\\ &u,v,w \geq 0, u+v+w=1,\end{aligned} \tag{2}$$

are the *generalized Bernstein polynomials* of degree n [12], and $\boldsymbol{T}^0_{ijk} = \boldsymbol{T}_{ijk}, i+j+k=n$ are the initial control points. By assigning the parameter (u_i, v_j, w_k) to each data point $\boldsymbol{T}_{ijk}$, a series of *difference vectors* can be produced as,

$$\boldsymbol{\Delta}^0_{ijk} = \boldsymbol{T}_{ijk} - \boldsymbol{T}^0(u_i,v_j,w_k), i+j+k=n. \tag{3}$$

Next, by adding the difference vectors (3) to the corresponding control points of the initial patch $\boldsymbol{T}^0(u,v,w)$ (1), we get the new control points, that is,

$$\boldsymbol{T}^1_{ijk} = \boldsymbol{T}^0_{ijk} + \boldsymbol{\Delta}^0_{ijk}, i+j+k=n. \tag{4}$$

978-0-7695-4497-7/11 $26.00 © 2011 IEEE
DOI 10.1109/CAD/Graphics.2011.73

Taking them as the new control points generates the new B-B patch $\boldsymbol{T}^1(u,v,w)$,

$$\begin{aligned}\boldsymbol{T}^1(u,v,w) = \sum_{i+j+k=n} B^n_{ijk}(u,v,w)\boldsymbol{T}^1_{ijk}, \\ u,v,w \geq 0, u+v+w=1.\end{aligned} \tag{5}$$

In this way, a B-B patch sequence $\{\boldsymbol{T}^l(u,v,w), l = 0,1,2,\cdots\}$ is constructed, namely,

$$\begin{aligned}\boldsymbol{T}^l(u,v,w) = \sum_{i+j+k=n} B^n_{ijk}(u,v,w)\boldsymbol{T}^l_{ijk}, u,v,w \geq 0, \\ u+v+w=1, l=0,1,\cdots\end{aligned} \tag{6}$$

where, $\boldsymbol{T}^l_{ijk} = \boldsymbol{T}^{l-1}_{ijk} + \boldsymbol{\Delta}^{l-1}_{ijk}$, and

$$\boldsymbol{\Delta}^{l-1}_{ijk} = \boldsymbol{T}_{ijk} - \boldsymbol{T}^{l-1}(u_i,v_j,w_k), \tag{7}$$

are the difference vectors.

The *progressive-iterative approximation* (abbr. PIA) property of the B-B patch means that the limit of the B-B patch sequence $\{\boldsymbol{T}^l(u,v,w), l = 0,1,2,\cdots\}$ interpolates the given data points $\boldsymbol{T}_{ijk}, i+j+k=n$, namely,

$$\lim_{l\to\infty} \boldsymbol{T}^l(u_i,v_j,w_k) = \boldsymbol{T}_{ijk}, i+j+k=n. \tag{8}$$

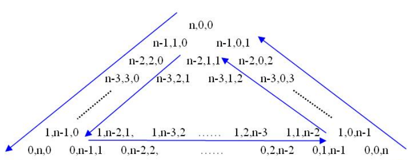

Fig. 1. The order of the control points of the B-B patch.

To show the convergence of the sequence $\boldsymbol{T}^l(u,v,w), l = 0,1,\cdots$ (8), the difference vectors in each iteration should be arranged into a one-dimensional sequence $\Delta^l, l = 0,1,\cdots$, in the order illustrated in Fig. II. In this order, the difference vectors whose subscripts are on the left boundary are first put in to the sequence Δ^l, followed by the bottom boundary, and finally the right boundary; this order is repeated in the inner rings. More clearly, in the one dimensional sequence Δ^l, $\boldsymbol{\Delta}^l_{n00}$ is the first element, $\cdots$, $\boldsymbol{\Delta}^l_{0n0}$ is the $(n+1)^{th}$ element, $\cdots$, $\boldsymbol{\Delta}^l_{00n}$ is the $(2n+1)^{th}$ element, $\cdots$, and $\boldsymbol{\Delta}^l_{n-1,0,1}$ is the $3n^{th}$ element; next, in the nearest inner ring, $\boldsymbol{\Delta}^l_{n-2,1,1}$ is the $(3n+1)^{th}$ element, $\cdots$, $\boldsymbol{\Delta}^l_{1,n-2,1}$ is the $(4n-2)^{th}$ element, $\cdots$ and $\boldsymbol{\Delta}^l_{1,1,n-2}$ is the $(5n-6)^{th}$ element; $\cdots$, and so on. In conclusion, the one-dimensional sequence Δ^l can be written as,

$$\begin{aligned}\Delta^l =& [\boldsymbol{\Delta}^l_{n00}, \boldsymbol{\Delta}^l_{n-1,1,0}, \cdots, \boldsymbol{\Delta}^l_{0n0}, \cdots, \boldsymbol{\Delta}^l_{00n}, \cdots, \\ & \boldsymbol{\Delta}^l_{n-1,0,1}, \cdots, \boldsymbol{\Delta}^l_{IJK}]^T,\end{aligned} \tag{9}$$

where, $l = 0,1,\cdots$. There are $\frac{(n+1)(n+2)}{2}$ elements in Δ^l, and the subscript of the last difference vector $\boldsymbol{\Delta}^l_{IJK}$ is determined by the degree of the B-B patch.

Therefore, the iterative format of the PIA for the degree n triangular patch is,

$$\Delta^{l+1} = D\Delta^l = (I-C)\Delta^l, l = 0,1,\cdots, \tag{10}$$

where, I is the identity matrix, and C can be written in block, namely,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{43} & C_{44} & C_{45} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ C_{61} & 0 & 0 & 0 & C_{65} & C_{66} & 0 \\ C_{71} & C_{72} & C_{73} & C_{74} & C_{75} & C_{76} & C_{77} \end{bmatrix}. \tag{11}$$

In the above matrix C (11), C_{22}, C_{44}, C_{66} are matrices of rank $(n-1)\times(n-1)$, and C_{77} are the matrix of rank $\frac{(n-1)(n-2)}{2} \times \frac{(n-1)(n-2)}{2}$. Then, we can present a sufficient and necessary condition for the convergence of the iterative format (10).

Theorem 1: Suppose $u_1 < u_2 < \cdots < u_{n-1}, v_1 < v_2 < \cdots < v_{n-1}$, and $w_1 < w_2 < \cdots < w_{n-1}$. The iterative format (10) of PIA is convergent for a degree n B-B patch, if and only if the matrices C_{22}, C_{44}, and C_{66} (11) are nonsingular, and the spectral radius of $I - C_{77}$ is less than 1, namely, $\rho(I - C_{77}) < 1$.

Proof: Based on Eq. (11), the characteristic polynomial of the matrix C is,

$$\begin{aligned}\det(\lambda I - C) =& (\lambda-1)^3 \det(\lambda I - C_{22}) \det(\lambda I - C_{44}) \\ & \times \det(\lambda I - C_{66}) \det(\lambda I - C_{77}),\end{aligned} \tag{12}$$

so, the eigenvalues of the matrix C includes triple $1s$, and others are determined by the matrices C_{22}, C_{44}, C_{66}, and C_{77}. Accordingly, the eigenvalues of the iterative matrix $D = I - C$ (Eq. (10)) have triple $0s$, and others are determined by the matrices $I - C_{22}, I - C_{44}$, and $I - C_{77}$.

Necessity: The necessity is evident. If the iterative format (10) is convergent, the spectral radius of the iterative matrix $D = I - C$ should satisfy $\rho(D) = \rho(I - C) < 1$. This means the matrices C_{22}, C_{44}, and C_{66} should be nonsingular, and $\rho(I - C_{77}) < 1$.

Sufficiency: Note that, the matrices C_{22}, C_{44}, and C_{66} are actually the collocation matrices of the univariate Bernstein polynomials, that is,

$$\begin{aligned} C_{22} &= \begin{pmatrix} B^n_1(v) & B^n_2(v) & \cdots & B^n_{n-1}(v) \\ v_1 & v_2 & \cdots & v_{n-1} \end{pmatrix}, \\ C_{44} &= \begin{pmatrix} B^n_1(w) & B^n_2(w) & \cdots & B^n_{n-1}(w) \\ w_1 & w_2 & \cdots & w_{n-1} \end{pmatrix}, \\ C_{66} &= \begin{pmatrix} B^n_1(u) & B^n_2(u) & \cdots & B^n_{n-1}(u) \\ u_1 & u_2 & \cdots & u_{n-1} \end{pmatrix}, \end{aligned} \tag{13}$$

where, $B^n_i(t) = \binom{n}{i} t^i (1-t)^{n-i}, t = u,v,w, i = 1,\cdots,n-1$ are the univariate Bernstein polynomials, and

$$\begin{pmatrix} B^n_1(t) & B^n_2(t) & \cdots & B^n_{n-1}(t) \\ t_1 & t_2 & \cdots & t_{n-1} \end{pmatrix} = \begin{bmatrix} B^n_1(t_1) & B^n_2(t_1) & \cdots & B^n_{n-1}(t_1) \\ B^n_1(t_2) & B^n_2(t_2) & \cdots & B^n_{n-1}(t_2) \\ \cdots & \cdots & \cdots & \cdots \\ B^n_1(t_{n-1}) & B^n_2(t_{n-1}) & \cdots & B^n_{n-1}(t_{n-1}) \end{bmatrix}.$$

Thus, if $u_1 < u_2 < \cdots < u_{n-1}, v_1 < v_2 < \cdots < v_{n-1}$, and $w_1 < w_2 < \cdots < w_{n-1}$, the matrices C_{22}, C_{44} and C_{66} (13) are all totally positive. Therefore, together with the ∞-norms of them are all less than 1, the eigenvalues of the three matrices are all real numbers and in the interval $(0, 1)$, since they are nonsingular. Consequently, the eigenvalues of $I - C_{22}, I - C_{44}$ and $I - C_{66}$ are all real and in $(0, 1)$, too.

Moreover, together with the spectral radius of the matrix C_{77} satisfies $\rho(I - C_{77}) < 1$, the spectral radius of the iterative matrix $D = I - C$ (10) fulfills $\rho(D) = \rho(I - C) < 1$. So, the PIA format (10) is convergent. That is,

$$\lim_{l \to \infty} \boldsymbol{\Delta}_{ijk}^l = 0,$$

equivalently,

$$\lim_{l \to \infty} \boldsymbol{T}^l(u_i, v_j, w_k) = \boldsymbol{T}_{ijk}, i + j + k = n. \square$$

The next theorem shows the convergence of the iterative format (10) for the quadratic and cubic B-B patches.

Theorem 2: Suppose $u_1 < u_2 < \cdots < u_{n-1}, v_1 < v_2 < \cdots < v_{n-1}$, and $w_1 < w_2 < \cdots < w_{n-1}$. The iterative format (10) is convergent for the quadratic and cubic B-B patch, if the matrices C_{22}, C_{44}, and C_{66} (Eq. (11)) are nonsingular.

Proof: For the quadratic B-B patch, the bottom row and the right-most column are lost from the matrix C (11), so the matrix C has triple $1s$ as its eigenvalues and the others are determined by C_{22}, C_{44} and C_{66} (see Eq. (12)), which are all real and in $(0, 1)$. Therefore, the eigenvalues of the iterative matrix D in Eq. (10) are all real and in $(0, 1)$. It means that the iterative format (10) is convergent to 0, namely, $\lim_{l \to \infty} \boldsymbol{T}^l(u_i, v_j, w_k) = \boldsymbol{T}_{ijk}, i + j + k = 2$.

On the other hand, for the cubic B-B patch, the sub-matrix C_{77} of the matrix C (11) is an single element matrix $C_{77} = [B_{111}^3(u_1, v_1, w_1)]$. Since $0 < B_{111}^3(u_1, v_1, w_1) < 1$, the eigenvalues of the matrix C (11) are all real and in $(0, 1)$. Thus, the eigenvalues of the iterative matrix D in the iterative format (10) are all real and in $(0, 1)$. And then, the iterative format (10) is convergent, namely, $\lim_{l \to \infty} \boldsymbol{T}^l(u_i, v_j, w_k) = \boldsymbol{T}_{ijk}, i + j + k = 3$. $\square$

III. The Convergence of the PIA Format for the Quartic Non-Uniform B-B Patch

Comparing to the cases of quadratic and cubic non-uniform B-B patches, the proof to the convergence of the PIA format for the quartic B-B patch is a bit complicated. In fact, there are three inner control points in the control net of the quartic B-B patch (See Fig. II), that is, $\boldsymbol{T}_{211}, \boldsymbol{T}_{121}$, and $\boldsymbol{T}_{112}$. To show the convergence of the PIA format for the quartic B-B patch, we need to check the eigenvalues of C_{77} (11), that is,

$$\begin{aligned} C_{77} &= \begin{bmatrix} B_{211}^4(u_2, v_1, w_1) & B_{121}^4(u_2, v_1, w_1) & B_{112}^4(u_2, v_1, w_1) \\ B_{211}^4(u_1, v_2, w_1) & B_{121}^4(u_1, v_2, w_1) & B_{112}^4(u_1, v_2, w_1) \\ B_{211}^4(u_1, v_1, w_2) & B_{121}^4(u_1, v_1, w_2) & B_{112}^4(u_1, v_1, w_2) \end{bmatrix} \\ &= 12 \begin{bmatrix} u_2^2 v_1 w_1 & u_2 v_1^2 w_1 & u_2 v_1 w_1^2 \\ u_1^2 v_2 w_1 & u_1 v_2^2 w_1 & u_1 v_2 w_1^2 \\ u_1^2 v_1 w_2 & u_1 v_1^2 w_2 & u_1 v_1 w_2^2 \end{bmatrix} = \begin{bmatrix} \alpha u_2 & \alpha v_1 & \alpha w_1 \\ \beta u_1 & \beta v_2 & \beta w_1 \\ \gamma u_1 & \gamma v_1 & \gamma w_2 \end{bmatrix} \end{aligned} \quad (14)$$

where, $\alpha = 12 u_2 v_1 w_1, \beta = 12 u_1 v_2 w_1$, and $\gamma = 12 u_1 v_1 w_2$.

It should be pointed out that, the parameters $u_i, v_i, w_i, i = 1, 2$ satisfy $u_2 > u_1, v_2 > v_1, w_2 > w_1$. Moreover, since $u_2 + v_1 + w_1 = u_1 + v_2 + w_1 = u_1 + v_1 + w_2 = 1$, we have,

$$u_2 - u_1 = v_2 - v_1 = w_2 - w_1 \triangleq h. \quad (15)$$

Evidently, the characteristic polynomial of the matrix C_{77} (14) is,

$$det(\lambda I - C_{77}) = \lambda^3 + a\lambda^2 + b\lambda + c = 0, \quad (16)$$

where,

$$\begin{aligned} a &= -(\alpha u_2 + \beta v_2 + \gamma v_2), \\ b &= \alpha\beta(u_2 v_2 - u_1 v_1) + \alpha\gamma(u_2 w_2 - u_1 w_1) + \beta\gamma(v_2 w_2 - v_1 w_1), \\ c &= -\alpha\beta\gamma h^2. \end{aligned} \quad (17)$$

To determine the roots of the characteristic polynomial (16), we need a lemma.

Lemma 1: If the roots of a polynomial with real coefficients are all real, the number of the positive roots (multiple roots count as its multiplicity) is equal to the sign changing number of its coefficient sequence [13].

Specifically, to compute the *sign changing number* of a sequence $\{a_1, a_2, \cdots, a_n\}$, zero elements should be deleted from the sequence, denoted as $\{b_1, b_2, \cdots, b_m\}$, and the sign changing number is defined as the number of negatives in the set $\{b_i b_{i+1} | 1 \le i \le m - 1\}$.

Therefore, since $a < 0, b > 0, c < 0$ (17), the sign changing number of the coefficients sequence $\{1, a, b, c\}$ of the polynomial (16) is 3. Thus, we have,

Corollary 1: If the three roots of the cubic polynomial (16) are all real, they are all positive.

Clearly, to prove the convergence of the PIA for the quartic B-B patch, we need to show that the spectral radius of the matrix $I - C_{77}$ (14) satisfies $\rho(I - C_{77}) < 1$. To this purpose, we require the lemmas as follows.

Lemma 2: Let $A = [a_{ij}] \in R^{n \times n}$ be a **nonnegative matrix**, that is, $a_{ij} \ge 0, (1 \le i, j \le n)$. Its spectral radius $\rho(A)$ is one of its eigenvalues [14].

Lemma 3: Let $A = [a_{ij}] \in R^{n \times n}$ is a nonnegative matrix. Then,

$$\min_{1 \le i \le n} \sum_{j=1}^n a_{ij} \le \rho(A) \le \max_{1 \le i \le n} \sum_{j=1}^n a_{ij}, \quad (18)$$

where, $\rho(A)$ denotes the spectral radius of the matrix A [14].

Now, we can show the key theorem on the spectral radius of the matrix $I - C_{77}$ (14).

Theorem 3: The spectral radius of the matrix $I - C_{77}$ (14) is less than 1, that is, $\rho(I - C_{77}) < 1$.

Proof. Suppose the eigenvalues of the matrix C_{77} (14) are λ_1, λ_2, and λ_3, respectively, which are the roots of the polynomial (16). Then, the eigenvalues of $I - C_{77}$ are, $1 - \lambda_i, i = 1, 2, 3$, respectively. According to Lemma 2, one of them, suppose λ_1, is the spectral radius of C_{77}, namely, $\lambda_1 = \rho(C_{77}) < \|C_{77}\|_\infty < 1$.

If the three eigenvalues $\lambda_i, i = 1, 2, 3$ are all real, according to Corollary 1, they are all positive. Moreover, they satisfy $0 <$

$\lambda_i < 1, i = 1, 2, 3$. Equivalently, $0 < 1 - \lambda_i < 1, i = 1, 2, 3$, and then, $\rho(I - C_{77}) < 1$.

On the other hand, if the characteristic polynomial (16) of the matrix C_{77} (14) has complex roots λ_2 and λ_3, then, they must be conjugate complex numbers, as well as $1 - \lambda_2$ and $1 - \lambda_3$. Thus,

$$|1-\lambda_2|^2 = |1-\lambda_3|^2 = (1-\lambda_2)(1-\lambda_3) = 1-(\lambda_2+\lambda_3)+\lambda_2\lambda_3.$$

According to Lemma 3, we have $\lambda_1 = \rho(C_{77}) > \min\{\alpha, \beta, \gamma\}$ (Eq. (14)). Without loss of generality, let $\gamma = \min\{\alpha, \beta, \gamma\}$.

Moreover, denoting $h = u_2 - u_1 = v_2 - v_1 = w_2 - w_1$ (15), we have,

$$\begin{aligned}\alpha\beta(u_2v_2 - u_1v_1) &= \alpha\beta((u_1 + h)(v_1 + h) - u_1v_1) \\ &= \alpha\beta((u_1 + v_1)h + h^2) > \alpha\beta h^2.\end{aligned}$$

Similarly, $\alpha\gamma(u_2w_2 - u_1w_1) > \alpha\gamma h^2$ and $\beta\gamma(v_2w_2 - v_1w_1) > \beta\gamma h^2$.

Therefore, together with the relation between the roots and the coefficients of the polynomial (16), and the fact $0 < \alpha, \beta, \gamma < 1$, we have (refer to (17)),

$$\begin{aligned}&\lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3 = |c| + \frac{|c|}{\rho(C_{77})} \leq |c| + \frac{|c|}{\gamma} = \alpha\beta\gamma h^2 + \alpha\beta h^2 \\ &< \alpha\beta h^2 + \alpha\gamma h^2 + \beta\gamma h^2 \\ &< \alpha\beta(u_2v_2 - u_1v_1) + \alpha\gamma(u_2w_2 - u_1w_1) + \beta\gamma(v_2w_2 - v_1w_1) \\ &= b = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3.\end{aligned}$$

Thus, $\lambda_2\lambda_3 < \lambda_2 + \lambda_3$, and $|1 - \lambda_2| = |1 - \lambda_3| < 1$. Together with $|1 - \lambda_1| = |1 - \rho(C_{77})| < 1$, we have $\rho(I - C_{77}) < 1$. □

Based on Theorems 1 and 3, follows,

Theorem 4: Suppose $u_1 < u_2 < u_3, v_1 < v_2 < v_3$, and $w_1 < w_2 < w_3$. The iterative format (10) of PIA is convergent for the quartic non-uniform B-B patch, if the matrices C_{22}, C_{44}, and C_{66} (11) are nonsingular.

Till now, we have shown that the PIA format (10) for the quadratic, cubic, and quartic non-uniform B-B patches is convergent. However, the convergence of the PIA format for higher degree B-B patch is not clear, because the eigen-structure of the matrix C_{77} (11) with higher rank is hard to analyze.

IV. Local PIA format for the Non-uniform Triangular B-B patches

The PIA format for a triangular B-B patch aforementioned is *global*, that is, all control points of the B-B patch are adjusted. Whereas, the *local* PIA format adjusts only a subset of the control points of a B-B patch, while other control points remain unchanged in the iterations.

Arranging the adjusted control points and the corresponding difference vectors in the order illustrated in Fig. II, we get the local iterative format,

$$\bar{\Delta}^{l+1} = \bar{D}\bar{\Delta}^l = (I - \bar{C})\bar{\Delta}^l, l = 0, 1, \cdots. \quad (19)$$

Note that the matrix $\bar{C}$ in (19) is a principal sub-matrix of C (11), containing the principal sub-matrices of C_{22}, C_{44}, C_{66} and C_{77}, denoted as $\bar{C}_{22}, \bar{C}_{44}, \bar{C}_{66}$, and $\bar{C}_{77}$, respectively.

Similar to the matrix C (11), the eigenvalues of its principal sub-matrix $\bar{C}$ are determined by the matrices $\bar{C}_{22}, \bar{C}_{44}, \bar{C}_{66}$, and $\bar{C}_{77}$. Because $\bar{C}_{22}, \bar{C}_{44}$, and $\bar{C}_{66}$ are sub-matrices of the stochastic and totally positive matrices, if they are nonsingular, their eigenvalues are all real numbers and in the interval $(0, 1)$. And then, the eigenvalues of the matrices $I - \bar{C}_{22}, I - \bar{C}_{44}$, and $I - \bar{C}_{66}$ are also real and in the interval $(0, 1)$. So, the convergence of the local PIA format depends on the eigenvalues of the matrix $\bar{C}_{77}$, which is the sub-matrix of the matrix C_{77} (11).

If the B-B patch is quadratic or cubic, the matrix C_{77} is either an empty matrix or a single-element matrix. Evidently, the spectral radius of the principal sub-matrix $\rho(I - \bar{C}_{77}) < 1$.

On the other hand, if the B-B patch is quartic, the matrix C_{77} is 3×3. First, since all of the elements of the matrix C_{77} (11) are greater than 0, and less than 1, the eigenvalues of its 1×1 rank principal sub-matrices satisfy $0 < \lambda(\bar{C}_{77}) < 1$; second, it is easy to show that, the eigenvalues of its 2×2 rank principal sub-matrices are all real and in the interval $(0, 1)$; finally, as shown in Theorem 3, $\rho(I - C_{77}) < 1$. It means that, for all principal sub-matrices $\bar{C}_{77}$ of the matrix C_{77}, $\rho(I - \bar{C}_{77}) < 1$.

In conclusion, the local PIA format for the quadratic, cubic, and quartic non-uniform B-B patches is convergent, if the matrices $\bar{C}_{22}, \bar{C}_{44}$, and $\bar{C}_{66}$ are nonsingular.

V. Conclusion

In this paper, we show that the quadratic, cubic, and quartic non-uniform triangular Bernstein-Bézier patches have the progressive-iterative approximation property. That is, by adjusting the control points of a B-B patch progressively, a sequence of B-B patches is generated, and the limit patch interpolates the initial control points. since the most often employed in geometric design are the low degree curves and patches, especially the cubic curves and patches, this result has practical significance for geometric design.

Acknowledgment

This paper is supported by Natural Science Foundation of China (Nos. 60970150, 60933008), and Zhejiang Provincial Natural Science Foundation of China (No. Y1090416).

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