



# On the derivative formula of a rational Bézier curve at a corner

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## ARTICLE INFO

### Keywords:

Arbitrary order derivative  
Derivative formula  
Rational Bézier curve

## ABSTRACT

In this paper, we develop the exact arbitrary order derivative formula of a rational Bézier curve at its corner. Some examples are presented to demonstrate the computation of the exact derivatives.

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## 1. Introduction

Derivative formula of a parametric curve has important applications in geometric design and applied mathematics. For example, to stitch two Bézier curves smoothly, we need the derivative formulae at their merged corners as constraints to configure the control points at either side of the corner.

Although computing the derivatives of a polynomial Bézier curve is straightforward, the derivative formula of a rational Bézier curve is very complicated and difficult to represent. To demonstrate its complexity, we develop the third order derivative at the corner  $t = 0$  of the rational curve (1) (refer to Section 2), that is,

$$\begin{aligned} \mathbf{R}^{(3)}(0) = & n(n-1)(n-2) \frac{\omega_3}{\omega_0} (\mathbf{P}_3 - \mathbf{P}_0) - \left( 3n(n-1)(n-2) \frac{\omega_2}{\omega_0} + 3n^2(n-1) \frac{(\omega_1 - \omega_0)\omega_2}{\omega_0^2} \right) (\mathbf{P}_2 - \mathbf{P}_0) \\ & + \left( 3n(n-1)(n-2) \frac{\omega_1}{\omega_0} + 6n^2(n-1) \frac{(\omega_1 - \omega_0)\omega_1}{\omega_0^2} + 6n^3 \frac{(\omega_1 - \omega_0)^2 \omega_1}{\omega_0^3} \right. \\ & \left. - 3n^2(n-1) \frac{(\omega_2 - 2\omega_1 + \omega_0)\omega_1}{\omega_0^2} \right) (\mathbf{P}_1 - \mathbf{P}_0). \end{aligned}$$

With the increasing of the derivative order, its formula is expanded rapidly in an uncontrollable manner. Thus, there is a little work on the exact high order derivative formula of a rational Bézier curve, while related work is mainly on how to approximate the exact derivatives [1–6].

In this paper, we present an effective method to develop the derivative formula of a rational curve at a corner in a well-regulated manner. So, the high order derivative formula can be computed easily.

This paper is organized as follows. In Section 2, we show the main result of this paper. Next, some examples are presented to demonstrate the computation of the derivative formula in Section 3. Section 4 concludes the paper.

## 2. Main result

Given a rational Bézier curve of degree  $n$ ,

$$\mathbf{R}(t) = \frac{\mathbf{P}(t)}{\omega(t)} = \frac{\sum_{i=0}^n \omega_i \mathbf{P}_i B_i^n(t)}{\sum_{i=0}^n \omega_i B_i^n(t)}, \quad t \in [0, 1], \quad (1)$$

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where  $\omega_0 \neq 0$ , and  $B_i^n(t) = \binom{n}{i} (1-t)^{n-i} t^i$  is the Bernstein polynomial, its exact arbitrary order derivative formula is presented in the following theorem.

**Theorem 1.** The  $k$ th derivative  $\mathbf{R}^{(k)}(0)$  of curve (1) at the corner  $t = 0$  is,

$$\mathbf{R}^{(k)}(0) = \sum_{i=1}^k C_i^k (\mathbf{P}_i - \mathbf{P}_0). \tag{2}$$

Here,

$$C_i^k = \begin{bmatrix} k \\ i \end{bmatrix} + I_i^{k-1} + I_i^{k-2} + \dots + I_i^1, \quad i = 1, 2, \dots, k, \tag{3}$$

$I_i^j = 0$ , when  $i > j$ ,  $j = 1, 2, \dots, k-1$ , and while  $i \leq j$ ,

$$I_i^j = \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \left( \begin{bmatrix} j \\ i \end{bmatrix} + \sum_{i \leq l_1 < j} \left\{ \begin{matrix} j \\ l_1 \end{matrix} \right\} \begin{bmatrix} l_1 \\ i \end{bmatrix} + \sum_{i \leq l_2 < l_1 < j} \left\{ \begin{matrix} j \\ l_1 \end{matrix} \right\} \left\{ \begin{matrix} l_1 \\ l_2 \end{matrix} \right\} \begin{bmatrix} l_2 \\ i \end{bmatrix} + \sum_{i \leq l_3 < l_2 < l_1 < j} \left\{ \begin{matrix} j \\ l_1 \end{matrix} \right\} \left\{ \begin{matrix} l_1 \\ l_2 \end{matrix} \right\} \left\{ \begin{matrix} l_2 \\ l_3 \end{matrix} \right\} \begin{bmatrix} l_3 \\ i \end{bmatrix} + \dots + \left\{ \begin{matrix} j \\ j-1 \end{matrix} \right\} \left\{ \begin{matrix} j-1 \\ j-2 \end{matrix} \right\} \dots \left\{ \begin{matrix} i+1 \\ i \end{matrix} \right\} \begin{bmatrix} i \\ i \end{bmatrix} \right), \tag{4}$$

where

$$\left\{ \begin{matrix} k \\ i \end{matrix} \right\} = (-1) \binom{k}{i} \frac{\omega^{(k-i)}(0)}{\omega_0}, \tag{5}$$

$$\begin{bmatrix} k \\ i \end{bmatrix} = \frac{n!}{(n-k)!} (-1)^{k-i} \binom{k}{i} \frac{\omega_i}{\omega_0}. \tag{6}$$

$\binom{k}{i}$  is the binomial coefficient,  $\omega^{(k-i)}(0)$  is the  $(k-i)$ th derivative of  $\omega(t)$  at  $t = 0$ ,  $n$  is the degree of the rational curve (1), and  $\left\{ \begin{matrix} k \\ i \end{matrix} \right\} = \begin{bmatrix} k \\ i \end{bmatrix} = 0$ , when  $k < i$ .

**Proof.** To prove the result, Eq. (1) is re-written as  $\mathbf{R}(t)\omega(t) = \mathbf{P}(t)$ . Computing the  $k$ th derivative of its both sides by Leibnitz rule produces:

$$\sum_{i=0}^k \binom{k}{i} \mathbf{R}^{(i)} \omega^{(k-i)}(t) = \mathbf{P}^{(k)}(t). \tag{7}$$

Therefore,

$$\mathbf{R}^{(k)}(t) = \frac{\mathbf{P}^{(k)}(t) - \sum_{i=0}^{k-1} \binom{k}{i} \mathbf{R}^{(i)}(t) \omega^{(k-i)}(t)}{\omega(t)}. \tag{8}$$

The formulae for exact derivative (2)–(4) can be proved by induction.

First, simple computation shows that these formulae are true when  $k = 1$  and  $k = 2$  (see Eqs. (10) and (11)).

Now suppose that the coefficients of  $\mathbf{R}^{(k)}(0)$ ,  $1 < k < n$  are also given by Eqs. (3) and (4). Then, the  $(k+1)$ th derivative  $\mathbf{R}^{(k+1)}(0)$  can be calculated by Eq. (8), namely,

$$\begin{aligned} \mathbf{R}^{(k+1)}(0) &= \frac{1}{\omega_0} \left( (\mathbf{P}^{(k+1)}(0) - \mathbf{R}(0)\omega^{(k+1)}(0)) - \binom{k+1}{k} \mathbf{R}^{(k)}(0)\omega^{(1)}(0) - \binom{k+1}{k-1} \mathbf{R}^{(k-1)}(0)\omega^{(2)}(0) \right. \\ &\quad \left. - \dots - \binom{k+1}{1} \mathbf{R}^{(1)}(0)\omega^{(k)}(0) \right) = \sum_{i=1}^{k+1} \frac{n!}{(n-k-1)!} \binom{k+1}{i} (-1)^{k+1-i} \frac{\omega_i}{\omega_0} (\mathbf{P}_i - \mathbf{P}_0) \\ &\quad + (-1) \binom{k+1}{k} \frac{\omega^{(1)}(0)}{\omega_0} \mathbf{R}^{(k)}(0) + (-1) \binom{k+1}{k-1} \frac{\omega^{(2)}(0)}{\omega_0} \mathbf{R}^{(k-1)}(0) + \dots + (-1) \binom{k+1}{1} \frac{\omega^{(k)}(0)}{\omega_0} \mathbf{R}^{(1)}(0) \\ &= \sum_{i=1}^{k+1} \begin{bmatrix} k+1 \\ i \end{bmatrix} (\mathbf{P}_i - \mathbf{P}_0) + \left\{ \begin{matrix} k+1 \\ k \end{matrix} \right\} \mathbf{R}^{(k)}(0) + \left\{ \begin{matrix} k+1 \\ k-1 \end{matrix} \right\} \mathbf{R}^{(k-1)}(0) + \dots + \left\{ \begin{matrix} k+1 \\ 1 \end{matrix} \right\} \mathbf{R}^{(1)}(0) = \sum_{i=1}^{k+1} C_i^{k+1} (\mathbf{P}_i - \mathbf{P}_0). \end{aligned}$$

The term  $\mathbf{P}_{k+1} - \mathbf{P}_0$  is contained only in the term  $\sum_{i=1}^{k+1} \begin{bmatrix} k+1 \\ i \end{bmatrix} (\mathbf{P}_i - \mathbf{P}_0)$ , so its coefficient is  $C_{k+1}^{k+1} = \begin{bmatrix} k+1 \\ k+1 \end{bmatrix}$ . The term  $\mathbf{P}_k - \mathbf{P}_0$  is contained in the terms  $\sum_{i=0}^{k+1} \begin{bmatrix} k+1 \\ i \end{bmatrix} (\mathbf{P}_i - \mathbf{P}_0)$  and  $\mathbf{R}^{(k)}(0)$ , therefore its coefficient is

$$C_k^{k+1} = \begin{bmatrix} k+1 \\ k \end{bmatrix} + \left\{ \begin{matrix} k+1 \\ k \end{matrix} \right\} C_k^k = \begin{bmatrix} k+1 \\ k \end{bmatrix} + \left\{ \begin{matrix} k+1 \\ k \end{matrix} \right\} \begin{bmatrix} k \\ k \end{bmatrix}.$$

Generally, the term  $\mathbf{P}_i - \mathbf{P}_0$ ,  $i = 1, 2, \dots, k$  comes from the terms  $\sum_{i=0}^{k+1} \binom{k+1}{i} (\mathbf{P}_i - \mathbf{P}_0), \mathbf{R}^{(k)}(0), \mathbf{R}^{(k-1)}(0), \dots, \mathbf{R}^{(i)}(0)$ , so

$$C_i^{k+1} = \binom{k+1}{i} + \left\{ \begin{matrix} k+1 \\ k \end{matrix} \right\} C_i^k + \left\{ \begin{matrix} k+1 \\ k-1 \end{matrix} \right\} C_i^{k-1} + \dots + \left\{ \begin{matrix} k+1 \\ i \end{matrix} \right\} C_i^i. \tag{9}$$

Substituting the expressions of  $C_i^k, C_i^{k-1}, \dots, C_i^i$  into Eq. (9), leads to

$$C_i^{k+1} = \binom{k+1}{i} + \left\{ \begin{matrix} k+1 \\ k \end{matrix} \right\} \left( \binom{k}{i} + \sum_{i \leq l_1 < k} \left\{ \begin{matrix} k \\ l_1 \end{matrix} \right\} \binom{l_1}{i} + \sum_{i \leq l_2 < l_1 < k} \left\{ \begin{matrix} k \\ l_1 \end{matrix} \right\} \left\{ \begin{matrix} l_1 \\ l_2 \end{matrix} \right\} \binom{l_2}{i} + \dots + \left\{ \begin{matrix} k \\ k-1 \end{matrix} \right\} \dots \left\{ \begin{matrix} i+1 \\ i \end{matrix} \right\} \binom{i}{i} \right) \\ + \left\{ \begin{matrix} k+1 \\ k-1 \end{matrix} \right\} \left( \binom{k-1}{i} + \sum_{i \leq l_1 < k-1} \left\{ \begin{matrix} k-1 \\ l_1 \end{matrix} \right\} \binom{l_1}{i} + \dots + \left\{ \begin{matrix} k-1 \\ k-2 \end{matrix} \right\} \dots \left\{ \begin{matrix} i+1 \\ i \end{matrix} \right\} \binom{i}{i} \right) + \dots + \left\{ \begin{matrix} k+1 \\ i \end{matrix} \right\} \binom{i}{i}, \\ i = 1, 2, \dots, k.$$

By induction, the derivative formulae (2)–(4) are true.  $\square$

It should be pointed out that, when  $\omega_0 = \omega_1 = \dots = \omega_n$ , the rational Bézier curve (1) degenerates to the polynomial Bézier curve, and then,

$$\omega^{(k)}(0) = \frac{n!}{(n-k)!} \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} \omega_i = \frac{n!}{(n-k)!} \sum_{i=1}^k (-1)^{k-i} \binom{k}{i} (\omega_i - \omega_0) = 0,$$

where  $k = 1, 2, \dots, n$ . Therefore,  $\left\{ \begin{matrix} k \\ i \end{matrix} \right\} = 0$ ,  $k - i = 1, 2, \dots, n$ , and  $C_i^k = \binom{k}{i}$ ,  $i = 1, 2, \dots, k$ . Hence, the  $k$ th derivative (8) degenerates to

$$\mathbf{R}^k(0) = \sum_{i=1}^k \binom{k}{i} (\mathbf{P}_i - \mathbf{P}_0) = \frac{n!}{(n-k)!} \sum_{i=1}^k (-1)^{k-i} \binom{k}{i} (\mathbf{P}_i - \mathbf{P}_0) = \frac{n!}{(n-k)!} \Delta^k \mathbf{P}_0,$$

where  $\Delta^k$  is the  $k$ th difference. It is just the  $k$ th derivative of the degree  $n$  polynomial Bézier curve at  $t = 0$ .

So far, we have developed the formula for the exact derivative at corner  $t = 0$  of the rational curve (1). By the symmetry property of the Bézier curve, the derivative formula at another corner  $t = 1$  is similar. Further more, the exact derivative at arbitrary parameter  $t$ ,  $0 < t < 1$  can be calculated by subdividing the rational curve (1) at  $t$  using de Casteljau algorithm, and then the point at  $t$  becoming the corner of the sub-curve.

### 3. Examples

By the main result presented in Section 2, the arbitrary order derivative of the rational curve (1) at a corner can be computed easily. For example, we list its derivatives from first order to fifth order in this section.

**Example 1.** The first order derivative of the rational curve (1) at  $t = 0$  is,

$$\mathbf{R}'(0) = n \frac{\omega_1}{\omega_0} (\mathbf{P}_1 - \mathbf{P}_0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (\mathbf{P}_1 - \mathbf{P}_0) = C_1^1 (\mathbf{P}_1 - \mathbf{P}_0), \tag{10}$$

namely,  $C_1^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

**Example 2.** The second order derivative of the rational curve (1) at  $t = 0$  is,

$$\mathbf{R}^{(2)}(0) = \frac{n!}{(n-2)!} \frac{\omega_2}{\omega_0} (\mathbf{P}_2 - \mathbf{P}_0) - \left( \frac{n!}{(n-2)!} \frac{2\omega_1}{\omega_0} + 2n \frac{\omega_1}{\omega_0} \frac{\omega'(0)}{\omega_0} \right) (\mathbf{P}_1 - \mathbf{P}_0) \\ = \begin{bmatrix} 2 \\ 2 \end{bmatrix} (\mathbf{P}_2 - \mathbf{P}_0) + \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) (\mathbf{P}_1 - \mathbf{P}_0) \\ = C_2^2 (\mathbf{P}_2 - \mathbf{P}_0) + C_1^2 (\mathbf{P}_1 - \mathbf{P}_0); \tag{11}$$

namely,  $C_2^2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ ,  $C_1^2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

**Example 3.** The third order derivative of the rational curve (1) at  $t = 0$  is,

$$\mathbf{R}^{(3)}(0) = C_3^3 (\mathbf{P}_3 - \mathbf{P}_0) + C_2^3 (\mathbf{P}_2 - \mathbf{P}_0) + C_1^3 (\mathbf{P}_1 - \mathbf{P}_0) \\ = \begin{bmatrix} 3 \\ 3 \end{bmatrix} (\mathbf{P}_3 - \mathbf{P}_0) + \left( \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right) (\mathbf{P}_2 - \mathbf{P}_0) \\ + \left( \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) (\mathbf{P}_1 - \mathbf{P}_0). \tag{12}$$

**Example 4.** The fourth order derivative of the rational curve (1) at  $t = 0$  is,

$$\mathbf{R}^{(4)}(0) = \sum_{i=1}^4 C_i^4 (\mathbf{P}_i - \mathbf{P}_0), \quad (13)$$

where

$$C_4^4 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}; \quad (14)$$

$$C_3^4 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} \begin{bmatrix} 3 \\ 3 \end{bmatrix}; \quad (15)$$

$$C_2^4 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} \begin{bmatrix} 2 \\ 2 \end{bmatrix}; \quad (16)$$

$$C_1^4 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 4 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (17)$$

**Example 5.** The fifth order derivative of the rational curve (1) at  $t = 0$  is,

$$\mathbf{R}^{(5)}(0) = \sum_{i=1}^5 C_i^5 (\mathbf{P}_i - \mathbf{P}_0), \quad (18)$$

where

$$C_5^5 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}; \quad (19)$$

$$C_4^5 = \begin{bmatrix} 5 \\ 4 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \begin{bmatrix} 4 \\ 4 \end{bmatrix}; \quad (20)$$

$$C_3^5 = \begin{bmatrix} 5 \\ 3 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\} \begin{bmatrix} 3 \\ 3 \end{bmatrix}; \quad (21)$$

$$C_2^5 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ + \left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\} \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 2 \end{matrix} \right\} \begin{bmatrix} 2 \\ 2 \end{bmatrix}; \quad (22)$$

$$C_1^5 = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \left\{ \begin{matrix} 4 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} \left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ + \left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\} \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\} \left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\} \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ + \left\{ \begin{matrix} 5 \\ 2 \end{matrix} \right\} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 2 \end{matrix} \right\} \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left\{ \begin{matrix} 5 \\ 1 \end{matrix} \right\} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (23)$$

#### 4. Conclusion

The well-regulated formula of arbitrary order derivative of a rational Bézier curve at a corner is developed in this paper. Additionally, some examples are presented to demonstrate the computation of the exact derivative. It can be used in constructing smooth connection of curves, and evaluating high order differential quantities, such as normal vector, curvature, and torsion.

#### Acknowledgements

We thank the reviewers for their valuable comments. This paper is supported by 973 program of China (No. 2004CB719400), NSF of China (No. 60503057), and Natural Science Foundation of Zhejiang Province (No. R106449).

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