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A simple method for approximating rational Bézier curve using Bézier curves

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Abstract

This paper presents a simple method for approximating a rational Bézier curve with Bézier curve sequence, whose control points are those of degree-elevated rational Bézier curves. It is proved that the derivatives with any given degree of the Bézier curve sequence constructed this way would uniformly converge to the corresponding derivatives of the original rational Bézier curve.

Keywords: Rational Bézier curve; Approximation with polynomial; Degree elevation; Uniform convergence

1. Introduction

Because rational function (rational curve, rational surface) has wide and important application in CAGD and its forms, especially the forms of its derivatives and integrals, are quite complicated, the problem of approximating rational function with polynomials has been raised and studied. For example, Sederberg and Kakimoto (1991) presented the hybrid polynomial approximation to rational function, this topic has been studied widely, e.g. Wang and Sederberg (1994) studied the areas bounded by rational Bézier curves, based on which some convergence conditions for the polynomial approximation of rational curves were obtained in later papers. Based on a hybrid expression of rational function, Wang et al. (1997) presented Hermite polynomial approximations to rational Bézier curves and investigated the convergence condition for the polynomial approximation of rational functions and rational curves. Wang and Zheng (1997) studied the bounds on the moving controlling points of hybrid curves. Liu and Wang (2000) presented the recursive formulae for Hermite polynomial approximations to rational Bézier curves. Floater (2006) showed that many rational parametric curves could be interpolated, in a Hermite sense, by polynomial curves with lower degree relative to the number of data being interpolated.

This paper presents another method of approximating rational Bézier curve with Bézier curves, which could be generalized easily to the case of common rational function or curve or surface. That is to degree-elevate a given rational Bézier curve continuously, and to construct Bézier curves with the controlling points of the degree-elevated rational

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Bézier curves as their controlling points. Obviously, this constructing method is simpler much more. Moreover, the sequence of derivatives with any given degree of the polynomial curves converges uniformly to the corresponding one of the original rational curve.

2. The main result

A given rational Bézier curve of degree n

$$R(t)=\frac{\sum_{i=0}^{n} w_i P_i B_{i,n}(t)}{\sum_{i=0}^{n} w_i B_{i,n}(t)},\quad w_0,w_n>0,\quad w_i\geqslant 0\quad (i=1,2,\dots,n-1),$$

could be elevated continuously, so n would increase simultaneously. In order to show clearly the relationship between n and the elevated rational Bézier curve $R(t)$, $R(t)$ is represented as

$$R(t)=R_n(t)=\frac{\sum_{i=0}^{n} w_{i,n} P_{i,n} B_{i,n}(t)}{\sum_{i=0}^{n} w_{i,n} B_{i,n}(t)},$$

where $P_{i,n}$ and $w_{i,n}$ represent the controlling points and weights of the rational Bézier curve $R_n(t)$.

$R_n(t)$ could be elevated in this way: Let

$$Q_{i,n}=w_{i,n}P_{i,n},$$

then $\{w_{i,n}\}, \{P_{i,n}\}, \{Q_{i,n}\}$ have the recursive formulae

$$w_{i,n+1}=\frac{i}{n+1}w_{i-1,n}+\frac{n+1-i}{n+1}w_{i,n},\quad i=0,1,\dots,n+1,$$
$$Q_{i,n+1}=\frac{i}{n+1}Q_{i-1,n}+\frac{n+1-i}{n+1}Q_{i,n},\quad i=0,1,\dots,n+1,$$
$$P_{i,n+1}=\frac{Q_{i,n+1}}{w_{i,n+1}}=\frac{iQ_{i-1,n}+(n+1-i)Q_{i,n}}{iw_{i-1,n}+(n+1-i)w_{i,n}},\quad i=0,1,\dots,n+1,$$

where for $i<0$ or $i>n$, the corresponding $w_{i,n}, Q_{i,n}, P_{i,n}$ do not exist, or they are regarded as zero.

Since $n\to\infty$ and the initial $w_0, w_n>0$, the formula for $w_{i,n}$ implies for n large enough all $w_{i,n}$ $(i=0,1,\dots,n)$ would be positive. Thus we can assume all $w_{i,n}>0$ in the posterior part of this paper.

If we construct Bézier curves $b_n(t)=\sum_{i=0}^{n} P_{i,n}B_{i,n}(t)$ using the controlling points of the elevated rational Bézier curves $\{P_{i,n}, i=0,1,\dots,n\}$, then the obtained Bézier curve sequence possesses the properties as mentioned in the following theorem:

Theorem. *For any given nonnegative integer number r, the rth derivatives of Bézier curve sequence $\{b_n(t)\}$ on $[0,1]$ converge uniformly to the derivative with the same degree of original rational Bézier curve $R(t)$, i.e.*

$$b_n^{(r)}(t)\rightrightarrows R^{(r)}(t)\quad (n\to\infty, t\in[0,1],\ \textit{for any integer } r\geqslant 0),$$

where '$\rightrightarrows$' stands for converging uniformly.

3. The proof of theorem

Firstly the following two lemmas are introduced:

Lemma 1. *(See Farin, 1999.) As $R_n(t)=\frac{\sum_{i=0}^{n} w_{i,n}P_{i,n}B_{i,n}(t)}{\sum_{i=0}^{n} w_{i,n}B_{i,n}(t)}$ being elevated, $\{P_{i,n}\}$ converge to $\{R(i/n)\}$ uniformly, i.e.*

$$P_{i,n}\rightrightarrows R(i/n)\quad \textit{or}\quad \left|P_{i,n}-R(i/n)\right|\rightrightarrows 0\quad (n\to\infty; i=0,1,\dots,n).$$

Lemma 2 *(A well-known property of Bernstein polynomials). Let $B_n(t)=\sum_{i=0}^{n} R(i/n)B_{i,n}(t)$, then for any fixed nonnegative integer r,*

$$\left|B_n^{(r)}(t)-R^{(r)}(t)\right|=\left|\sum_{i=0}^{n} R(i/n)B_{i,n}^{(r)}(t)-R^{(r)}(t)\right|\rightrightarrows 0\quad (n\to\infty, \forall t\in[0,1]).$$

The lemmas complete the proof of theorem immediately:

$$\begin{aligned}\left|b_n^{(r)}(t) - R^{(r)}(t)\right| &= \left|\sum_{i=0}^{n} P_{i,n} B_{i,n}^{(r)}(t) - R^{(r)}(t)\right| \\ &\leqslant \left|\sum_{i=0}^{n}\left[P_{i,n} - R(i/n)\right] B_{i,n}^{(r)}(t)\right| + \left|\sum_{i=0}^{n} R(i/n) B_{i,n}^{(r)}(t) - R^{(r)}(t)\right| \\ &\leqslant \sum_{i=0}^{n}\left|P_{i,n} - R(i/n)\right| B_{i,n}^{(r)}(t) + \left|\sum_{i=0}^{n} R(i/n) B_{i,n}^{(r)}(t) - R^{(r)}(t)\right| \\ &\rightrightarrows 0 \quad (n \to \infty).\end{aligned}$$

4. Conclusion

This paper presents a simple method for approximating a rational Bézier curve with Bézier curve sequence. The method can be generalized to the cases of general rational functions, including the case of rational B-spline, furthermore, to the case of rational surface.

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