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• **HIGHLIGHT** •

# Trivariate B-spline Solid Construction by Pillow Operation and Geometric Iterative Fitting

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While the traditional geometric design community focuses on the design of curves and surfaces, the advent of Isogeometric Analysis (IGA) [1] has made the development of methods for designing Trivariate B-spline Solids (TBSs) imperative. In IGA, a *valid* TBS should have a positive Jacobian value at every point in its domain. A negative Jacobian value at any point in the domain of the TBS can render the IGA invalid.

In this paper, we developed a method that can generate a TBS with a guaranteed positive Jacobian, if the initial TBS is valid. Using a tetrahedral (tet) mesh with six surfaces segmented on its boundary mesh as the input, we first partition the tet mesh model into seven sub-volumes using the *pillow* operation [2], a method originally developed for improving the quality of hexahedral meshes. After each of them is parameterized into a cubic parameter domain, seven initial valid TBSs are constructed. Moreover, starting with the initial valid TBSs, the boundary curves, boundary surfaces and the TBSs are fitted by a geometric iterative fitting algorithm, known as the Geometric Feasible Direction (GFD) algorithm. In each iteration of the GFD algorithm, the movements of the control points are restricted inside a feasible region to ensure the validity. Finally, the smoothness between two adjacent TBSs is improved by the GFD algorithm. In this way, the validity of the generated TBSs is guaranteed with desirable smoothness between adjacent TBSs.

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The authors declare that they have no conflict of interest.

In this highlight, we will list the main results and main algorithmic steps for generating the valid TBSs. For more detail, please refer to the supporting information to this highlight.

**Validity conditions:** What we want to generate is a composition of valid TBSs with as desirable as possible smoothness between adjacent TBSs. So, the validity conditions and geometric continuity definition between TBSs should be clarified.

Given a B-spline curve of degree $d$ with control points $\boldsymbol{P}_i, i = 0, 1, \cdots, m$, and denoting the difference vectors as,

$$\boldsymbol{T}_i = \frac{\boldsymbol{P}_{i+1} - \boldsymbol{P}_i}{\|\boldsymbol{P}_{i+1} - \boldsymbol{P}_i\|},\ i = 0, 1, \cdots, m-1, \qquad (1)$$

the validity condition for the B-spline curve is:

**Proposition 1** (Validity condition for B-spline curves)**.** A B-spline curve of degree $d$ is valid if the apertures of the minimum circular cones enclosing the difference vectors (1)

$$\{\boldsymbol{T}_i, \boldsymbol{T}_{i+1}, \cdots, \boldsymbol{T}_{i+d-1}\},\ i = 0, 1, \cdots, m-d,$$

respectively, are all less than $\pi$. □

Moreover, suppose we are given a B-spline surface of degree $d_u \times d_v$ with control points

$$\boldsymbol{S}_{ij}, i = 0, 1, \cdots, m,\ j = 0, 1, \cdots, n,$$

and denote the difference vectors as

$$\boldsymbol{T}^u_{ij} = \frac{\boldsymbol{S}_{i+1,j} - \boldsymbol{S}_{ij}}{\|\boldsymbol{S}_{i+1,j} - \boldsymbol{S}_{ij}\|}, \qquad (2)$$

and,

$$\boldsymbol{T}^v_{kl} = \frac{\boldsymbol{S}_{k,l+1} - \boldsymbol{S}_{kl}}{\|\boldsymbol{S}_{k,l+1} - \boldsymbol{S}_{kl}\|}. \qquad (3)$$

Let $\mathcal{M}_{IJ}, I = 0, 1, \cdots, m-d_u, J = 0, 1, \cdots, n-d_v$ be the sub-control-polygon constituted by the control points,

$$\begin{matrix} \boldsymbol{S}_{IJ} & \boldsymbol{S}_{I,J+1} & \cdots & \boldsymbol{S}_{I,J+d_v} \\ \boldsymbol{S}_{I+1,J} & \boldsymbol{S}_{I+1,J+1} & \cdots & \boldsymbol{S}_{I+1,J+d_v} \\ \vdots & \vdots & & \vdots \\ \boldsymbol{S}_{I+d_u,J} & \boldsymbol{S}_{I+d_u,J+1} & \cdots & \boldsymbol{S}_{I+d_u,J+d_v} \end{matrix}$$

Moreover, suppose $\boldsymbol{U}_{IJ}$ and $\boldsymbol{V}_{IJ}$ are the unit axis vectors of the minimum circular cones $\mathcal{C}^u_{IJ}$ and $\mathcal{C}^v_{IJ}$ enclosing the difference vectors $\boldsymbol{T}^u_{ij}$ (2) and $\boldsymbol{T}^v_{kl}$ (3) of the sub-control-polygon $\mathcal{M}_{IJ}$, starting from the apexes of cones, respectively. Then, a sufficient condition for the validity of a B-spline surface is presented as follows:

**Proposition 2** (Validity condition for B-spline surfaces)**.** If $\boldsymbol{T}^u_{ij} \cdot \boldsymbol{U}_{IJ} > \boldsymbol{T}^u_{ij} \cdot \boldsymbol{V}_{IJ} \geqslant 0$, and, $\boldsymbol{T}^v_{kl} \cdot \boldsymbol{V}_{IJ} > \boldsymbol{T}^v_{kl} \cdot \boldsymbol{U}_{IJ} \geqslant 0$, where $\boldsymbol{T}^u_{ij}$ and $\boldsymbol{T}^v_{kl}$ are defined on each sub-control-polygon $\mathcal{M}_{IJ}$, $I = 0, 1, \cdots, m-d_u, J = 0, 1, \cdots, n-d_v$, the B-spline surface is valid. □

Similarly, we can develop a sufficient condition for determining the validity of a TBS $\boldsymbol{H}(u, v, w)$ of degree $d_u \times d_v \times d_w$, with control points,

$$\boldsymbol{H}_{ijk}, i = 0, 1, \cdots, m, j = 0, 1, \cdots, n, k = 0, 1, \cdots, l.$$

Denote the difference vectors as

$$\boldsymbol{T}^u_{ijk} = \frac{\boldsymbol{H}_{i+1,j,k} - \boldsymbol{H}_{ijk}}{\|\boldsymbol{H}_{i+1,j,k} - \boldsymbol{H}_{ijk}\|},$$
$$\boldsymbol{T}^v_{ijk} = \frac{\boldsymbol{H}_{i,j+1,k} - \boldsymbol{H}_{ijk}}{\|\boldsymbol{H}_{i,j+1,k} - \boldsymbol{H}_{ijk}\|},$$
$$\boldsymbol{T}^w_{ijk} = \frac{\boldsymbol{H}_{i,j,k+1} - \boldsymbol{H}_{ijk}}{\|\boldsymbol{H}_{i,j,k+1} - \boldsymbol{H}_{ijk}\|}.$$

Moreover, letting

$$\mathcal{G}_{IJK}, I = 0, 1, \cdots, m-d_u, \ J = 0, 1, \cdots, n-d_v, \ K = 0, 1, \cdots, l-d_w,$$

be the sub-grid constituted by the control points

$$\boldsymbol{H}_{ijk}, \ i = I, I+1, \cdots, I+d_u, \ j = J, J+1, \cdots, J+d_v, k = K, K+1, \cdots, K+d_w,$$

we have,

**Proposition 3** (Validity condition for TBSs)**.** If

$$\boldsymbol{T}^u_{i_u j_u k_u} \cdot (\boldsymbol{T}^v_{i_v j_v k_v} \times \boldsymbol{T}^w_{i_w j_w k_w}) > 0,$$

where $\boldsymbol{T}^u_{i_u j_u k_u}, \boldsymbol{T}^v_{i_v j_v k_v}$ and $\boldsymbol{T}^w_{i_w j_w k_w}$ are defined on each sub-grid $\mathcal{G}_{IJK}$, the TBS $\boldsymbol{H}(u, v, w)$ is valid. □

The following proposition presents a sufficient condition for the $G^1$ geometric continuity between two TBSs $\boldsymbol{P}(u, v, w)$, with control points $\boldsymbol{P}_{ijk}$, and $\boldsymbol{Q}(\mu, \nu, \omega)$, with control points $\boldsymbol{Q}_{ijk}$, along their common boundary surface $\boldsymbol{P}(u, v, 1) = \boldsymbol{Q}(\mu, \nu, 0)$.

**Proposition 4.** Suppose the two TBSs $\boldsymbol{P}(u, v, w)$ and $\boldsymbol{Q}(\mu, \nu, \omega)$ have uniform knot vectors with Bézier end condition, respectively. If

$$\boldsymbol{P}_{i,j,l_p} = \boldsymbol{Q}_{i,j,0}, \text{ and,}$$
$$\boldsymbol{P}_{i,j,l_p} - \boldsymbol{P}_{i,j,l_p-1} = \alpha(\boldsymbol{Q}_{i,j,1} - \boldsymbol{Q}_{i,j,0}),$$

where $\alpha > 0$ is a positive constant, the two TBSs $\boldsymbol{P}(u, v, w)$ and $\boldsymbol{Q}(\mu, \nu, \omega)$ are $G^1$ geometric continuous along their common boundary surface $\boldsymbol{P}(u, v, 1) = \boldsymbol{Q}(\mu, \nu, 0)$. □

**Partition of the tet mesh model by pillow operation:** In order to perform the pillow operation on the input tet mesh, the tet mesh model (Fig. 1(a)) is first parameterized into a cubic parameter domain $\Omega = [0, 1] \times [0, 1] \times [0, 1]$ by the volume parameterization method [3] (Fig. 1(b)). Then, the parameter domain $\Omega$ shrinks to the sub-domain,

$$\Omega_c = [\frac{1}{3}, \frac{2}{3}] \times [\frac{1}{3}, \frac{2}{3}] \times [\frac{1}{3}, \frac{2}{3}].$$

As illustrated in Fig. 1(b), the vertices of the cubes $\Omega$ and $\Omega_c$ are denoted as $v_0, v_1, \cdots, v_7$, and $v^c_0, v^c_1, \cdots, v^c_7$, respectively. Connecting $v_i$ to $v^c_i, i = 0, 1, \cdots, 7$ generates six sub-domains $\Omega_u, \Omega_d, \Omega_l, \Omega_r, \Omega_f$, and $\Omega_b$. For example, the sub-domain $\Omega_u$ is enclosed by the six faces

$$v_0v_1v_2v_3, \ v^c_0v^c_1v^c_2v^c_3, \ v_0v^c_0v^c_1v_1,$$
$$v_1v^c_1v^c_2v_2, \ v_2v^c_2v^c_3v_3, \ v_3v^c_3v^c_0v_0.$$

Mapping the seven sub-domains into the original tet mesh model produces seven partitioned sub-volumes (Fig. 1(c)).

**Construction of the initial TBSs:** After the input tet mesh model is partitioned into seven sub-volumes. Each of them are parameterized into the cubic parameter domain by the volume parameterization method developed in [3]. Note that the parameterization on the common boundary curves and common boundary surfaces of adjacent sub-volumes should conform with each other. Moreover, each cubic parameter domain is sampled into a $(M+1) \times (N+1) \times (K+1)$ grid. Similar to the parameterization, the grid on the common boundary curves and common boundary surfaces of adjacent sub-volumes should be the same. Mapping the grids into the corresponding sub-volumes leads to the control grids of the initial TBSs, whose knot vectors are uniformly distributed in $[0, 1] \times [0, 1] \times [0, 1]$ with Bézier end

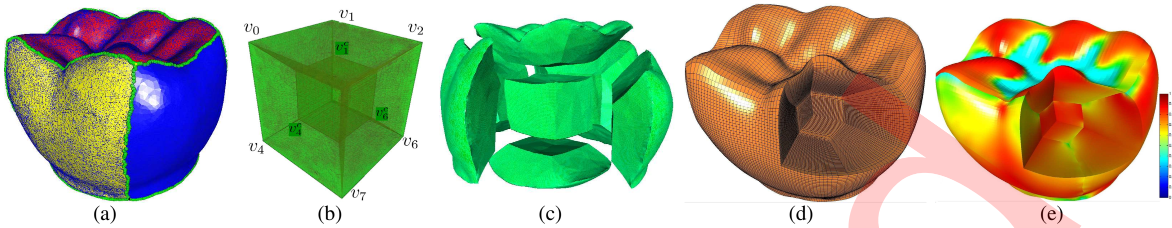


**Figure 1** Generation of the trivariate B-spline solid by the pillowing operation and geometric iterative fitting. (a) The input to the developed algorithm is a tet mesh with six surfaces segmented on its boundary mesh. (b) The tet mesh is parameterized into the cubic domain $[0,1] \times [0,1] \times [0,1]$, which is partitioned into seven sub-domains. (c) Mapping the seven sub-domains into the tet mesh model leads to the seven partitioned sub-volume meshes. (d) Cut-away view of the generated TBSs. (e) Distribution of the scaled Jacobian values on the TBSs.

conditions. Owing to the conformity of the parameterization and control grids on the common boundary curves and boundary surfaces, there is the unique control grid on each boundary curve or boundary surface.

**Geometric iterative fitting:** The result generated by the developed method is a composition of seven valid TBSs. As stated above, the objective function for guaranteeing the validity of a TBS is highly nonlinear with a large number of unknowns, so the optimization is prone to fail. Even if we can find a solution, the computation for solving the optimization problem is complicated, owing to a significant number of unknowns. To reduce the difficulty in guaranteeing the validity of the TBSs, we solve this problem step by step, in the order of,

(1) boundary curve fitting,

(2) boundary surface fitting, and,

(3) TBS fitting.

As mentioned above, the first objective we want to reach is that the generated TBSs should be valid, that is, the Jacobian value at any point of each TBS should be greater than 0. This means that the boundary curves and boundary surfaces of TBSs should also be valid. After the TBS fitting is completed, the smoothness between two adjacent TBSs, and the fairness of each TBS are improved. The boundary curve fitting, boundary surface fittig, TBS fitting and the smoothness and fairness improvement are performed by solving the corresponding constrained minimization problems.

**Results:** With the devloped method, we generated six TBS models. All of them are valid. Actually, in each of the six TBS models, the Jacobian values are larger than 0.5 in over 80% region. In Fig. 1(d), the cut-away views of the TBS models are illustrated. It can be seen that the iso-parametric curves vary smoothly not only inside a single TBS, but between two adjacent TBSs as well. Moreover, in Fig. 1(e), the distribution of the scaled Jacobian values [4] of the TBSs is visualized with different colors. The darker the red color, the higher the scaled Jacobian values. As shown in Fig. 1(e), the scaled Jacobian values of the model are all positive. A majority of region of the TBS model is in red.

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**Supporting information** The supporting information is available online at info.scichina.com and link.springer.com. The supporting materials are published as submitted, without typesetting or editing. The responsibility for scientific accuracy and content remains entirely with the authors.

## References

1 Hughes J, Cottrell J, Bazilevs Y. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. Computer methods in applied mechanics and engineering, 2005, 194 (39) : 4135–4195.

2 Mitchell S A, Tautges T J, Pillowing doublets: refining a mesh to ensure that faces share at most one edge. In: Proceedings of the 4th International Meshing Roundtable, Citeseer, 1995. 231–240.

3 Lin H, Jin S, Hu Q, et al. Constructing B-spline solids from tetrahedral meshes for isogeometric analysis. Computer Aided Geometric Design, 2015, 35 : 109–120.

4 Knupp P. A method for hexahedral mesh shape optimization. International Journal for Numerical Methods in Engineering, 2003, 58 (2) : 319–332.